Perturbatively including inhomogeneities in axion inflation

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Based on 2310.09186

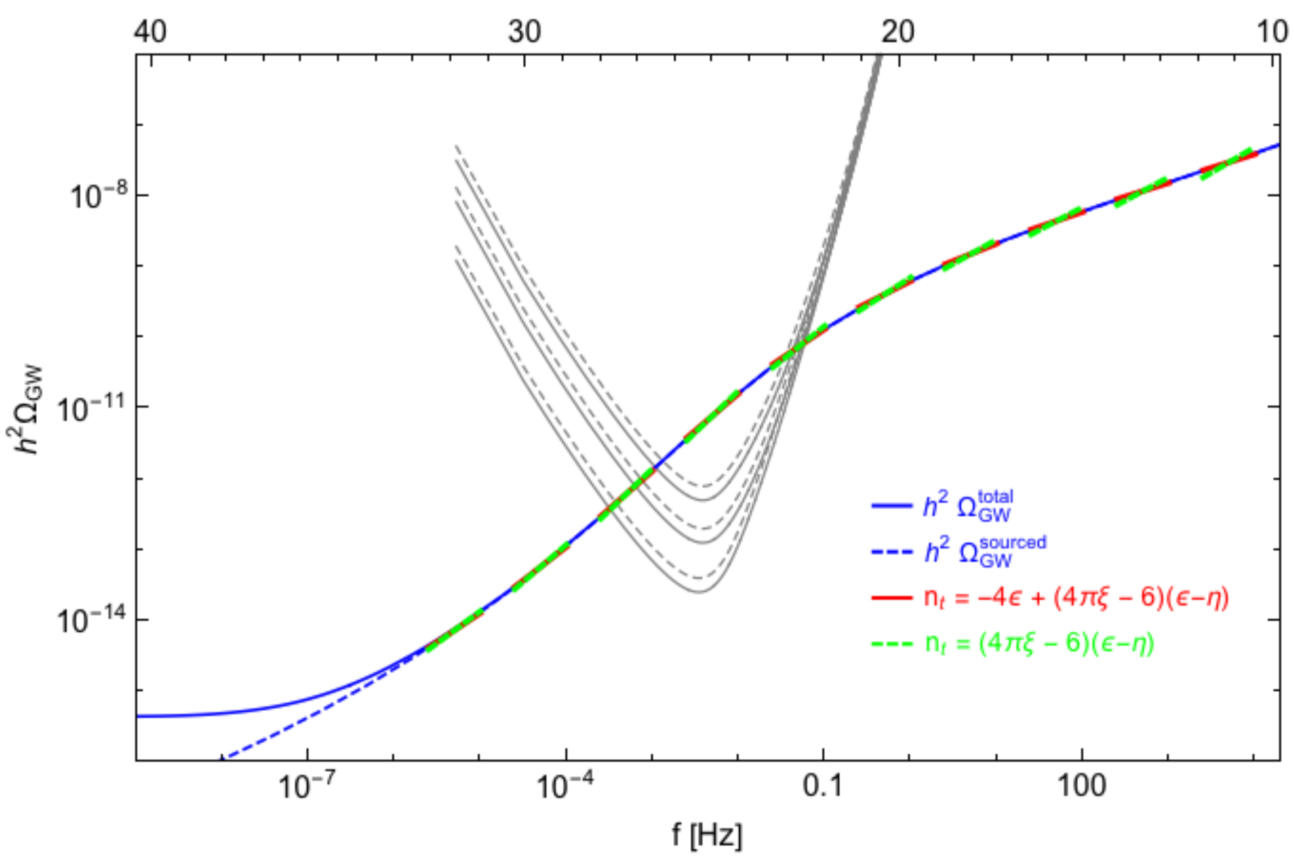
with Valerie Domcke and Stefan Sandner

Axion inflation

- Axion inflation: originally motivated by shift symmetry. [Freese+90]
- Shift symmetric coupling $\phi F\tilde{F}$ tachyonically produce gauge bosons.

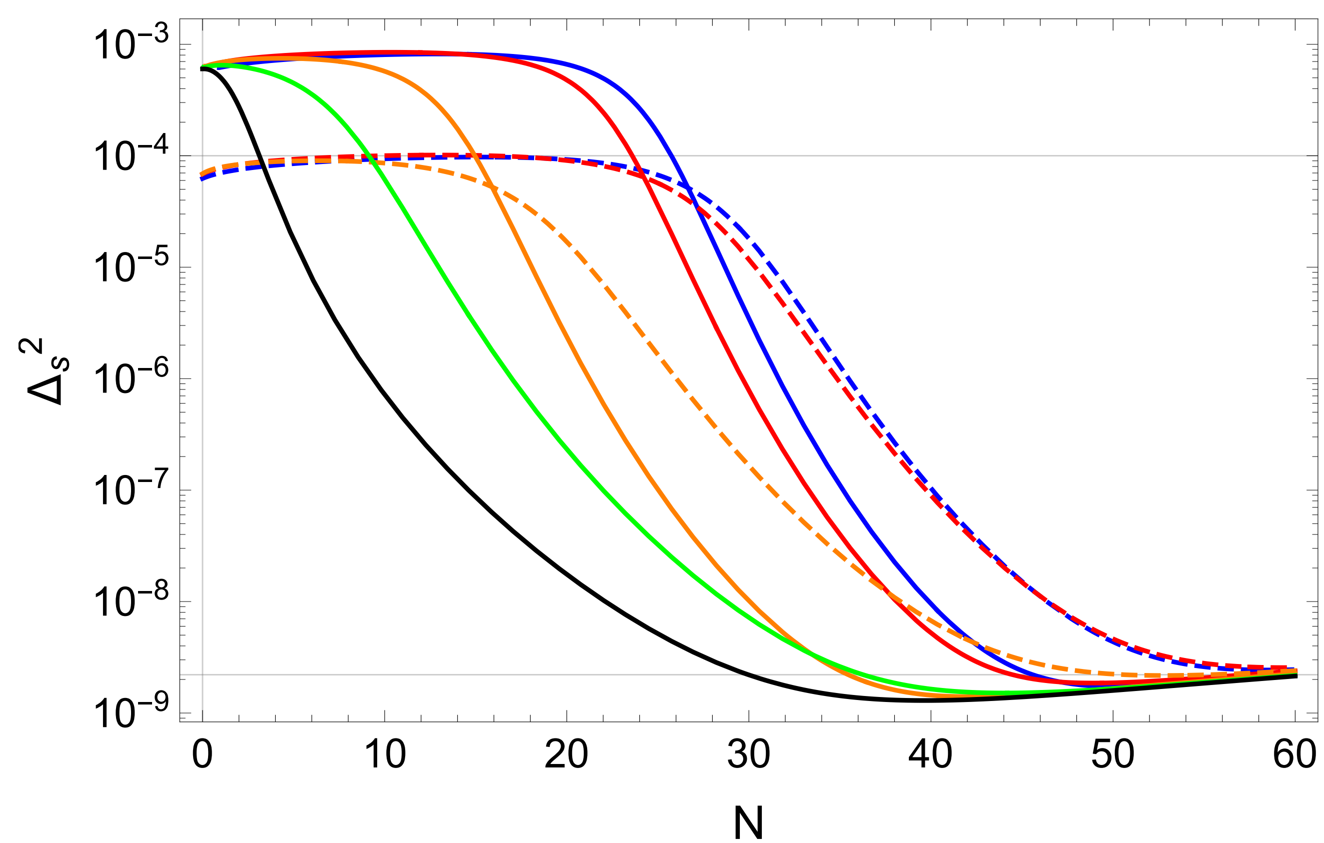
  ➡ Rich and interesting phenomenology.

Chiral GW production



[Bartolo+16]

PBH production



[Domcke+17]

Gauge field production

[Anber, Sorbo 09; ...]

- Axion inflation with $\mathrm{U}(1)$ gauge boson coupling:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{\beta\phi}{4M_P}F_{\mu\nu}\tilde{F}^{\mu\nu}\right].$$

- Axion velocity modifies the dispersion relation:

$$0 = \left[\frac{d^2}{d\eta^2} + k\left(k \pm 2aH\xi\right)\right]A_\pm(\eta,\vec{k}), \quad \xi = \frac{\beta\dot{\phi}}{2HM_P}.$$

➡ exponential enhancement: $A_- \simeq \frac{1}{\sqrt{2k}}\left(\frac{k}{2\xi aH}\right)^{1/4} e^{\pi\xi - 2\sqrt{2k\xi/aH}}$ for $\dot{\phi} > 0$.

(Expected to be Gaussian.)

- Helical gauge boson production towards the end of inflation:

$$\langle\vec{E}^2\rangle \simeq 2.6\times10^{-4}\frac{e^{2\pi\xi}}{\xi^3}H^4, \quad \langle\vec{B}^2\rangle \simeq 3.0\times10^{-4}\frac{e^{2\pi\xi}}{\xi^5}H^4, \quad \langle\vec{E}\cdot\vec{B}\rangle \simeq 2.6\times10^{-4}\frac{e^{2\pi\xi}}{\xi^4}H^4.$$

(ignoring backreaction)

Strong backreaction regime

- Backreaction necessarily included to make a prediction for large β.

- Two (or more) methods in the market:

  1) Classical lattice simulation [Caravano+22; Figueroa+23]

     ✔ No approximation (except for classical approximation which is fairly good).

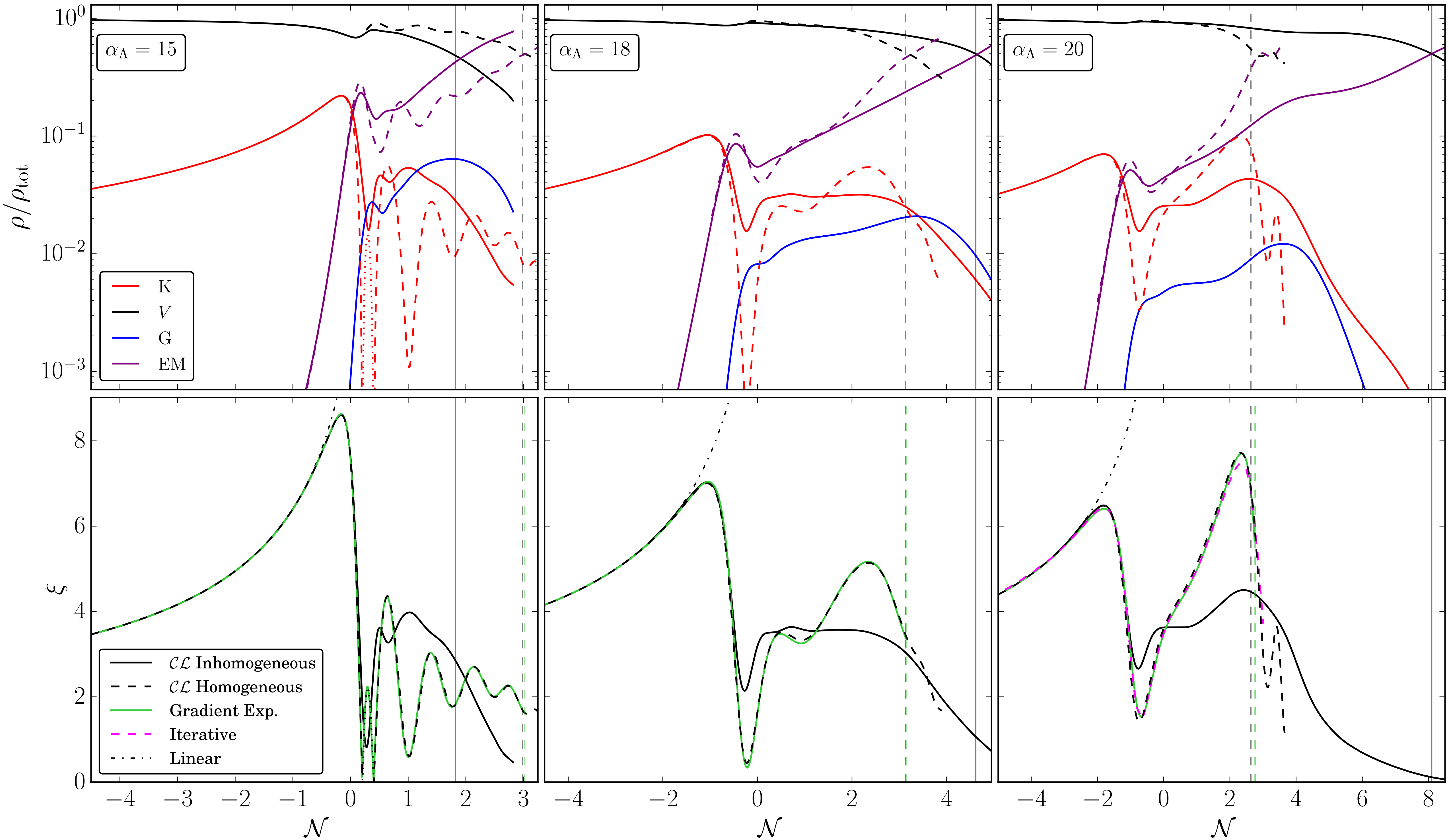
     ✘ Numerically expensive and time consuming. (Note that this is during inflation.)

  2) Gradient expansion formalism (GEF) [Sobol+ 19, 20; Gorbar+21]

     ✔ Numerically cheap and fast.

     ✘ Ignoring axion inhomogeneities $\rightarrow$ quantitative difference from lattice.

Lattice vs. GEF



From [Figueroa, Lizarraga, Urio, Urrestilla 23]

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     ➡ Is there a way to include axion inhomogeneties in GEF?

Outline

1. Introduction
2. Gradient expansion: 2pt functions
3. Gradient expansion: 3pt functions
4. Summary

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"Linearized" EoM

- Equation of motion given by

$$0 = \ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2 \phi}{a^2} + m_\phi^2 \phi - \frac{\beta}{M_P} \vec{E} \cdot \vec{B} \ ,$$

$$0 = \dot{\vec{E}} + 2H\vec{E} - \frac{1}{a}\vec{\nabla} \times \vec{B} + \frac{\beta}{M_P}\dot{\phi}\vec{B} + \frac{\beta}{M_P}\frac{1}{a}\vec{\nabla}\phi \times \vec{E},$$

$$0 = \dot{\vec{B}} + 2H\vec{B} + \frac{1}{a}\vec{\nabla} \times \vec{E},$$

$$0 = \vec{\nabla} \cdot \vec{E} + \frac{\beta}{M_P}\vec{\nabla}\phi \cdot \vec{B}, \quad 0 = \vec{\nabla} \cdot \vec{B} \,.$$

- Gradient expansion formalism: ignore axion inhomogeneity $\phi = \phi(t)$.

EoM: linear in inhomogeneous quantities, i.e. $\vec{E}$ and $\vec{B}$.

System closed within 2pt functions:

$$\langle \vec{E} \cdot (\vec{\nabla} \times)^n \vec{E} \rangle, \quad \langle \vec{B} \cdot (\vec{\nabla} \times)^n \vec{B} \rangle, \quad \langle \vec{E} \cdot (\vec{\nabla} \times)^n \vec{B} \rangle \,.$$

[Sobol+19, 20; Gorbar+21]

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[Sobol+19, 20; Gorbar+21]

Tower of 2pt functions

- System closed within 2pt functions $\mathscr{P}^{(n)}_{XY} = a^{-n-4}\langle \vec{X} \cdot (\vec{\nabla} \times)^n \vec{Y} \rangle$: [Sobol+19, 20; Gorbar+21]

$$\dot{\mathscr{P}}^{(n)}_E + (n+4)H\mathscr{P}^{(n)}_E - \frac{2\beta\dot{\phi}}{M_P}\mathscr{P}^{(n)}_{EB} + 2\mathscr{P}^{(n+1)}_{EB} = \left[\dot{\mathscr{P}}^{(n)}_E\right]_b ,$$

$$\dot{\mathscr{P}}^{(n)}_B + (n+4)H\mathscr{P}^{(n)}_B - 2\mathscr{P}^{(n+1)}_{EB} = \left[\dot{\mathscr{P}}^{(n)}_B\right]_b ,$$

$$\dot{\mathscr{P}}^{(n)}_{EB} + (n+4)H\mathscr{P}^{(n)}_{EB} - \mathscr{P}^{(n+1)}_E + \mathscr{P}^{(n+1)}_B - \frac{\beta\dot{\phi}}{M_P}\mathscr{P}^{(n)}_B = \left[\dot{\mathscr{P}}^{(n)}_{EB}\right]_b ,$$

- Boundary term originating from sub-horizon quantum fluctuations

e.g. $\left[\dot{\mathscr{P}}^{(n)}_E\right]_b = \frac{\dot{k}_h}{a^{n+4}}\frac{k_h^2}{2\pi^2}\sum_{\sigma=\pm}(\sigma k_h)^n \left|\frac{dA_\sigma}{d\tau}\right|^2_{k=k_h}$ where $k_h = \max_{t' \le t}(2aH\xi)$: "horizon".

- Infinite tower in derivatives truncated by extrapolation at $n_{\max} \sim \mathcal{O}(100)$.

$$\bar{\mathscr{P}}^{(n_{\max}+1)}_X = \sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l}\bar{\mathscr{P}}^{(n_{\max}+1-2l)}_X ,$$ where $\bar{\mathscr{P}}^{(n)}_X = \mathscr{P}^{(n)}_X / H_0^4 (k_h/a)^n$.

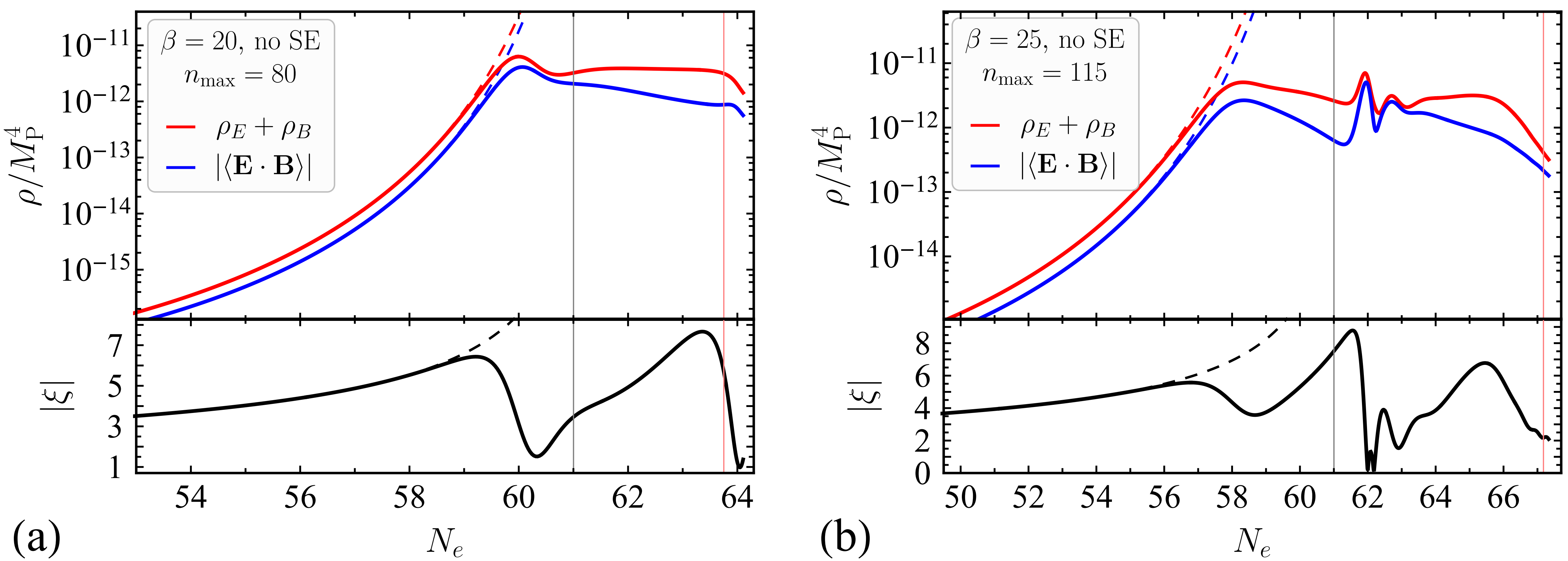
$L=1$ used in original ref, $L>1$ increases the stability of the system [Domcke, YE, Sandner 23].

Numerical result

- Showing oscillatory features after backreaction.

  (confirming [Domcke+20])

- Numerically extremely cheap (a few min at most with my laptop).



From [Gorbar, Schmitz, Sobol, Vilchinskii 21]

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Including inhomogeneities

- Equation of motion with inhomogeneity $\phi = \phi(t) + \chi$:

$$0 = \ddot{\phi} + 3H\dot{\phi} + m_\phi^2 \phi - \frac{\beta}{M_P}\left\langle \vec{E}\cdot\vec{B} \right\rangle ,$$

$$0 = \ddot{\chi} + 3H\dot{\chi} - \frac{\nabla^2\chi}{a^2} + m_\phi^2\chi - \frac{\beta}{M_P}\left(\vec{E}\cdot\vec{B} - \left\langle \vec{E}\cdot\vec{B} \right\rangle\right) ,$$

$$0 = \dot{\vec{E}} + 2H\vec{E} - \frac{1}{a}\vec{\nabla}\times\vec{B} + \frac{\beta}{M_P}\left(\dot{\phi} + \dot{\chi}\right)\vec{B} + \frac{\beta}{M_P}\frac{1}{a}\vec{\nabla}\chi\times\vec{E} ,$$

$$0 = \dot{\vec{B}} + 2H\vec{B} + \frac{1}{a}\vec{\nabla}\times\vec{E} , \quad 0 = \vec{\nabla}\cdot\vec{E} + \frac{\beta}{M_P}\vec{\nabla}\chi\cdot\vec{B} , \quad 0 = \vec{\nabla}\cdot\vec{B} ,$$

➡ EoM non-linear in inhomogeneous quantities.

$$\frac{d}{dt}(2\text{pt}) = \ldots + (3\text{pt}), \;\; \frac{d}{dt}(3\text{pt}) = \ldots + (4\text{pt}), \quad \cdots .$$

- One can truncate the tower of p-pt functions by factorization:

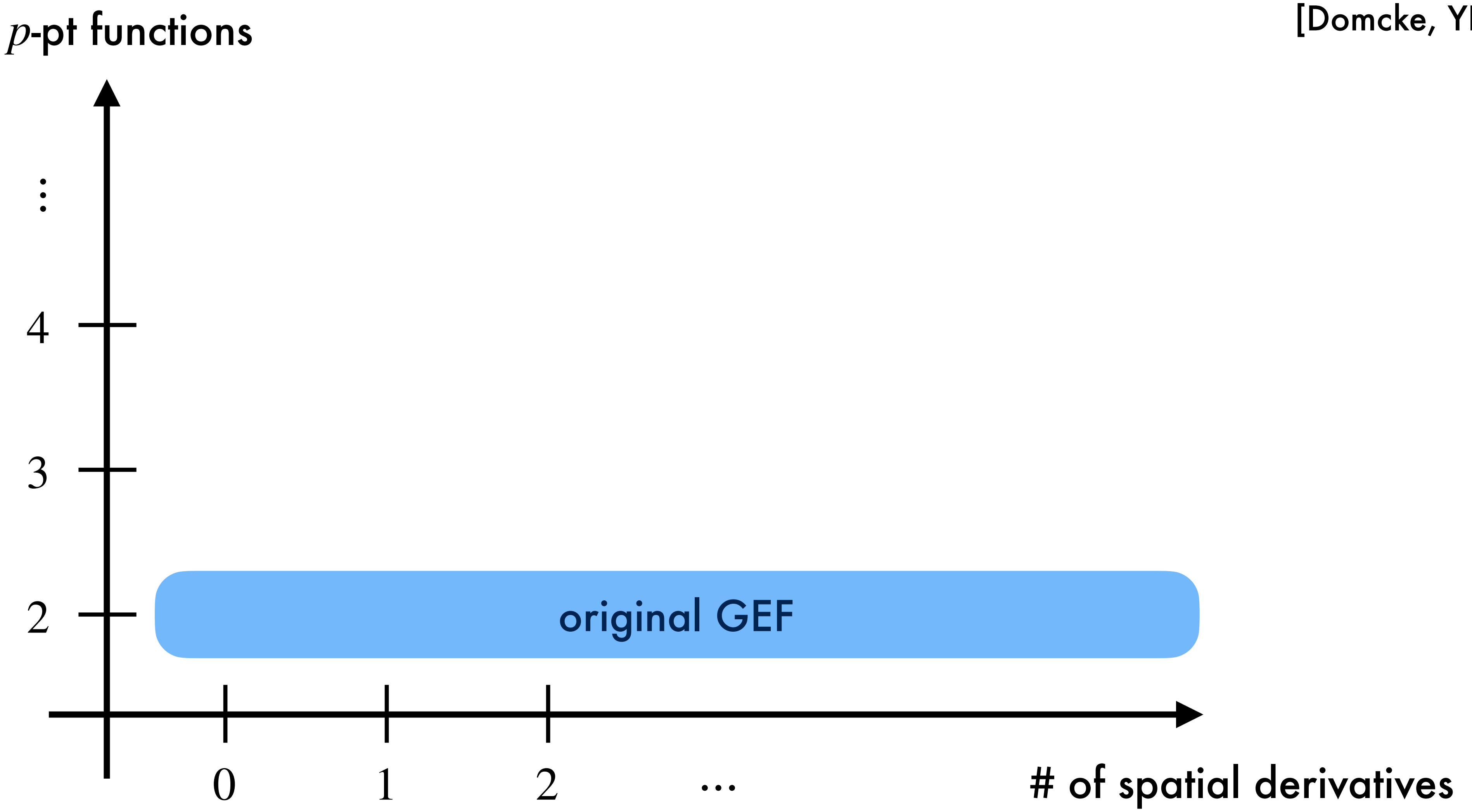
$$\langle(\vec{X}\cdot\vec{Y})(\vec{Z}\cdot\vec{W})\rangle \to \langle\vec{X}\cdot\vec{Y}\rangle\langle\vec{Z}\cdot\vec{W}\rangle + \frac{1}{3}\langle\vec{X}\cdot\vec{Z}\rangle\langle\vec{Y}\cdot\vec{W}\rangle + \frac{1}{3}\langle\vec{X}\cdot\vec{W}\rangle\langle\vec{Y}\cdot\vec{Z}\rangle$$

(Wick contraction: works when the fluctuations are close to Gaussian.)

Truncation of infinite towers

- Include 3pt fn with up to one spatial derivative (as the lowest order approx)

[Domcke, YE, Sandner 23]



Can be extended to higher orders.
# of correlators eventually diverges, but not a problem unless going to very high order.

- Monitor axion gradient energy to check the validity of our approx.

$$R_\chi \equiv \left| \frac{\langle (\nabla \chi)^2 \rangle}{\dot{\phi}^2 + \langle \dot{\chi}^2 \rangle} \right| .$$

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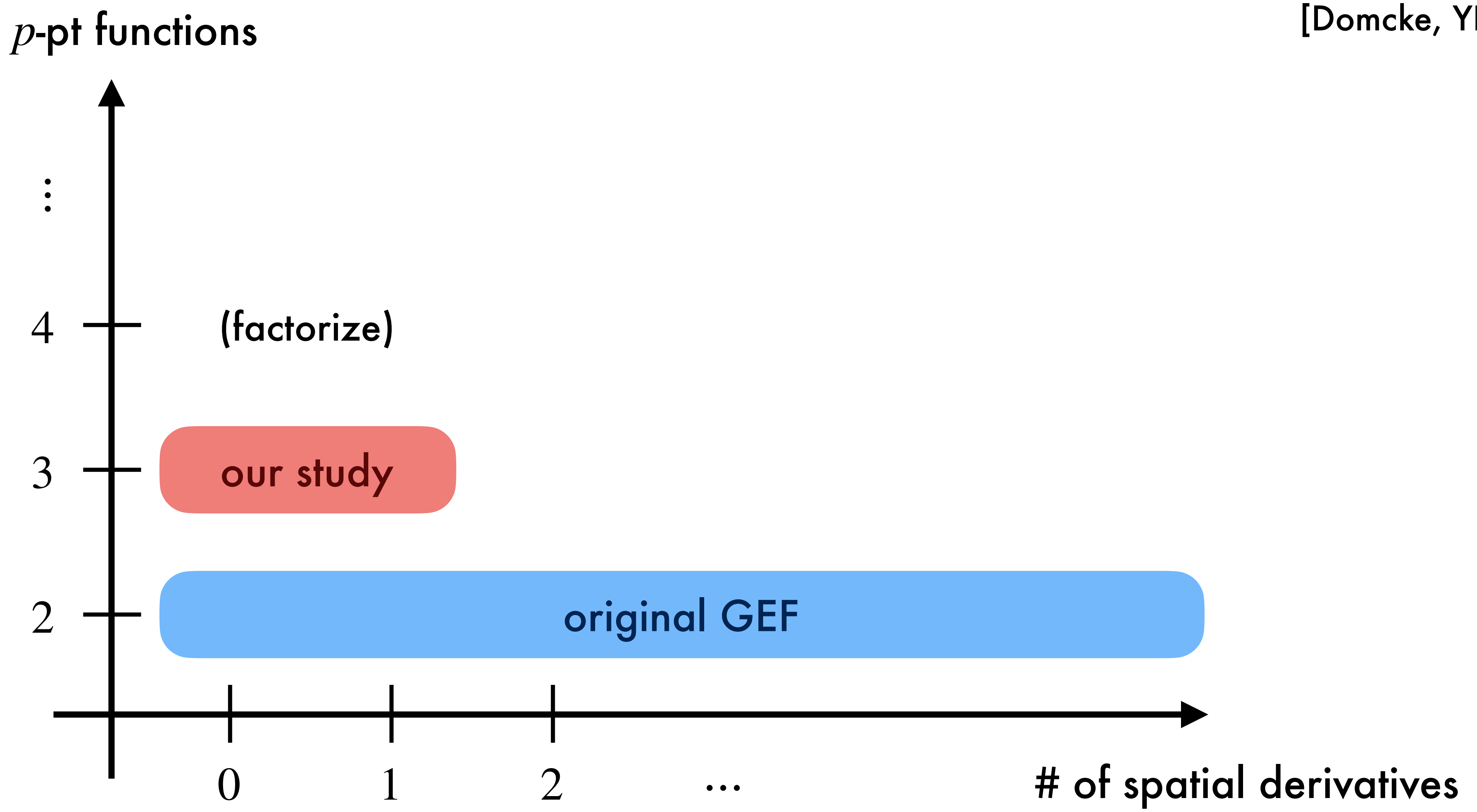

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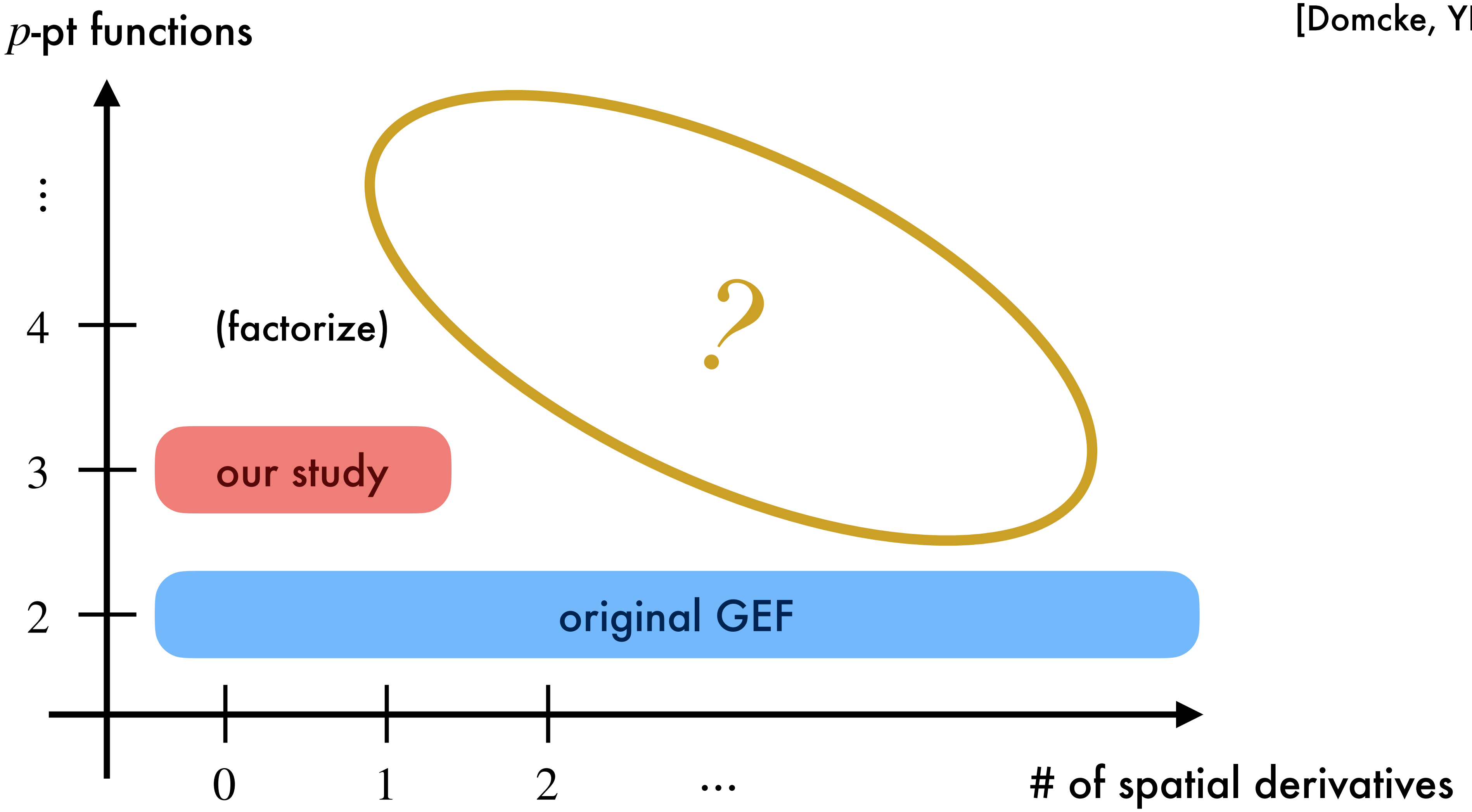

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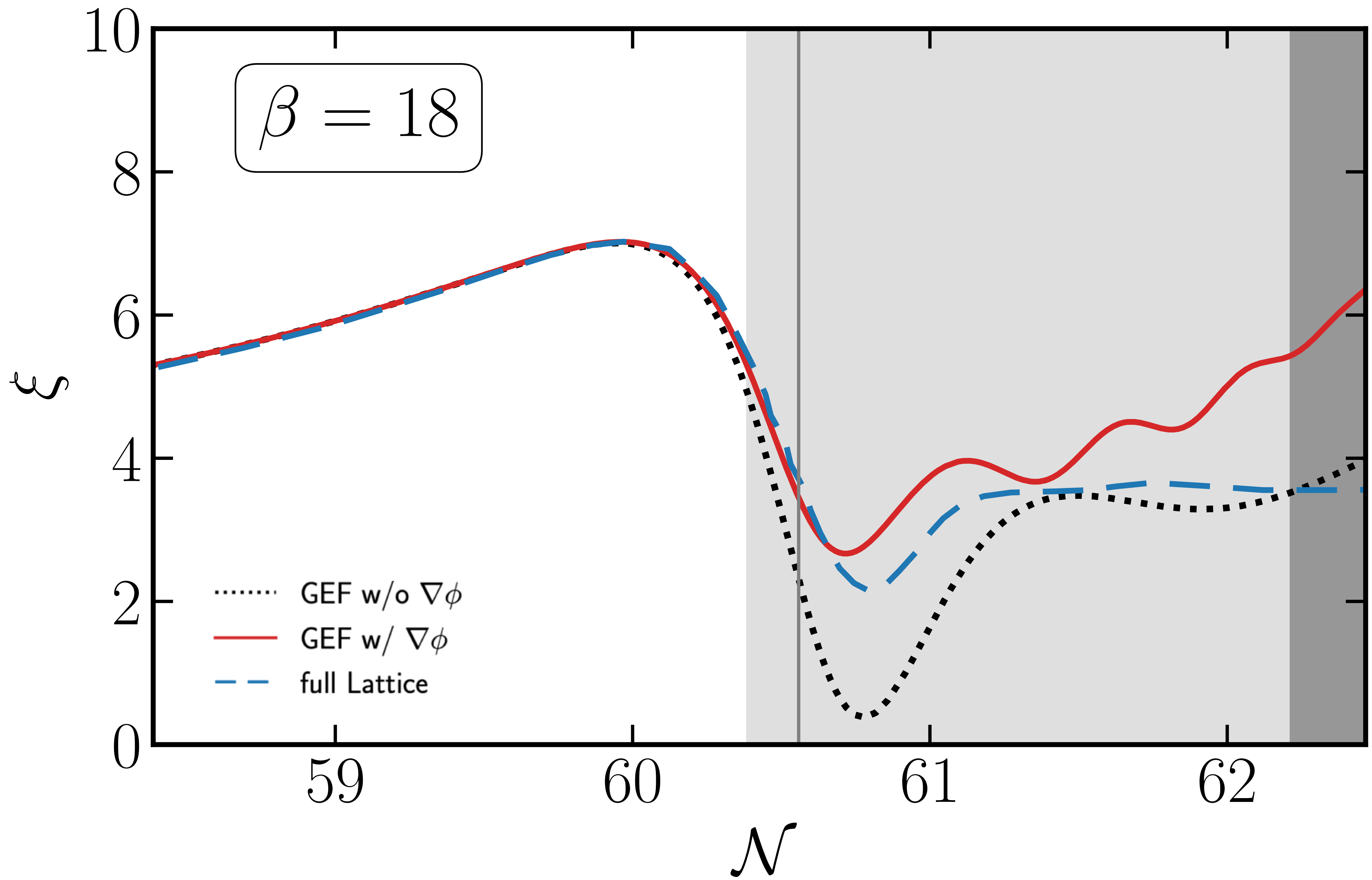
Numerical result

Time evolution of $\xi = \beta|\dot{\phi}|/2HM_P$.

[Domcke, YE, Sandner 23]



Light gray: $1\,\%$ axion gradient energy, dark gray: $50\,\%$.

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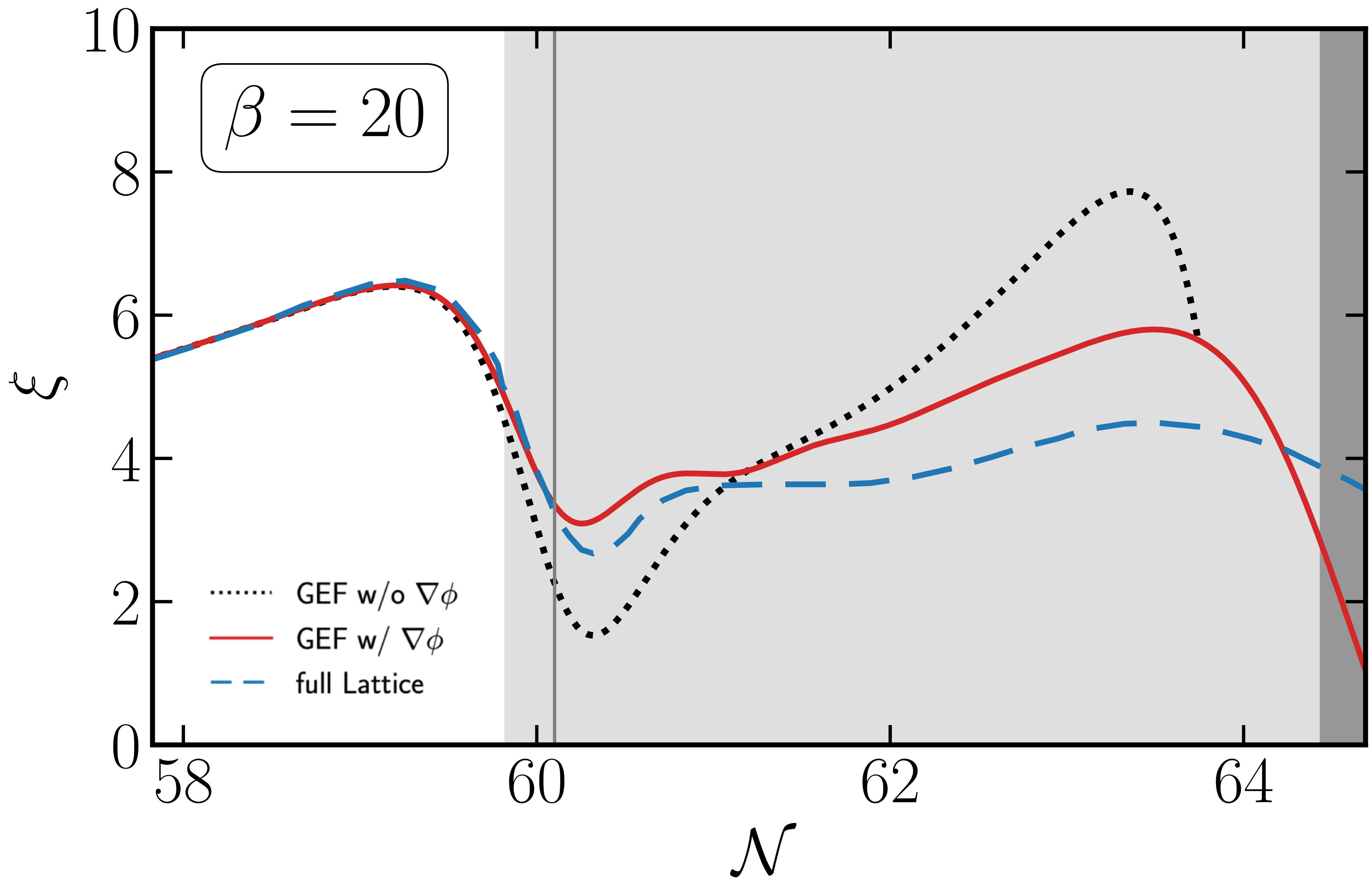

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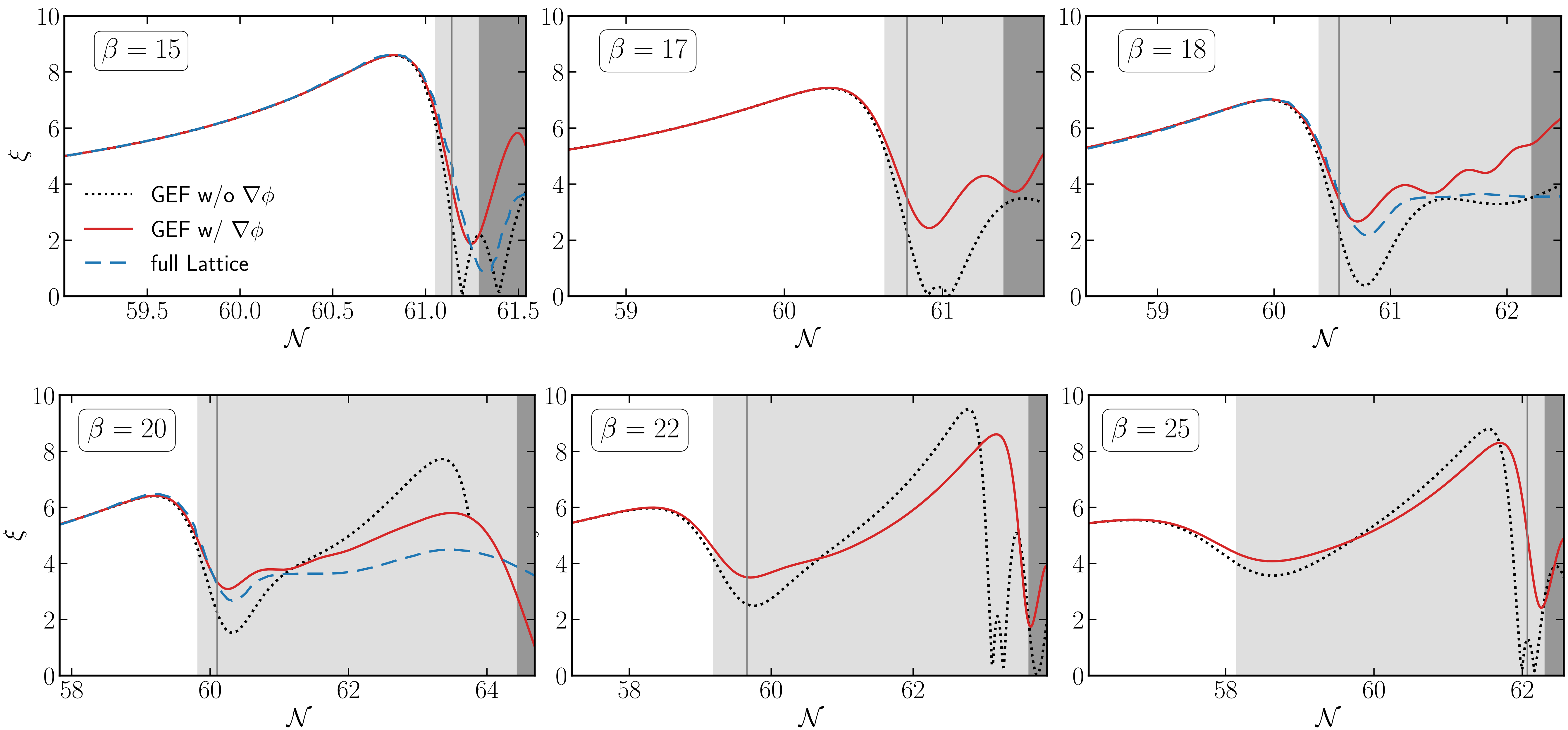

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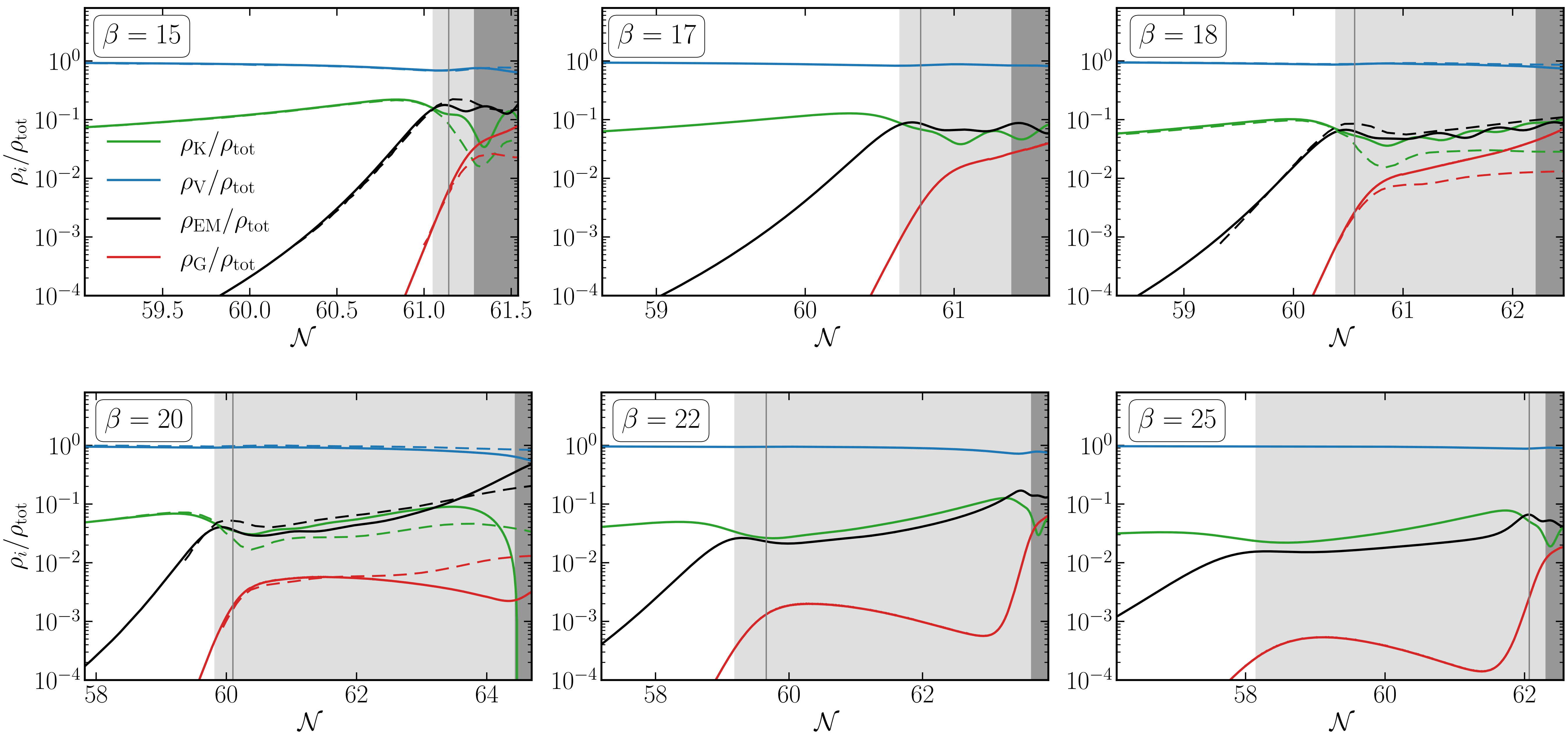
Numerical result

Time evolution of energy densities.

[Domcke, YE, Sandner 23]



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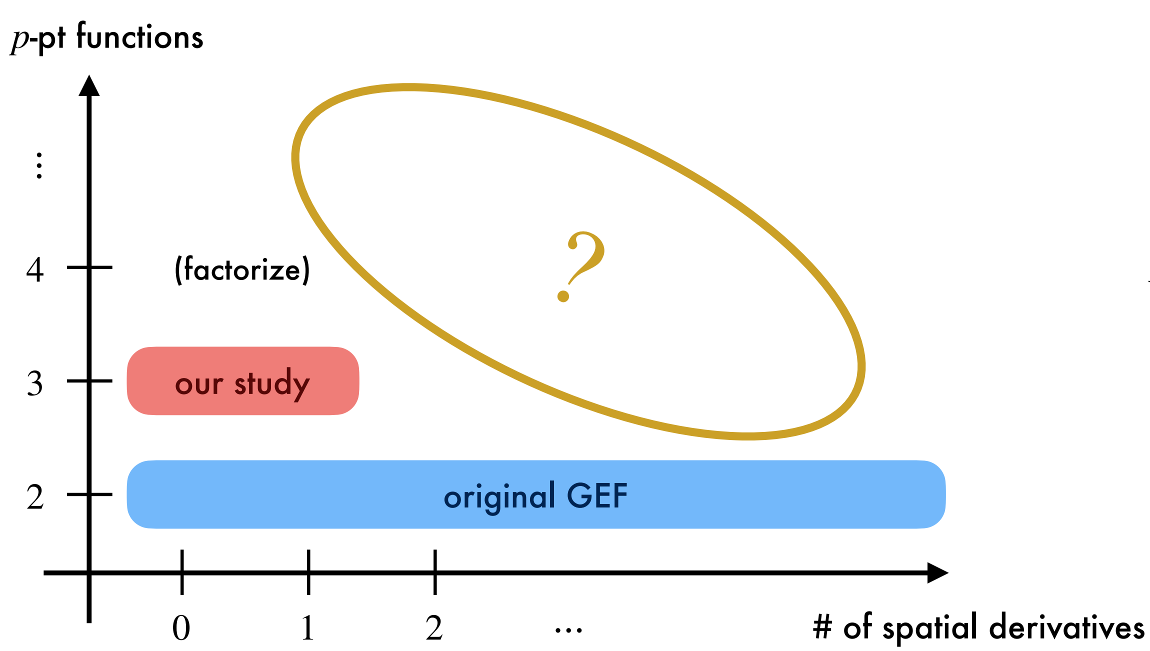
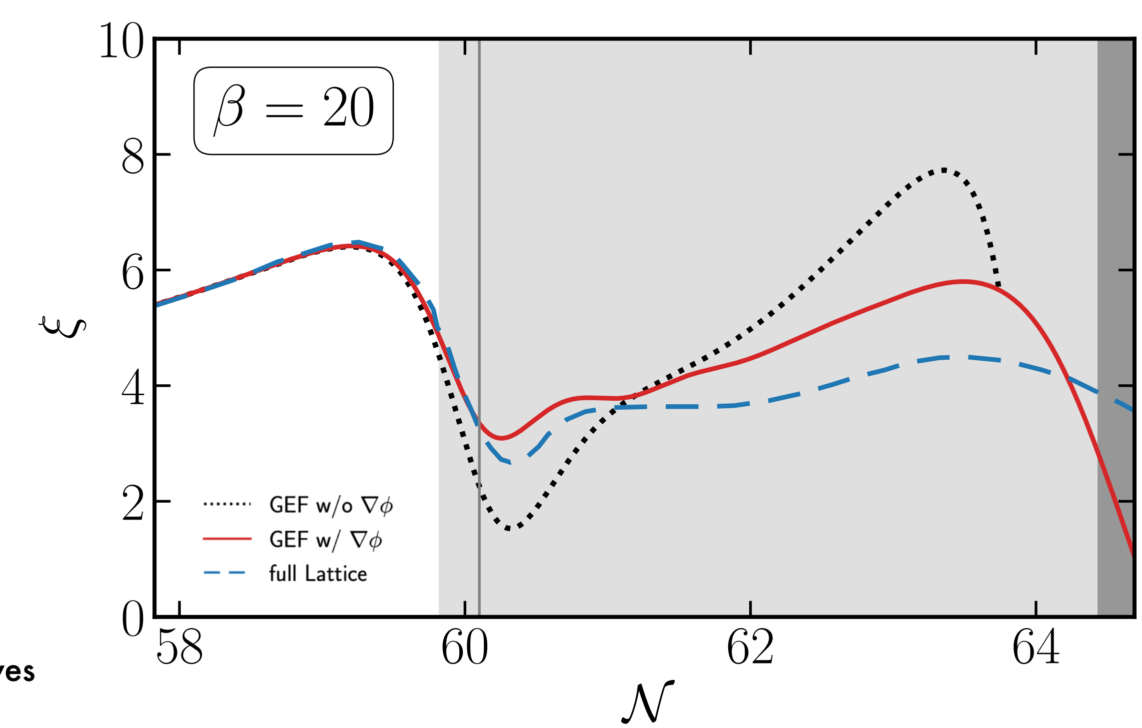
Summary

- Axion inflation with $\phi F\tilde{F}$: great pheno interest (GW, PBH, ...).
- Strong backreaction regime for large β.
  - Classical lattice: precise but numerically expensive.
  - GEF: numerically cheap but axion inhomogeneity ignored.
- We propose a way of including backreaction with axion inhomogeneity.

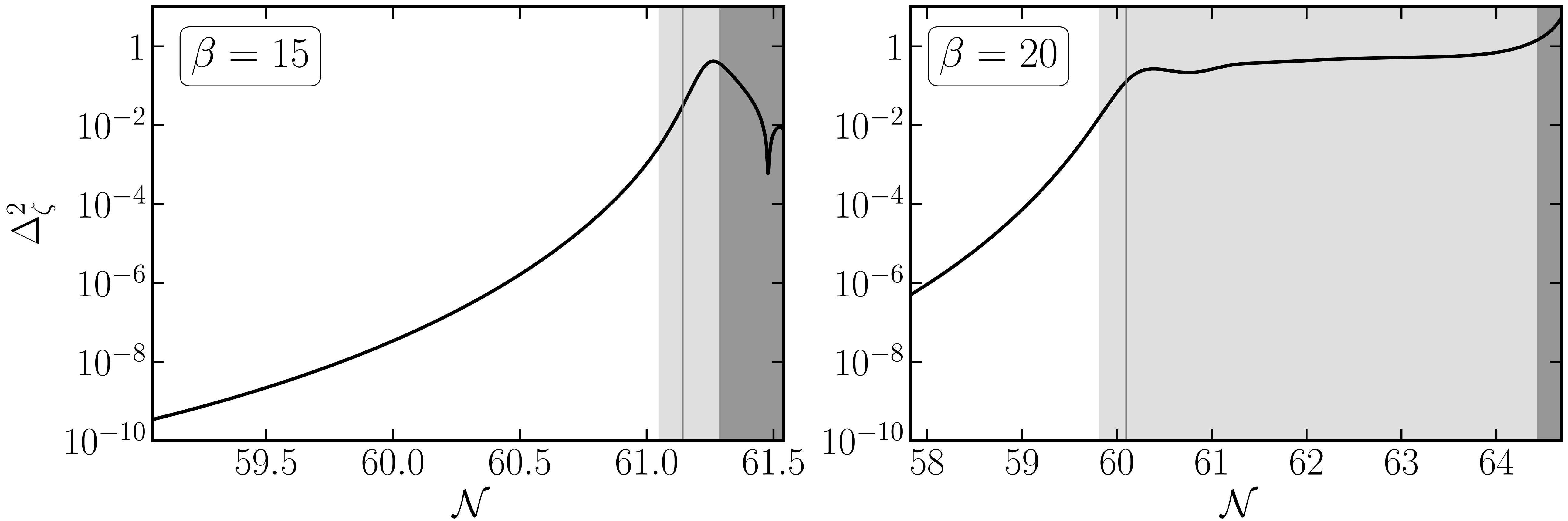




Back up

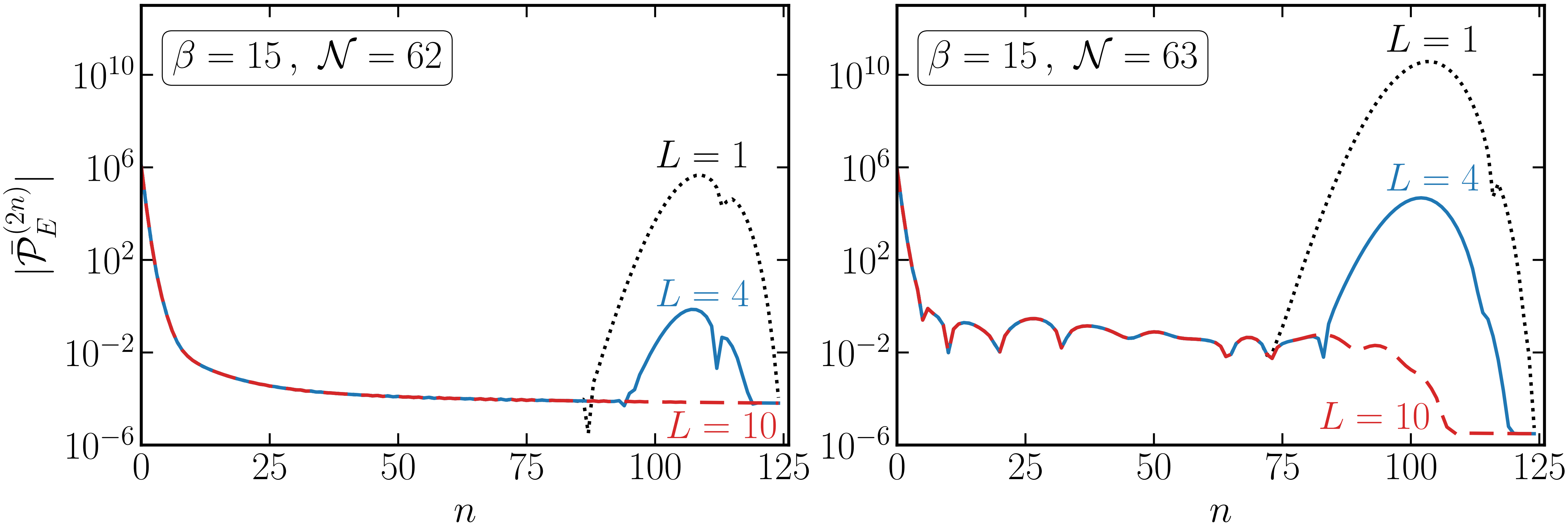
Power spectrum

We can compute the power spectrum within our formalism: $\Delta_\zeta^2 = (H/\dot{\phi})^2 \langle \chi^2 \rangle$.



Improved truncation condition

$$\bar{\mathscr{P}}_X^{(n_{\max}+1)} = \sum_{l=1}^{L} (-1)^{l-1} \binom{L}{l} \bar{\mathscr{P}}_X^{(n_{\max}+1-2l)}\,,$$ where $\bar{\mathscr{P}}_X^{(n)} = \mathscr{P}_X^{(n)}/H_0^4(k_h/a)^n$.



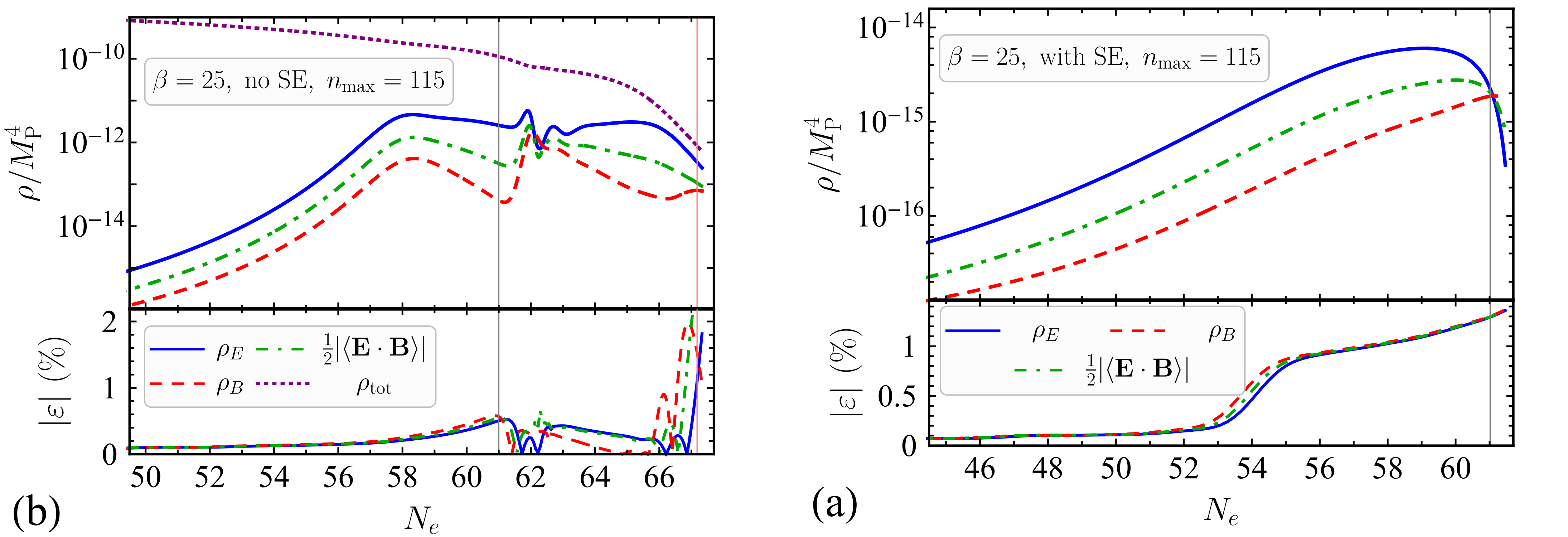
$\mathcal{N} = 63$ still shows instability for high n (even in 2pt case).

Phenomenology

- Axion-gauge field coupling induces tachyonic gauge field production.

$$\langle \vec{E}^2 \rangle \simeq 10^{-4} \frac{e^{2\pi\xi}}{\xi^3} H^4, \quad \langle \vec{B}^2 \rangle \simeq 10^{-4} \frac{e^{2\pi\xi}}{\xi^5} H^4, \quad \xi = \frac{\beta \dot{\phi}}{2HM_P}.$$

- Fermion suppresses gauge boson production, even without axion coupling.



"Gradient expansion method" [Gorbar, Schmitz, Sobol, Vilchinskii 21]

- Axion coupling enhances induced current, can be more effective:

$$g\langle J_z \rangle \sim \tau \times \frac{g^3 E^2}{2\pi^2} e^{-\frac{\pi m^2}{gE}} \times \max\left[\frac{B}{E} \coth\left(\frac{\pi B}{E} \right), \frac{\dot{\theta}_{5+m}^2}{\pi m^2} \right],$$

τ : duration of electric field.