Higher operations in TQFTs
(or Massey products for lc PFAs)

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based on 2307.04856 w/ S. Bruinsma & A. Schenkel

Motivation:

- Prefactorisation algebras (PFAs) axiomatize algebraic structures of observables in QFTs. [Costello, Gwilliam]
- Locally constant (lc) PFAs $\longleftrightarrow$ Topological QFTs.
- lc PFAs on $\mathbb{R}^m$ $\equiv$ $\mathbb{E}_m$-algebras [Lurie]

  $\equiv$ $H\mathbb{E}_m$-algebras [formality of $\mathbb{E}_m$ operad]

  $\equiv$ $\mathbb{P}_m$-algebras

(in $m=2$ dim)

lc PFAs on $\mathbb{R}^2$ $\equiv$ Gerstenhaber algebras

Massey product

[graded commutative algebra with -1 shifted Poisson bracket]

- see also [Beem, Ben-Zvi, Bullimore, Dimofte, Neitzke '18]

Definition: A prefactorisation algebra (PFA) $\mathcal{F}$ on $\mathbb{R}^m$, $m \geq 1$ assigns:

(i) to each disc $D \subseteq \mathbb{R}^m$ a cochain complex:

$$\longmapsto \mathcal{F}(D) = \left(\cdots \xrightarrow{d^{i-1}} \mathcal{F}^i(D) \xrightarrow{d^i} \mathcal{F}^{i+1}(D) \xrightarrow{d^{i+1}} \cdots \right)$$

vector spaces

$(d^{i+1} d^i = 0)$

[**Recall:** cohomology groups $H\mathcal{F}^i(D) = \ker d^i / \operatorname{im} d^{i-1}$ form a trivial cochain complex (i.e. graded vector space):

$$H\mathcal{F}(D) = \left(\cdots \xrightarrow{0} H\mathcal{F}^i(D) \xrightarrow{0} H\mathcal{F}^{i+1}(D) \xrightarrow{0} \cdots \right)$$
]

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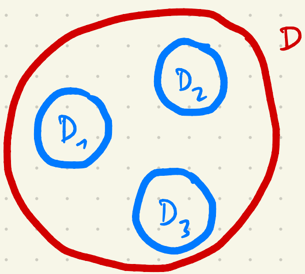
vector spaces

$(d^{i+1} d^{i} = 0)$

(ii) to each inclusion $D_1 \sqcup \dots \sqcup D_n \subset D$ a **cochain map**:



$$\longmapsto \left(\mathcal{F}(\iota_{\underline{D}}^{D}) : \bigotimes_{i=1}^{n} \mathcal{F}(D_i) \longrightarrow \mathcal{F}(D) \right)$$

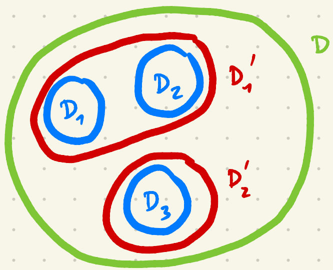
$\underline{D} = (D_1, \dots, D_n)$

$d = \sum_{i=1}^{n} d^{(i)}$

d

such that it:

(1) preserves compositions:



$$\begin{tikzcd}
\bigotimes_i F(D_i) \arrow[rr, "F(\iota^{D'}_{\underline{D}})"] \arrow[dr, "F(\iota^{D}_{\underline{D}})"'] & & \bigotimes_j F(D_j') \arrow[dl, "F(\iota^{D}_{\underline{D'}})"] \\
& F(D) &
\end{tikzcd}$$

(2) preserves units: $F(\iota^{D}_{D}) = id_{F(D)} : F(D) \longrightarrow F(D)$

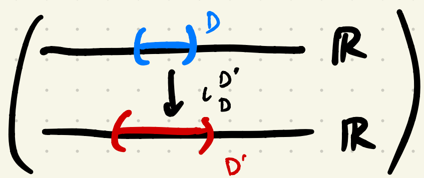
(3) is equivariant under permutations:

$$\begin{tikzcd}
\bigotimes_i F(D_i) \arrow[rr, "\tau_\sigma"] \arrow[dr, "F(\iota^{D}_{\underline{D}})"'] & & \bigotimes_i F(D_{\sigma(i)}) \arrow[dl, "F(\iota^{D}_{\underline{D}\sigma})"] \\
& F(D) &
\end{tikzcd}$$

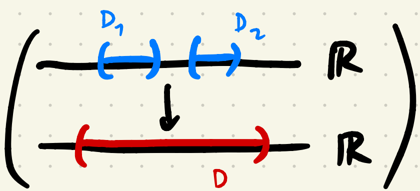
Example: Every associative unital algebra A defines a PFA $\mathcal{F} = A^{\text{fact}}$ on $\mathbb{R}$:

(i) 

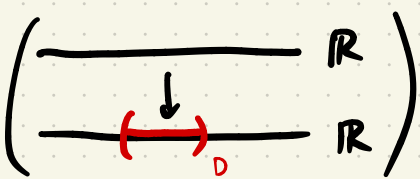
$$\longmapsto \quad \mathcal{F}(D) := A = (\cdots \to 0 \to \overset{(0)}{A} \to 0 \to \cdots)$$

(ii) 

$$\longmapsto \quad \left(\mathcal{F}(\iota_D^{D'}) = \mathrm{id}_A : A \longrightarrow A\right)$$

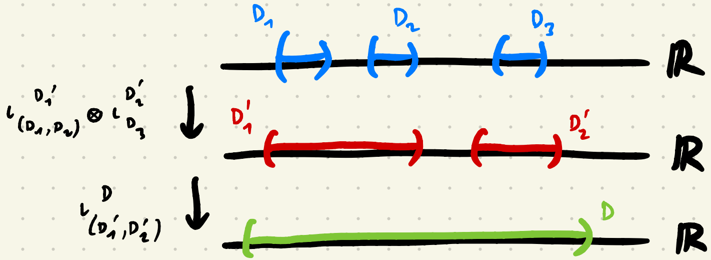


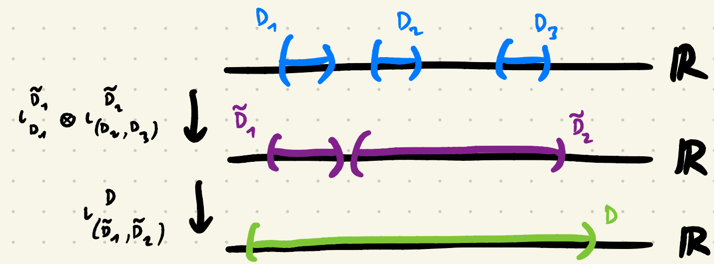
$$\longmapsto \quad \left(\mathcal{F}(\iota_{(D_1,D_2)}^{D}) = \mu : A \otimes A \longrightarrow A\right)$$



$$\longmapsto \quad \left(\mathcal{F}(\iota_{\emptyset}^{D}) = \eta : \mathbb{K} \longrightarrow A\right)$$

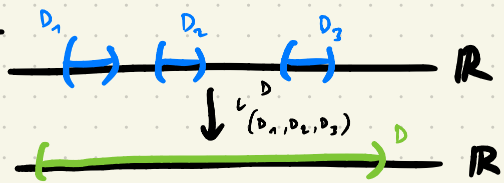
[$\emptyset$ = empty tuple]



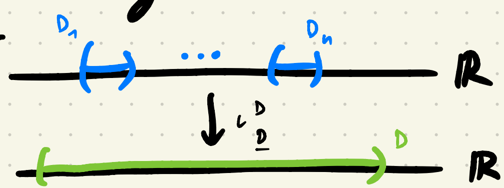


$$\mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu)$$

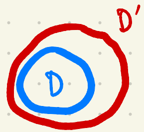
Defines:

() $\longmapsto F\left(\iota^{D}_{(D_1,D_2,D_3)}\right) : A \otimes A \otimes A \longrightarrow A$

Similarly define all

() $\longmapsto F\left(\iota^{D}_{\underline{D}}\right) : A^{\otimes n} \longrightarrow A$

Definition: A PFA $\mathcal{F}$ is locally constant if it assigns to each $D \subseteq D'$ a quasi-isomorphism



$$\longmapsto \left(\mathcal{F}(\iota_D^{D'}) : \mathcal{F}(D) \xrightarrow{\simeq} \mathcal{F}(D') \right)$$

[induced map on cohomology $H\mathcal{F}(\iota_D^{D'}) : H\mathcal{F}(D) \xrightarrow{\cong} H\mathcal{F}(D')$ is an isomorphism]

$$\begin{tikzcd} H\mathcal{F}(D) \arrow[r, "\cong"] \arrow[d, dashed] & H\mathcal{F}(D') \\ \mathcal{F}(D) \arrow[r, "\mathcal{F}(\iota_D^{D'})"'] & \mathcal{F}(D') \arrow[u, dashed] \end{tikzcd}$$

In this case $\boxed{H\mathcal{F}(D) \cong H\mathcal{F}(\mathbb{R}^m)}$ for all disc $D \subseteq \mathbb{R}^m$.

Q: What algebraic structure do factorisation products

$$F(\iota_{\underline{D}}^{D}) : \bigotimes_{i=1}^{n} F(D_i) \longrightarrow F(D)$$

induce on the cohomology $HF(\mathbb{R}^m)$?

Idea: Consider a strong deformation retract (SDR)

$$\begin{tikzcd} HF(\mathbb{R}^m) \arrow[r, shift left, "i_D"] & F(D) \arrow[l, shift left, "p_D"] \arrow[loop right, "h_D"] \end{tikzcd}$$

for each disc $D \subseteq \mathbb{R}^m$ i.e. such that:

$$p_D i_D = id, \qquad i_D p_D = id + \partial h_D = id + dh_D + h_D d$$

(+ additional conditions)

Will obtain algebraic structure on $HF(\mathbb{R}^m)$ by homotopy transfer of PFA structure on F along SDR.

Homotopy transfer (a simple example):

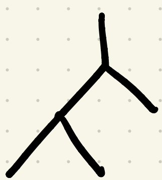
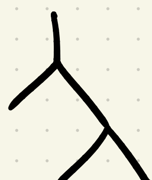
Let A be an associative dg algebra, i.e.

$$A = \left(\cdots \xrightarrow{d^{i-1}} A^{i} \xrightarrow{d^{i}} A^{i+1} \xrightarrow{d^{i+1}} \cdots \right) \in Ch$$

equipped with product

 $:= \mu : \overset{d\otimes 1 + 1 \otimes d}{\overset{\circlearrowleft}{A \otimes A}} \longrightarrow \overset{d}{\overset{\circlearrowleft}{A}}$ (degree 0),

satisfying associativity relation

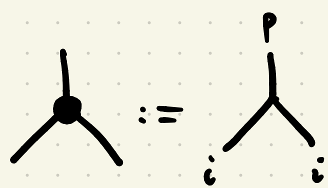
 $-$  $= 0$ $\quad$ (i.e. $\mu(\mu \otimes id) - \mu(id \otimes \mu) = 0$)

Suppose we have a strong deformation retract

$$V \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowright h$$

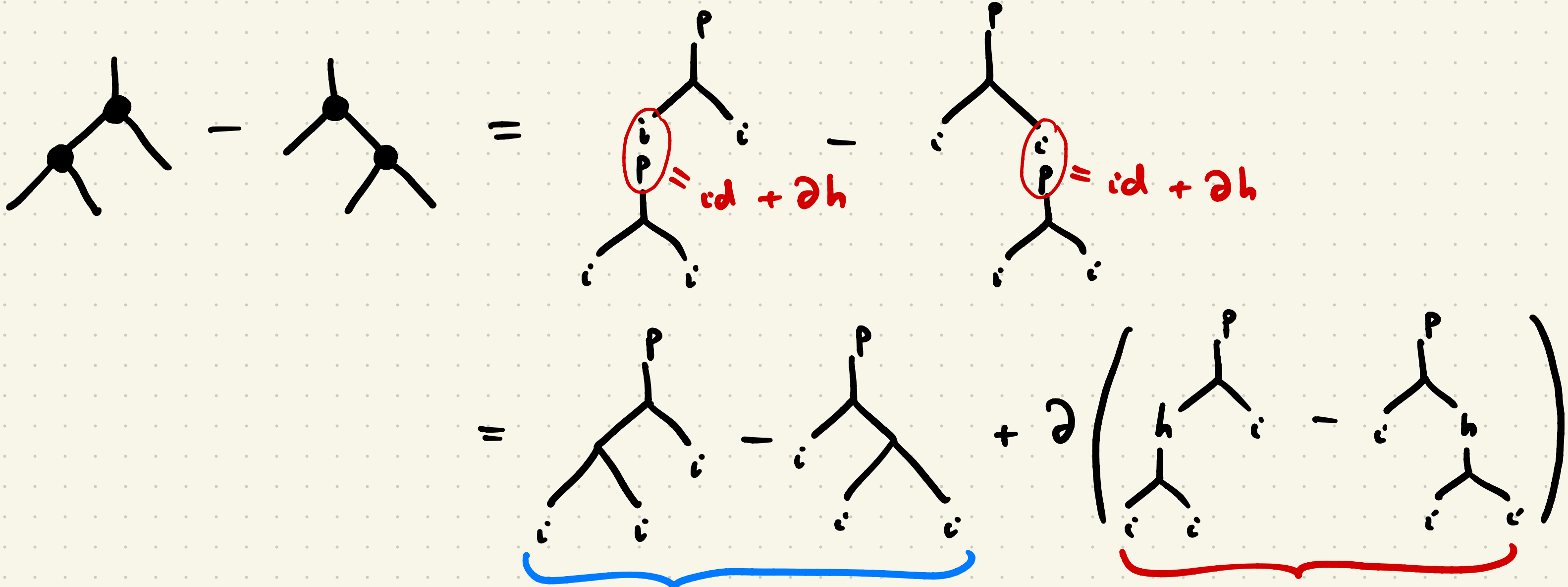
to some $V \in \mathrm{Ch}$. $(pi = \mathrm{id},\ ip = \mathrm{id} + \partial h,\ ph = hi = h^2 = 0)$

Product $\mu : A \otimes A \to A$ induces an obvious product on V:

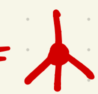
$\mu_2 =$  $: V \otimes V \to V$

(i.e. $\mu_2 := p\,\mu\,(i \otimes i)$)

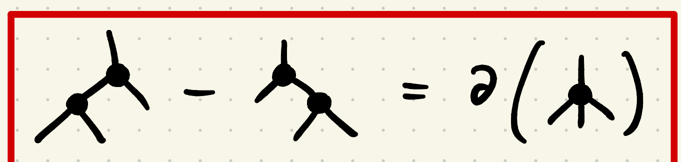
Is μ_2 associative?



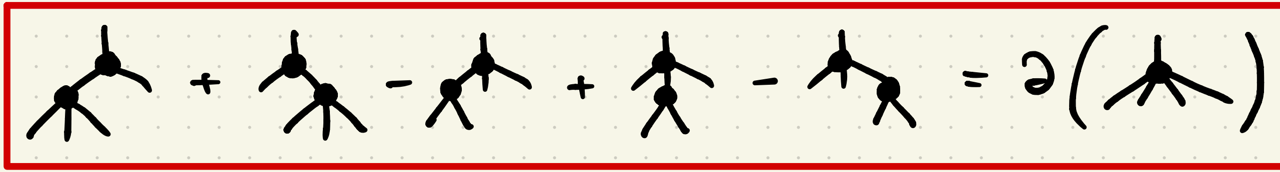
$=0$ by associativity of μ

$\mu_3 =$  $: V^{\otimes 3} \to V$ (degree -1)

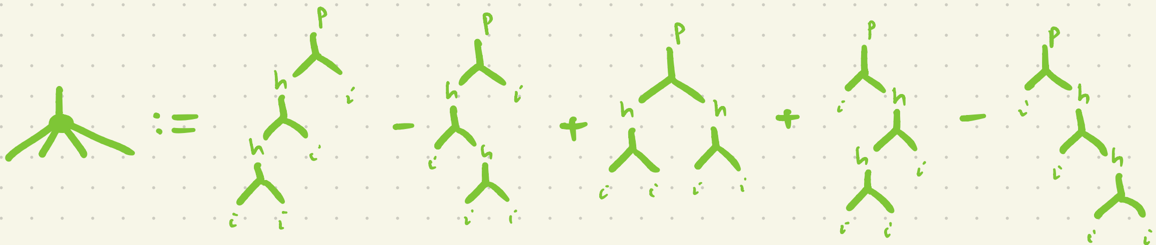
So μ_2 is associative <u>only up to homotopy</u> μ_3 :



μ_2 and μ_3 satisfy a relation up to a new homotopy:



where

$\mu_4 =$  $: V^{\otimes 4} \to V$

(degree -2)

And so on...

$V \in Ch$ carries structure of a homotopy associative algebra, or A_∞-algebra, with operations

$$\mu_n : V^{\otimes n} \to V \qquad \text{(degree } 2-n\text{)} \qquad \forall n \geq 2,$$

satisfying certain relations. Note: $(V, \{\mu_n\}_{n\geq 2}) \rightsquigarrow (A, \mu)$.

If $V = HA$, i.e. consider strong deformation retract

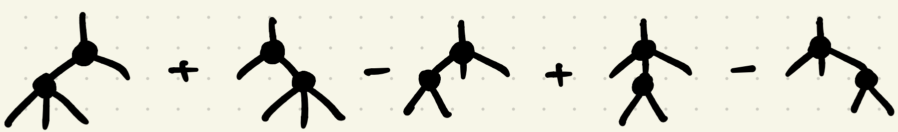
$$HA \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h$$

then:

(a) $\mu_2 : HA \otimes HA \longrightarrow HA$ (degree 0) is (strictly) associative,

$= 0$

(b) $\mu_n : HA^{\otimes n} \longrightarrow HA$ (degree $2-n$) for $n \geq 3$ give higher operations called Massey products, satisfying (strict) relations:



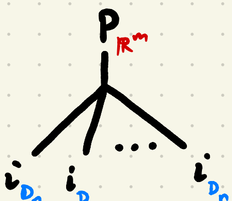
$= 0$, ...

Homotopy transfer in PFA case:

Had strong deformation retract, for each disc $D \subseteq \mathbb{R}^m$,

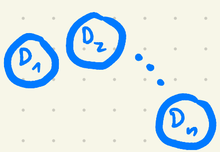
$$\begin{tikzcd} H\mathcal{F}(\mathbb{R}^m) \arrow[r, shift left, "i_D"] & \mathcal{F}(D) \arrow[l, shift left, "p_D"] \arrow[loop right, "h_D"] \end{tikzcd}$$

(a) Define (degree 0) transferred products:



$:$

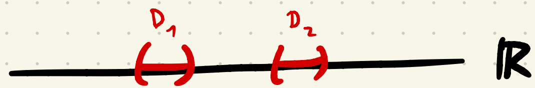
$$\begin{tikzcd}[column sep=huge, row sep=huge] H\mathcal{F}(\mathbb{R}^m)^{\otimes n} \arrow[r, "\mu_{\underline{D}}"] \arrow[d, "\bigotimes_{i=1}^{n} i_{D_i}"'] & H\mathcal{F}(\mathbb{R}^m) \\ \bigotimes_{i=1}^{n} \mathcal{F}(D_i) \arrow[r, "\mathcal{F}(\iota^{\mathbb{R}^m}_{\underline{D}})"'] & \mathcal{F}(\mathbb{R}^m) \arrow[u, "p_{\mathbb{R}^m}"'] \end{tikzcd}$$



Proposition:

- **In $m=1$ dim:** $\mu_{\underline{D}} = \mu_\sigma$ only depends on permutation $\sigma \in \Sigma_n$ s.t. $\underline{D}\sigma = (D_{\sigma(1)}, \dots, D_{\sigma(n)})$ is ordered on $\mathbb{R}$.

$\Rightarrow (HF(\mathbb{R}), \mu_\sigma)$ is associative unital algebra.

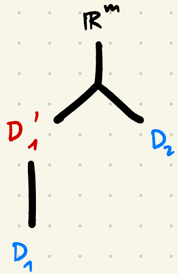


- **In $m \geq 2$ dim:** $\mu_{\underline{D}} = \mu_n$ only depends on length $n \geq 1$ of tuple $\underline{D} = (D_1, \dots, D_n)$.

$\Rightarrow (HF(\mathbb{R}^m), \mu_n)$ is <u>commutative</u> associative unital algebra.

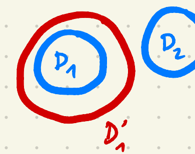
(b) Also get "higher" transferred operations on $HF(\mathbb{R}^m)$, labelled by trees of disc inclusions,

e.g.

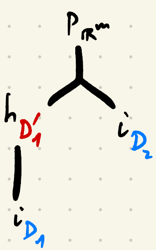


$: HF(\mathbb{R}^m)^{\otimes 2} \longrightarrow HF(\mathbb{R}^m)$

(degree -1)



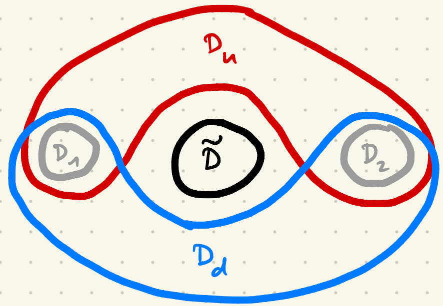
More explicitly,



$$= p_{\mathbb{R}^m} F\left(\iota^{\mathbb{R}^m}_{(D_1', D_2)}\right)\left(h_{D_1'} \otimes id\right)\left(F\left(\iota^{D_1'}_{D_1}\right) \otimes id\right)\left(i_{D_1} \otimes i_{D_2}\right).$$

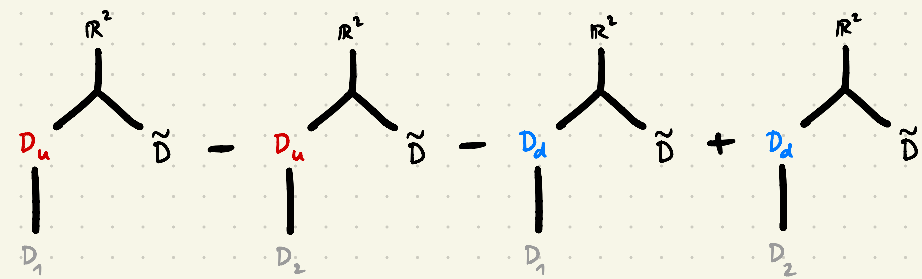
Problem: Such "higher" operations depend on many choices: SDR, discs involved, ...

In $m=2$ dimensions: Consider disc configuration



Define (degree -1) operation $\{\cdot,\cdot\} : HF(\mathbb{R}^2)^{\otimes 2} \to HF(\mathbb{R}^2)$ as

$\{\cdot,\cdot\} :=$



Proposition: $\{\cdot,\cdot\}$ is independent of the various choices.

Theorem: $\{\cdot,\cdot\}$ defines a -1 shifted Poisson bracket on commutative associative unital algebra $HF(\mathbb{R}^2)$.

Explicitly, $\forall a,b,c \in HF(\mathbb{R}^2)$ it satisfies:

(1) Symmetry: $\{a,b\} = (-1)^{|a||b|}\{b,a\}$

(2) Derivation: (write $ab := \mu_2(a,b)$)

$$\{a,bc\} = \{a,b\}c + (-1)^{(|a|-1)|b|} b\{a,c\}$$

(3) Jacobi:

$$(-1)^{(|c|-1)|a|}\{a,\{b,c\}\} + (-1)^{(|a|-1)|b|}\{b,\{c,a\}\} + (-1)^{(|b|-1)|c|}\{c,\{a,b\}\} = 0$$

$\Rightarrow (HF(\mathbb{R}^2), \mu_2, \{\cdot,\cdot\})$ is a Gerstenhaber algebra.

Example (Factorisation envelopes):

Finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.

Consider $\mathfrak{g}$-valued compactly supported de Rham complex on $D \subseteq \mathbb{R}^m$:

$$\begin{tikzcd} \mathfrak{g}^{\mathbb{R}^m}(D) := \Big(\underset{(0)}{\mathfrak{g} \otimes \Omega_c^0(D)} \arrow[r, "d_{dR}"] & \cdots \arrow[r, "d_{dR}"] & \underset{(m-1)}{\mathfrak{g} \otimes \Omega_c^{m-1}(D)} \arrow[r, "d_{dR}"] & \underset{(m)}{\mathfrak{g} \otimes \Omega_c^m(D)} \Big) \arrow[d, "\int_D"] \\ & & & \mathfrak{g} \end{tikzcd}$$

Cohomology:

$$H\left(\mathfrak{g}^{\mathbb{R}^m}(D)\right) = \left(0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathfrak{g} \right)$$

Symmetric algebra

$$\mathrm{Sym}\left(\mathfrak{g}^{\mathbb{R}^m}(D)\,[1]\right)$$

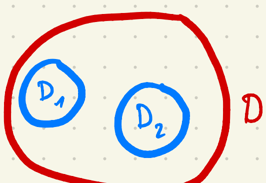
is a locally constant PFA.

shift degrees down by 1.

$$\left(\overset{(-1)}{\mathfrak{g}\otimes\Omega^0_c(D)} \xrightarrow{d_{dR}} \cdots \xrightarrow{d_{dR}} \overset{(m-2)}{\mathfrak{g}\otimes\Omega^{m-1}_c(D)} \xrightarrow{d_{dR}} \overset{(m-1)}{\mathfrak{g}\otimes\Omega^m_c(D)} \right)$$

Factorisation product:

$$\mathrm{Sym}\left(\mathfrak{g}^{\mathbb{R}^m}(D_1)[1]\right)\otimes \mathrm{Sym}\left(\mathfrak{g}^{\mathbb{R}^m}(D_2)[1]\right)$$



$$\longrightarrow \mathrm{Sym}\left(\mathfrak{g}^{\mathbb{R}^m}(D)[1]\right)$$

Lie algebra structure on $\mathfrak{g}$ induces differential

$$d_{CE} : \mathrm{Sym}\left(\mathfrak{g}^{\mathbb{R}^m}(D)[1]\right) \longrightarrow \mathrm{Sym}\left(\mathfrak{g}^{\mathbb{R}^m}(D)[1]\right)$$

defined by degrees:

degree 1 map

$$d_{CE}\left(\overbrace{X\otimes\omega}^{|\omega|-1}\cdot\overbrace{Y\otimes\eta}^{|\eta|-1}\right) = (-1)^{|\omega|}\,\overbrace{[X,Y]\otimes\omega\wedge\eta}^{|\omega|+|\eta|-1}.$$

Factorisation envelope is perturbation:

$$\mathcal{U}\mathfrak{g}^{\mathbb{R}^m}(D) := \left(\mathrm{Sym}\left(\mathfrak{g}^{\mathbb{R}^m}(D)[1]\right),\ d_{dR} + d_{CE}\right)$$

This is a Locally constant PFA with cohomology

$$H\,\mathcal{U}\mathfrak{g}^{\mathbb{R}^m}(D) = \mathrm{Sym}\left(\mathfrak{g}[1-m]\right). \qquad \left(\cdots \to 0 \to \overset{(m-1)}{\mathfrak{g}} \to 0 \to \cdots\right)$$

Proposition:

- In $m=1$ dim: $\mathcal{U}\mathfrak{g}^{\mathbb{R}}(D) = \mathrm{Sym}(\mathfrak{g})$ and

$\mu_2 : \mathrm{Sym}(\mathfrak{g}) \otimes \mathrm{Sym}(\mathfrak{g}) \longrightarrow \mathrm{Sym}(\mathfrak{g})$ is Gutt star product $\star$

$\Rightarrow \boxed{(\mathrm{Sym}(\mathfrak{g}), \star) \cong U(\mathfrak{g})}$. No Massey products.

- In $m=2$ dim: $\mathcal{U}\mathfrak{g}^{\mathbb{R}^2}(D) = \mathrm{Sym}(\mathfrak{g}[-1]) \cong \Lambda \mathfrak{g}$ and

$\mu_2 : \Lambda\mathfrak{g} \otimes \Lambda\mathfrak{g} \longrightarrow \Lambda\mathfrak{g}$ is graded symmetric product $\cdot$,

$\{\cdot,\cdot\} : \Lambda\mathfrak{g} \otimes \Lambda\mathfrak{g} \longrightarrow \Lambda\mathfrak{g}$ is $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ extended to $\Lambda\mathfrak{g}$ by graded Leibniz.

$\boxed{(\Lambda\mathfrak{g}, \cdot, \{\cdot,\cdot\})} \longrightarrow$ a Gerstenhaber algebra.

• <u>In $m \geq 3$ dim:</u>

$$\mathcal{U}\mathfrak{g}^{\mathbb{R}^m}(D) = \mathrm{Sym}(\mathfrak{g}[1-m]) \cong \begin{cases} \mathrm{Sym}\, \mathfrak{g} & \text{for odd } m, \\ \Lambda \mathfrak{g} & \text{for even } m. \end{cases}$$

$$\mu_2 : \mathrm{Sym}(\mathfrak{g}[1-m]) \otimes \mathrm{Sym}(\mathfrak{g}[1-m]) \longrightarrow \mathrm{Sym}(\mathfrak{g}[1-m])$$

is (degree 0) graded symmetric product.

Note that

$$\mathrm{Sym}(\mathfrak{g}[1-m]) = \Big(\overset{(0)}{1} \to 0 \to \dots \to 0 \to \overset{(m-1)}{\mathfrak{g}} \to 0 \to \dots \to 0 \to \overset{(2m-2)}{\mathfrak{g}^{\otimes 2}} \to 0 \to \cdots \Big)$$

so <u>first non-trivial Massey product</u> expected in degree $1-m$.

Conclusion & Outlook

- Given any lcPFA $\mathcal{F}$ (observables of a TQFT) on $\mathbb{R}^2$, explicit construction of Gerstenhaber ($\mathbb{P}_2$-) algebra structure on it cohomology $H\mathcal{F}$ (gauge invariant observables).
  ↳ generates full ∞-lcPFA structure on $H\mathcal{F}$ [Lurie].
- For lcPFA $\mathcal{F}$ on $\mathbb{R}^m$, $m \geq 3$, construct degree $1-m$ invariant Massey product on $H\mathcal{F}$.
  ⇝ explicit $\mathbb{P}_m$-algebra structure
- Generalise to non-locally constant PFAs, e.g. holomorphic PFAs ⇝ higher vertex algebras [Budzik, Gaiotto, Kulp, Williams, Wu, Yu]