The strange world of AdS_2 integrability

Alessandro Torrielli

University of Surrey, UK

Durham

August 2023

AdS_5 Review with Beisert et al. 1012.3982

AdS_2 S-matrix with Hoare and Pittelli
1407.0303 and 1509.07587

AdS_2 Massless dressing factor and Bethe ansatz with Fontanella
1706.02634 and 1903.10759

AdS_2 Massless transfer matrix and Yangian
1708.09598 and 2203.15367

Free-fermion description with de Leeuw, Paletta, Pribytok and Retore 2011.08217

AdS_2 Form factors with Bielli and Gautam 2302.08491

AdS_2 Deformation with Nieto Garcia and Ruiz 2303.16108

BETHE ANSATZ [Bethe '31]

Extra conservation laws

Infinite spin-chain limit: 2-particle state

$$|\psi> = \sum_{n_1<n_2} \psi(n_1, n_2)\, |Z...Z\, V\, Z...Z\, W\, Z...\rangle$$

$$\qquad\qquad\qquad\qquad {n_1}^{\uparrow} \qquad {n_2}^{\uparrow}$$

$$\psi(n_1, n_2) = e^{ip_1 n_1 + ip_2 n_2} + S(p_1, p_2)\, e^{ip_2 n_1 + ip_1 n_2}$$

$S(p_1, p_2)$ S-matrix: *magnon* scattering

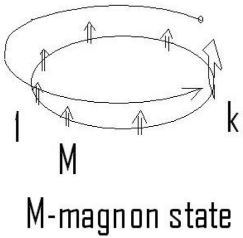
$$\downarrow \dots \downarrow \;\Uparrow\; \downarrow \dots \downarrow \dots \downarrow \dots \downarrow \;\Uparrow\; \downarrow \dots \downarrow$$

[Minahan-Zarembo '03]

Periodicity restored by Bethe Equations

$$e^{i p_k L} = \prod_{\substack{j=1 \\ j \neq k}}^{M} S_{kj}$$

$$S_{kj} = S(p_k, p_j)$$



AdS/CFT [Arutyunov-Frolov-Staudacher '04; Beisert-Staudacher '05]

EXACT S-MATRICES

{*review*} [P. Dorey '98]

2D integrable massive S-matrices

- No particle production/annihilation

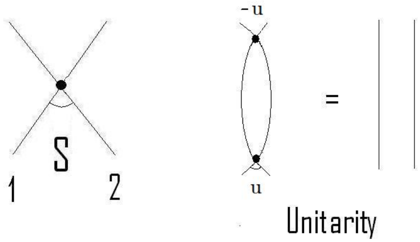
- Equality of initial and final sets of momenta

- Factorisation: $S_{M \longrightarrow M} = \prod S_{2 \longrightarrow 2}$

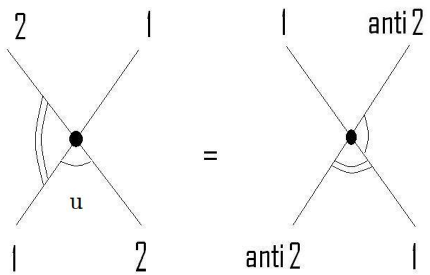
  (all info in 2-body processes)

Extrapolate from relativistic properties / Feynman graphs

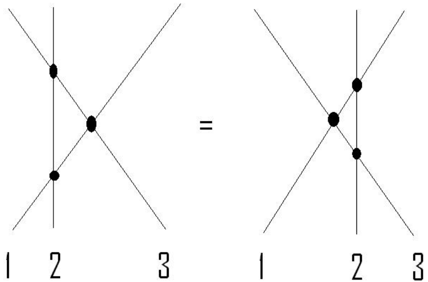
$$S_{2\longrightarrow 2} = S(u_1 - u_2) \equiv S(u) \qquad [E_i = m_i \cosh u_i,\ p_i = m_i \sinh u_i]$$



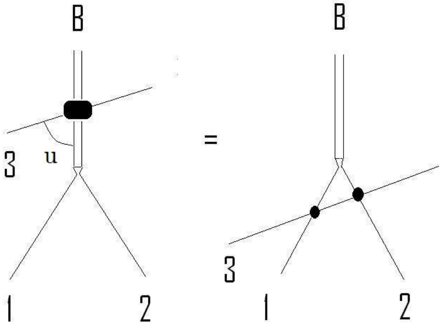
$$S_{12}(u)\ S_{21}(-u) = 1$$



Crossing symmetry $S_{12}(u) = S_{\bar{2}1}(i\pi - u)$



Yang-Baxter Equation (YBE) $S_{12}\, S_{13}\, S_{23} \;=\; S_{23}\, S_{13}\, S_{12}$



Bootstrap $S_{3B} = S_{32}\, S_{31}$ Fusion

[Zamolodchikov-Zamolodchikov '79]

RESIDUAL ALGEBRAS

Spin-chain sites: reps of the superconformal algebra

After fixing (BMN) vacuum the magnon (spin-wave) excitations carry reps of smaller subalgebras

Reduced scattering problem

Vast literature - please ask me for references

Basic building block:

- Five and four dimensions: $psu(2|2)$
- Three dim: $psu(1|1)_L \times psu(1|1)_R$
- Two dimensions: $psu(1|1)$ Abbott-Murugan-Sundin-Wulff 1308.1370

$\longrightarrow$ all centrally extended [following path indicated by Beisert in 5D]

- Quantum inverse scattering method

Massless magnons

New feature in three and two dimensions:
integrability with massless particles

Separate story: to be discussed later

Apart from this, remarkable universality of structures

- Centrally-extended superalgebras (also with vanishing Killing form)
- "Braided" coproduct
- Zhukovsky variables massive / γ-variable massless (pseudo-rel)
- Yangian-like symmetry
- Secret (bonus) symmetry

review de Leeuw *et al* 1204.2366

$\longrightarrow$ underlying non-standard quantum group

AdS_2: a different[1] central extension of $su(1|1)^2$ Hoare-Pittelli-AT 1407.0303

Let us begin with $\mathfrak{gl}(1|1)$: algebra relations and rep theory (focus on 1 copy)
from Gotz-Quella-Schomerus 0504234

Kac module $\langle H,\nu\rangle$ (long)

$$
\begin{array}{lll}
\{\mathbb{Q},\mathbb{G}\} = \mathbb{H} & \mathbb{Q}|\phi\rangle = |\psi\rangle & \mathbb{Q}|\psi\rangle = 0 \\
\{\mathbb{Q},\mathbb{Q}\} = 0 & \mathbb{S}|\phi\rangle = 0 & \mathbb{S}|\psi\rangle = H\,|\phi\rangle \\
\{\mathbb{S},\mathbb{S}\} = 0 & \mathbb{H}|.\rangle = H|.\rangle & \\
\{\mathbb{B},\mathbb{Q}\} = -\mathbb{Q} & \mathbb{B}|\phi\rangle = \nu|\phi\rangle & \\
\{\mathbb{B},\mathbb{S}\} = \mathbb{S} & \mathbb{B}|\psi\rangle = (\nu-1)|\psi\rangle & H \neq 0
\end{array}
$$

anti-Kac module $\overline{\langle H,\nu\rangle}$ (long)

$$
\begin{array}{lll}
\{\mathbb{Q},\mathbb{G}\} = \mathbb{H} & \mathbb{Q}|\psi\rangle = H\,|\phi\rangle & \mathbb{Q}|\phi\rangle = 0 \\
\{\mathbb{Q},\mathbb{Q}\} = 0 & \mathbb{S}|\psi\rangle = 0 & \mathbb{S}|\phi\rangle = |\psi\rangle \\
\{\mathbb{S},\mathbb{S}\} = 0 & \mathbb{H}|.\rangle = H|.\rangle & \\
\{\mathbb{B},\mathbb{Q}\} = -\mathbb{Q} & \mathbb{B}|\phi\rangle = (\nu-1)|\phi\rangle & \\
\{\mathbb{B},\mathbb{S}\} = \mathbb{S} & \mathbb{B}|\psi\rangle = \nu|\psi\rangle & H \neq 0
\end{array}
$$

As long as $H \neq 0$, Kac is isomorphic to anti-Kac

[1] Different from AdS_3

Short modules: $H = 0$

- Kac module becomes reducible but indecomposable, with $|\psi\rangle$ being the sub-rep $\langle \nu - 1 \rangle$

$$\langle \nu - 1 \rangle \longleftarrow \langle \nu \rangle$$

- anti-Kac module becomes reducible but indecomposable, with $|\psi\rangle$ being the sub-rep $\langle \nu \rangle$

$$\langle \nu - 1 \rangle \longrightarrow \langle \nu \rangle$$

The two indecomposables are not isomorphic

Tensor product rules

$$\langle H_1, \nu_1 \rangle \otimes \langle H_2, \nu_2 \rangle = \langle H_1 + H_2, \nu_1 + \nu_2 - 1 \rangle \oplus \langle H_1 + H_2, \nu_1 + \nu_2 \rangle$$
$$\langle H_1, \nu_1 \rangle \otimes \langle -H_1, \nu_2 \rangle = P_{\nu_1 + \nu_2}$$

where the *projective module*

$$P_\nu = \langle \nu \rangle \longrightarrow \langle \nu + 1 \rangle \oplus \langle \nu - 1 \rangle \longrightarrow \langle \nu \rangle.$$

the righmost short subrep is known as "socle"

Central extensions evade Iohara-Koga theorem, since $psu(1|1)$ (modding out $\mathbb{B}$ to get su and then modding out H to get psu) still not simple

AdS_2: a different central extension of $su(1|1)^2$

Hoare-Pittelli-AT 1407.0303

We choose the Kac module and central extend each copy

$$
\begin{array}{lll}
\{\mathbb{Q},\mathbb{G}\} = 2\mathbb{C} & \mathbb{Q}|\phi\rangle = a|\psi\rangle & \mathbb{Q}|\psi\rangle = b|\phi\rangle \\
\{\mathbb{Q},\mathbb{Q}\} = 2\mathbb{P} & \mathbb{S}|\phi\rangle = c|\psi\rangle & \mathbb{S}|\psi\rangle = d\,|\phi\rangle \\
\{\mathbb{S},\mathbb{S}\} = 2\mathbb{K} & \mathbb{C}|.\rangle = C|.\rangle & 2C = ad + bc \\
P = ab & \mathbb{P}|.\rangle = P|.\rangle & \\
K = cd & \mathbb{K}|.\rangle = K|.\rangle &
\end{array}
$$

Closure of this long rep imposes

$$C^2 - PK = \frac{(ad-bc)^2}{4}$$

but $m \equiv ad - bc$ is *a priori* a free parameter (real for unitary rep)

at odds with all other AdS's

$m = 0$ works as a shortening condition: *e.g.* $\frac{\mathbb{Q}}{a} - \frac{\mathbb{S}}{c}$ kills both states

For sake of comparison we keep the traditional parameterisation:

$$a \sim \alpha\sqrt{\frac{h}{2}}\sqrt{i(x^- - x^+)} \qquad b \sim \alpha^{-1}\sqrt{\frac{h}{2}}e^{-\frac{ip}{2}}\frac{\sqrt{i(x^- - x^+)}}{x^-}$$

$$c \sim \alpha\sqrt{\frac{h}{2}}e^{\frac{ip}{2}}\frac{\sqrt{i(x^- - x^+)}}{x^+} \qquad d \sim \alpha^{-1}\sqrt{\frac{h}{2}}\sqrt{i(x^- - x^+)}$$

"Dispersion relation"

$$C^2 = \frac{1}{4}\left(m^2 + 4h^2 \sin\frac{p}{2}\right) = \frac{E^2}{4}$$

Zhukovsky variables

$$x^\pm = e^{\pm\frac{ip}{2}}\frac{E+m}{2h\sin\frac{p}{2}}$$

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2im}{h}$$

We will set $\alpha = 1$ (natural in comparison with string theory)

Hopf algebra is shaped on the higher dimensional counterparts

$$\Delta(\mathbb{Q}) = \mathbb{Q} \otimes \mathbb{1} + e^{\frac{ip}{2}} \otimes \mathbb{Q} \qquad \Delta(\mathbb{S}) = \mathbb{S} \otimes \mathbb{1} + e^{\frac{-ip}{2}} \otimes \mathbb{S}$$
$$\Delta(\mathbb{P}) = \mathbb{P} \otimes \mathbb{1} + e^{ip} \otimes \mathbb{P}, \qquad \Delta(\mathbb{K}) = \mathbb{K} \otimes \mathbb{1} + e^{-ip} \otimes \mathbb{K}$$
$$\Delta(\mathbb{H}) = \mathbb{H} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{H}.$$

This will be used to obtain the R (S) matrix

$$\Delta^{op}(.)\, R = R\, \Delta(.)$$

Tensor product of two long 2D reps is genericaly fully reducible in two long 2D reps $\longrightarrow$ extra (unit. cross.) requirements to fix all coefficients

Still true that tensoring "m" and "$-m$" gives reducible but indecomposable ("projective")

R (S) matrix is of 8-vertex type

(another difference w.r.t. higher dimensions)

$$\begin{pmatrix} S_1 & 0 & 0 & Q_1 \\ 0 & T_1 & R_1 & 0 \\ 0 & R_2 & T_2 & 0 \\ Q_2 & 0 & 0 & S_2 \end{pmatrix}$$

It satisfies the free fermion condition $\longrightarrow$ later

Problem The entries are particularly difficult to simplify

Much more than in higher dimensions

Crossing and unitarity can be formulated and constraints for the dressing phase can be written down, but very little hope to solve them

Massless case

$x^{\pm} \to e^{\pm i\frac{p}{2}} \operatorname{sign}\left[\sin\frac{p}{2}\right]$

One function appears eveywhere with interesting massless limit

$$f = \frac{\sqrt{\frac{x_1^+}{x_1^-}}\left(x_1^- - \frac{1}{x_1^+}\right) - \sqrt{\frac{x_2^+}{x_2^-}}\left(x_2^- - \frac{1}{x_2^+}\right)}{1 - \frac{1}{x_1^+ x_1^- x_2^+ x_2^-}}$$

becoming $\frac{0}{0}$: regularise by sending one mass to zero at a time

- For mixed right-left and left-right - order does not matter: $f \to 1$, resp. $f \to -1$
- For right-right and left-left ambiguous, both cases either $f \to \pm 1$

curious observation: ambiguity also in Fendley's "2^{nd}" $\mathcal{N} = 1$ S-matrices

Fendley 1990

Let us focus on right-right: ambiguity of limit (hystorically) dubbed *solution 3* and *5*

Relativistic variable $\to$ difference form

Fontanella-AT 1903.10759

Fontanella-Stefanski-Ohlsson Sax -AT 1905.00757, Frolov-Sfondrini 2112.08895

$$\gamma = \log \tan \frac{p}{4} \qquad p \in (0, \pi)$$

Define $\theta = \gamma_1 - \gamma_2$ (facilitates BMN limit)

Solution 3:

$$\Omega_3(\theta) \begin{pmatrix} 1 & 0 & 0 & -e^{-\frac{\theta}{2}} \\ 0 & -1 & e^{-\frac{\theta}{2}} & 0 \\ 0 & e^{-\frac{\theta}{2}} & 1 & 0 \\ -e^{-\frac{\theta}{2}} & 0 & 0 & -1 \end{pmatrix}$$

Solution 5:

$$\Omega_5(\theta) \begin{pmatrix} 1 & 0 & 0 & e^{\frac{\theta}{2}} \\ 0 & 1 & e^{\frac{\theta}{2}} & 0 \\ 0 & e^{\frac{\theta}{2}} & -1 & 0 \\ e^{\frac{\theta}{2}} & 0 & 0 & -1 \end{pmatrix}$$

Massless sector controlled by 8-vertex type S-matrix $\rightarrow$
minimal dressing factors are found

Fontanella-AT 1706.02634, 1903.10759

$$\Omega_3(\theta) = \frac{e^{\frac{\gamma}{2}-\frac{\pi i}{8}+\frac{\theta}{4}}}{\sqrt{2\pi}} \prod_{j=1}^{\infty} e^{-\frac{1}{2j}} j \frac{\Gamma\left(j-\frac{1}{2}+\frac{\theta}{2\pi i}\right)\Gamma\left(j-\frac{\theta}{2\pi i}\right)}{\Gamma\left(j+\frac{1}{2}-\frac{\theta}{2\pi i}\right)\Gamma\left(j+\frac{\theta}{2\pi i}\right)}$$

satisfies

$$\Omega_3(\theta)\Omega_3(\theta+i\pi) = \frac{e^{\frac{\theta}{2}}}{2\cosh\frac{\theta}{2}}$$

γ is Euler's constant. $\Omega(\theta)$ has no poles in the physical strip - all poles on imaginary axis

$$\Omega_5(\theta) = \Omega_3(i\pi-\theta)$$

curious braiding unitarity $\qquad \Omega_5(-\theta)\Omega_3(\theta) = \frac{1}{1+e^{-\theta}}$

2D $\mathcal{N}=1$ supersymmetry

the theory is at the critical point in BMN limit

Physical unitarity $\quad \Omega_3^*(\theta)\Omega_3(\theta) = \frac{1}{1+e^{-\theta}} \qquad \theta$ real

Using Malmstén rep of Γ one finds

$$\Omega_3(\theta) = \frac{e^{-i\frac{\pi}{8}}}{\sqrt{2}} \exp\left[\frac{\theta}{4} + \frac{1}{2}\int_0^\infty \frac{dx}{x} \frac{\cosh[x(1-\frac{2\theta}{i\pi})] - \cosh x}{\cosh x \sinh 2x}\right]$$

in a suitable region of θ-plane

All these relations have an $\mathcal{N} = 1$ (Fendley) origin: Fontanella-AT 1706.02634

both S-matrices and the conjectured dressing factors limits at $\theta \to \pm\infty$ of Fendley's "2^{nd}" susy S-matrix with De Martino - Moriconi improved dressing

Fendley 1990, De Martino - Moriconi 9803136

dressing factor checked numerically Bielli-Gautam-AT 2302.08491

- Sol. 3 from Fendley $p = \frac{1}{2}$ at $\theta \to +\infty$, or Fendley $p = -\frac{3}{2}$ at $\theta \to -\infty$
- Sol. 5 from Fendley $p = \frac{1}{2}$ at $\theta \to -\infty$, or Fendley $p = -\frac{3}{2}$ at $\theta \to +\infty$

the curious braiding unitarity is then obvious, as $-\theta \to -\infty$ if $\theta \to \infty$, and Fendley is braiding unitary

Bethe Ansatz via Inversion Relations

Felderhof 1973, Zamolodchikov 1991

Alternative method to Algebraic Bethe Ansatz (no pseudo-vacuum)

works thanks to the free fermion condition

It is illustrative to do it for The $N = 1$ super-Sine-Gordon S-matrix

Ahn 1991

SSG coupling $\alpha \equiv \frac{\beta^2/4\pi}{1-\beta^2/4\pi}$

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$A = a_+ \, \mathrm{E}_{11} + b_+ \, \mathrm{E}_{22} \qquad B = d_+ \, \mathrm{E}_{12} + c_- \, \mathrm{E}_{21}$$
$$C = c_+ \, \mathrm{E}_{12} + d_- \, \mathrm{E}_{21} \qquad D = b_- \, \mathrm{E}_{11} + a_- \, \mathrm{E}_{22}$$

$$a_\pm = \pm 1 + \frac{2i \sin \alpha\pi}{\sinh \theta} \qquad b_- = b_+ = 1$$
$$c_+ = c_- = \frac{i \sin \alpha\pi}{\sinh \frac{\theta}{2}} \qquad d_- = d_+ = \frac{\sin \alpha\pi}{\cosh \frac{\theta}{2}}$$

The entries satisfy the *free-fermion condition*:

$$a_+ a_- + b_+ b_- = c_+ c_- + d_+ d_-$$

Define a new S-matrix $S^{(1)}$ same form as S but replacing

$$a_\pm \to a_\pm^{(1)} = -b_\pm \qquad b_\pm \to b_\pm^{(1)} = a_\pm$$
$$c_\pm \to c_\pm^{(1)} = c_\pm \qquad d_\pm \to d_\pm^{(1)} = -d_\pm$$

$S^{(1)}$ still satisfies the free-fermion condition

$$\begin{aligned} T\, T^{(1)} &= \mathrm{tr}_0 \left[\mathrm{S}_{01}(\theta - \theta_1) \ldots \mathrm{S}_{0\mathrm{N}}(\theta - \theta_\mathrm{N}) \right] \mathrm{tr}_{0'} \left[\mathrm{S}^{(1)}_{0'1}(\theta - \theta_1) \ldots \mathrm{S}^{(1)}_{0'\mathrm{N}}(\theta - \theta_\mathrm{N}) \right] \\ &= \mathrm{tr}_{0 \otimes 0'} \prod_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{S}_{0\mathrm{i}}(\theta - \theta_\mathrm{i}) \otimes \mathrm{S}^{(1)}_{0'\mathrm{i}}(\theta - \theta_\mathrm{i}) \end{aligned}$$

tensor product is between the two auxiliary spaces 0 and $0'$

Trick: find a similarity transformation on $S_{0i}(\theta-\theta_i)\otimes S^{(1)}_{0'i}(\theta-\theta_i)$ putting it into upper triangular form.
Transformation performed at each site but indep. of θ_i $\rightarrow$ similarity matrices cancel and trace is straightforward

works thanks to free-fermion condition

$$X=\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \cosh\tau & 0 & 0 & -\sinh\tau \\ -\sinh\tau & 0 & 0 & \cosh\tau \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad \tanh 2\tau=\sin\alpha\pi$$

is such that

$$XS_{0i}\otimes S^{(1)}_{0'i}X^{-1} = X\begin{pmatrix} AA^{(1)} & AB^{(1)} & BA^{(1)} & BB^{(1)} \\ AC^{(1)} & AD^{(1)} & BC^{(1)} & BD^{(1)} \\ CA^{(1)} & CB^{(1)} & DA^{(1)} & DB^{(1)} \\ CC^{(1)} & CD^{(1)} & DC^{(1)} & DD^{(1)} \end{pmatrix}X^{-1}=\begin{pmatrix} m_+ & * & * & * \\ 0 & n_+ & * & * \\ 0 & 0 & n_- & * \\ 0 & 0 & 0 & m_- \end{pmatrix}$$

with

$$m_+ = -\frac{\sinh(\theta/2+i\alpha\pi)}{\sinh\theta/2}\frac{\sinh(\theta/2-i\alpha\pi)}{\sinh\theta/2}\mathbb{1} \qquad m_- = -\frac{\cosh(\theta/2+i\alpha\pi)}{\cosh\theta/2}\frac{\cosh(\theta/2-i\alpha\pi)}{\cosh\theta/2}\mathbb{1}$$

$$n_+ = -\frac{\sinh(\theta/2+i\alpha\pi)}{\sinh\theta/2}\frac{\cosh(\theta/2-i\alpha\pi)}{\cosh\theta/2}\sigma_3 \qquad m_- = -\frac{\cosh(\theta/2+i\alpha\pi)}{\cosh\theta/2}\frac{\sinh(\theta/2-i\alpha\pi)}{\sinh\theta/2}\sigma_3$$

$\mathbb{1}=\mathrm{E}_{11}+\mathrm{E}_{22}$ and $\sigma_3=\mathrm{E}_{11}-\mathrm{E}_{22}$. Since $\mathrm{tr}_{0\otimes 0'}=\mathrm{tr}_4$

$$T\,T^{(1)} = \prod_{i=1}^{N} m_+(\theta-\theta_i)+\prod_{i=1}^{N} m_-(\theta-\theta_i)+\prod_{i=1}^{N} n_+(\theta-\theta_i)+\prod_{i=1}^{N} n_-(\theta-\theta_i)$$

Use relation between S and $S^{(1)}$:

$$S^{(1)}_{0'i} = -\sigma_1 \, S_{0'i}(\theta - \theta_i + i\pi)\sigma_1$$
$$\sigma_1 \equiv \mathsf{E}_{12} + \mathsf{E}_{21}$$

similarity transformation performed in space $0'$. Therefore

$$T^{(1)}(\theta) = (-)^N \, T(\theta + i\pi)$$

which turns our problem into a crossing-type equation referred to as *inversion relation*.

The eigenvalues of will be given by the same expression, with $\prod \sigma_3$ replaced by the fermionic number of the particular eigenstate

The final task is to factorise as $f(\theta)f(\theta+i\pi)$

$$TT^{(1)} = \left[\prod_{i=1}^{N} A(\theta-\theta_i) + F\prod_{i=1}^{N} B(\theta-\theta_i)\right] \times \left[\prod_{i=1}^{N} C(\theta-\theta_i) + F\prod_{i=1}^{N} D(\theta-\theta_i)\right] \frac{1}{\prod_{i=1}^{N} 2\cosh^2(\theta-\theta_i)}$$

where $F=\pm$ is the fermionic degree of the particular state one considers, and (assuming $\alpha>0$)

$$A(\theta-\theta_i) = \frac{\cosh(\theta_{0i}/2+i\alpha\pi)}{\cosh\theta_{0i}/2} \qquad B(\theta-\theta_i) = \frac{\sinh(\theta_{0i}/2+i\alpha\pi)}{\sinh\theta_{0i}/2}$$

$$C(\theta-\theta_i) = \frac{\cosh(\theta_{0i}/2-i\alpha\pi)}{\cosh\theta_{0i}/2} \qquad D(\theta-\theta_i) = \frac{\sinh(\theta_{0i}/2-i\alpha\pi)}{\sinh\theta_{0i}/2}$$

One can show $T(\theta+2i\pi)=T(\theta)$, hence can restrict to strip $\theta\in[-\pi,\pi)$. Potential zeroes can come from

$$\prod_{i=1}^{N}\frac{A(z_k-\theta_i)}{B(z_k-\theta_i)} = -F \qquad \prod_{i=1}^{N}\frac{C(z_k-\theta_i)}{D(z_k-\theta_i)} = -F$$

namely

$$\prod_{i=1}^{N}\frac{\tanh\left[\frac{z_j-\theta_i}{2}+i\alpha\pi\right]}{\tanh\frac{z_j-\theta_i}{2}} = -F \qquad \prod_{i=1}^{N}\frac{\tanh\left[\frac{z_j-\theta_i}{2}-i\alpha\pi\right]}{\tanh\frac{z_j-\theta_i}{2}} = -F$$

for any fixed value of the fermionic number $F=0,1$ of the state under consideration

To extract the actual zeros of T one needs some extra work

In AdS_2 the auxiliary Bethe equations known in the literature reduce in the massless limit to the square of the ones we obtain via this method

Sorokin-Tseytlin-Wulff-Zarembo 1104.1793

Momentum carrying equations conjectured in AT 1708.09598

not easily comparable to known ones

Massless transfer matrix brute force up to 5 sites and Yangian

special "conformal Yangian": $u = 0$ in Drinfeld 1^{st} realis.

Creutzig 1011.6424

Away from BMN - use γ-variable but non-conformal

Great need for tests (perturbation theory, holographic dual)

Free-fermion descript. wth de Leeuw, Paletta, Pribytok and Retore

"pseudo-pseudo-vacuum"

2011.08217

AdS_2 Form factors with Bielli and Gautam

Mussardo's trick and De Martino Moriconi dressing phase

2302.08491

AdS_2 Deformation with Nieto Garcia and Ruiz

infinite dimensional generalisation

2302.08491

Several directions of investigation $\longrightarrow$ future work

QSC with Volin-Ekhammar - in discussion

$AdS_2 - AdS_3$ interpolating S-matrix

de Leeuw-Pribytok-Retore-Ryan 2109.00017

see also deformations Hoare-Seibold 1811.07841, Bocconcello-Masuda-Seibold-Sfondrini 2008.07603

Thanks to EPSRC/SFI grant *Solving Spins and Strings* with Marius de Leeuw - past postdocs Ana Retore and Juan Miguel Nieto Garcia

Stay tuned - thank you very much!

Bonus material

FIVE DIMENSIONS: $\text{AdS}_5/\text{CFT}_4$

enormous literature - people in this room

$AdS_5 \times S^5$ dual to $\mathcal{N} = 4$ Super Yang Mills in 4D

Superalgebra from psu family: $psu(2,2|4)$

- Coset: $PSU(2,2|4)/[SO(4,1) \times SO(5)]$
- Relation among parameters: $g_s = g_{YM}^2$, $\lambda = g_{YM}^2 N \propto R^4/\alpha'^2$
- Central charge balanced by coset

$\longrightarrow$

- Spectral problem "solved" **TBA - Quantum Spectral Curve**

Arutyunov-Frolov 0903.0141 Bombardelli-Fioravanti-Tateo 0902.3930

Gromov-Kazakov-Kozak-Vieira 0902.4458 Gromov-Kazakov-Leurent-Volin 1305.1939

FOUR DIMENSIONS: $\mathrm{AdS}_4/\mathrm{CFT}_3$

$AdS_4 \times \mathbb{CP}^3$ dual to $\mathcal{N} = 6$ Chern-Simons in 3D

Superalgebra from osp family: $osp(6|4)$

- Coset: $OSP(6|4)/[SO(3,1) \times U(3)]$
- Central charge balanced by coset

$\longrightarrow$

- Spectral problem "similar" (in a way) to five-dimensions - $h(\lambda)$

THREE DIMENSIONS: $\mathrm{AdS}_3/\mathrm{CFT}_2$

1. $AdS_3 \times S^3 \times S^3 \times S^1$ dual to mystery CFT_2 [Tong 1402.5135]

Superalgebra from osp family (deformation): $d(2,1;\alpha)^2$

- Coset: $D(2,1;\alpha)^2/[SU(1,1) \times SU(2) \times SU(2)]$
- Relation among parameters: $R^2_{AdS}/R^2_{S_+} = \alpha = 1 - R^2_{AdS}/R^2_{S_-}$
- Central charge balance needs extra S^1 CFT

$\longrightarrow$ Babichenko-Stefanski-Zarembo 0912.1723

- Massless modes appear

Borsato-Sfondrini-Sax-Stefanski-AT - people in this room and many others

THREE DIMENSIONS: AdS_3/CFT_2

2. $AdS_3 \times S^3 \times T^4$ dual to CFT_2 [Ohlsson Sax, Sfondrini, Stefanski 1411.3676]

Superalgebra from psu family: $psu(1,1|2)^2$

- Coset: $PSU(1,1|2)^2/[SU(1,1) \times SU(2)]$
- Relation among parameters: $R_{AdS} = R_S$
- Central charge balance needs extra T^4 CFT

$\longrightarrow$ *Symmetric orbifold, Higher spins* *Gaberdiel, Gopakumar et al*

- More massless modes

Borsato-Sfondrini-Sax-Stefanski-Frolov-AT - people in this room and many others

By now huge literature: please ask me for references