Non-perturbative partition functions in susy QFT

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a rather elegant picture is emerging

for the non-perturbative topological string:

BPS quivers

Riemann-Hilbert problems /

DT invariants

[Bridgeland et al]

[Alim, Saha, Teschner, Tulli]

topological string / spectral theory

[Grassi, Marino, Gu, ...]

non-perturbative

topological string

susy gauge thy

Ω- background

[Nekrasov, Shatashvili]

[Gaiotto, Moore, Neitzke]

wall-crossing

holomorphic Floer thy

[Kontsevich-Soibelman]

BPS states

spectral networks

[H, Neitzke]

TBA-like integral eqns

[H, Rüter, Szabo]

[Grassi, Hao, Neitzke]

exact WKB analysis

abelianization

[Alim, H, Tulli]

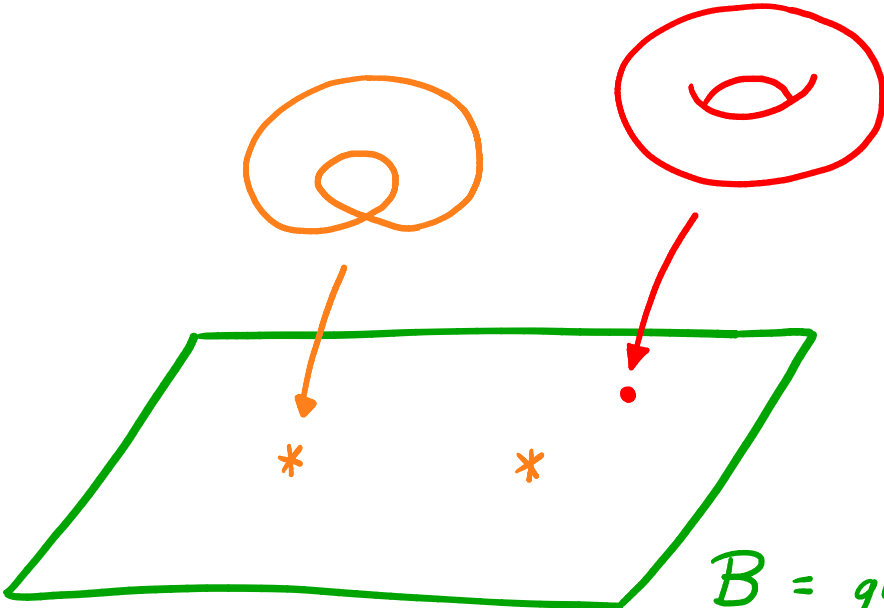
4d $\mathcal{N}=2$ theories and Seiberg-Witten geometry

beautiful structure

+ clear path for 5d generalization

Start in 4d with $\mathcal{N}=2$ thy, e.g. pure $SU(2)$ SYM

Low energy description:



Σ = Seiberg-Witten curve

B = quantum Coulomb branch

[Seiberg-Witten]

4d $\mathcal{N}=2$ thys of class $\mathcal{S}$

rank K $\leftrightarrow$ $su(K)$

e.g. pure $SU(2)$ gauge thy

but also intrinsically strongly

coupled thy MN E_6

$\updownarrow$

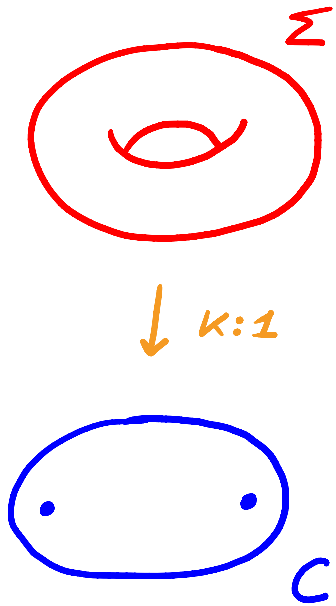
Seiberg-Witten curve $\Sigma \subset T^*_y C_x$

$\|$

$\{\det(y - \varphi(x)) = 0\}$ $\lambda = y\,dx$

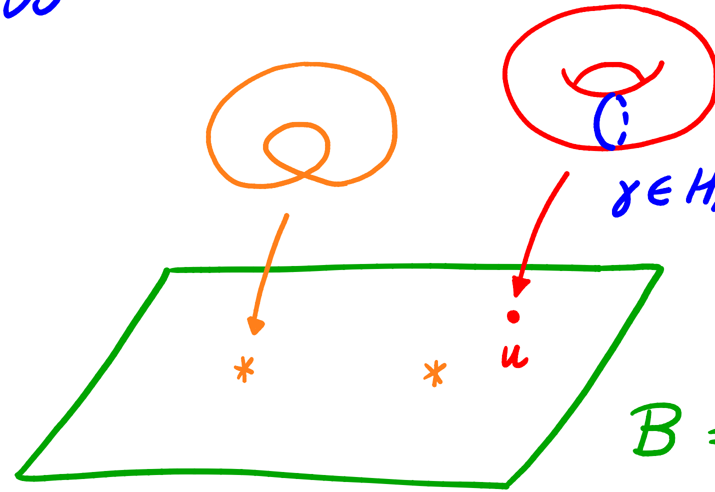
spectral curve in

Hitchin integrable system



[Gaiotto], [Gaiotto-Moore-Neitzke]

Low energy $\mathcal{I}$:



Σ = Seiberg-Witten curve

$\gamma \in H_1(\bar{\Sigma})$

$\mathcal{B}$ = quantum Coulomb branch

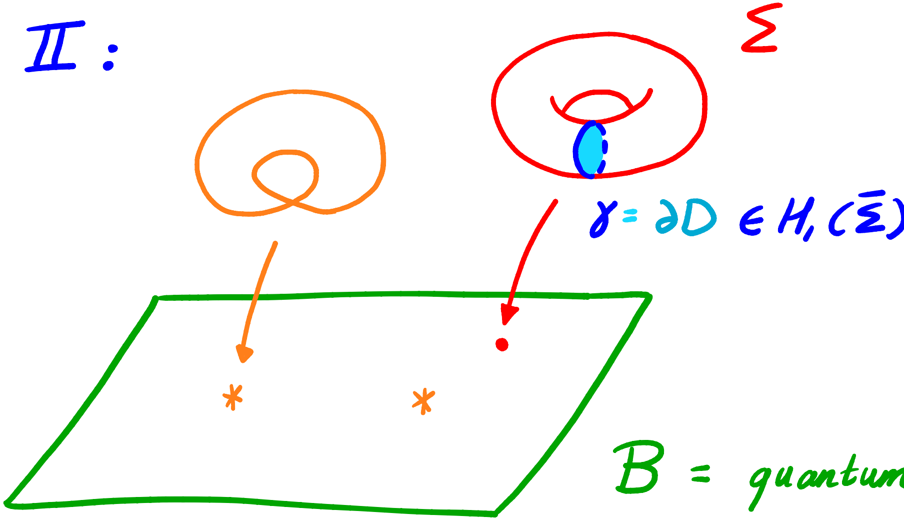
hol'c central charge function $\quad Z: H_1(\bar{\Sigma}_u) \to \mathbb{C}$

$$Z(\gamma) = \oint_\gamma \lambda$$

$$\left.\begin{aligned} Z(\alpha^I) &\equiv a^I \\ Z(\beta_I) &\equiv a_{D,I} \end{aligned}\right\} \quad \frac{\partial \mathcal{F}_0}{\partial a^I} = a_{D,I}$$

hol'c prepotential

Low energy II :



B = quantum Coulomb branch

BPS particles

$$\mathcal{H}^{BPS}_{u,\gamma} = \{ \psi \in \mathcal{H}_{u,\gamma} \mid H\psi = |Z_\gamma(u)|\psi \}$$

↑ Hilbert space in vacuum u

[Klemm, Lerche, Mayr, Vafa]

BPS condition $\quad \left| \oint_\gamma \lambda \right| = \oint_\gamma |\lambda|$

(calibration)

↑ phase λ "constant along γ"

hol'c prepotential

$$Z(\alpha^I) = a^I$$
$$Z(\beta_I) = a_{D,I}$$

$$\frac{\partial F_0}{\partial a^I} = a_{D,I}$$

$$F_0 = F_0^{cl} + F_0^{1\text{-loop}} + F_0^{inst}$$

$$\tau \underline{a}^2 \qquad \underline{a}^2 \log \underline{a}^2 \qquad \sum_{k=1}^{\infty} F_k(\underline{a}) \Lambda^{4k}$$

vs $\updownarrow$

BPS particles

$$\mathcal{H}^{BPS}_{u,\gamma} = \{ \psi \in \mathcal{H}_{u,\gamma} \mid H\psi = |Z_\gamma(u)|\psi \}$$

Hilbert space in vacuum u

beautiful relation between these objects

$\rightsquigarrow$ if we consider the $\mathcal{N}=2$ theory

in the Ω-background

[LH, Neitzke][2]

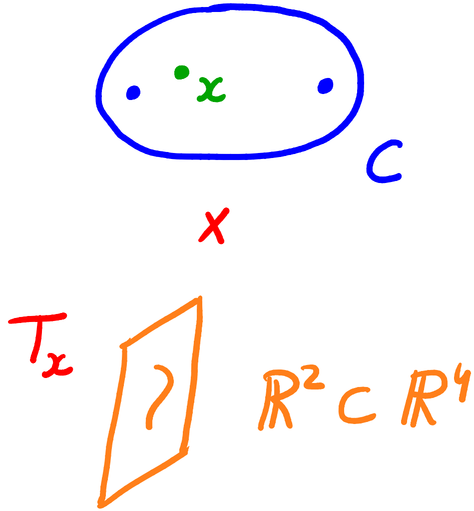
[LH, Rüter, Szabo]

[Grassi, Hao, Neitzke][2]

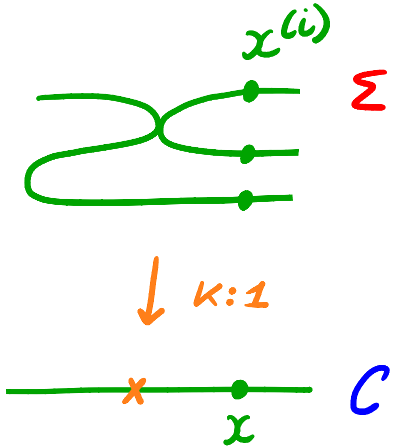
[Alim, LH, Tulli]

Surface defects and BPS states

UV curve C plays another important role in the $4d$ $\mathcal{N}=2$ theory:



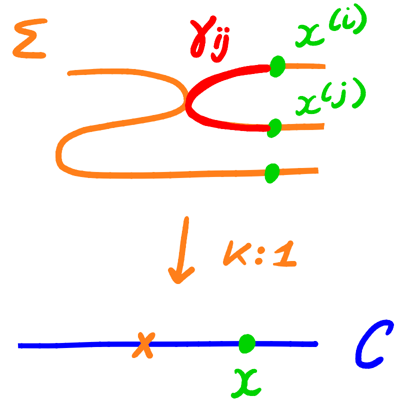
C = moduli space of canonical surface defect S_x



$x^{(i)}$ parametrize the vacua of the $2d$ $\mathcal{N}=(2,2)$ worldvolume theory T_x on the defect

[Cecotti-Vafa]

The 2d theory T_x has a spectrum of 2d BPS states that describe the tunneling between the vacua $x^{(i)}$



BPS condition: $\left| \oint_{\gamma_{ij}} \lambda \right| = \oint_{\gamma_{ij}} |\lambda|$

Both 2d and 4d BPS states may be visualized using the technology of spectral networks $\mathcal{W}_{u,\vartheta}$:

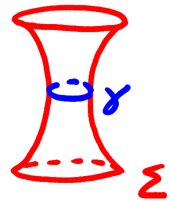
Fix phase ϑ, draw all real 1-dim'l trajectories on C with

$$(\lambda_j - \lambda_k) \cdot v \in e^{i\vartheta} \mathbb{R}$$

[Gaiotto-Moore-Neitzke]

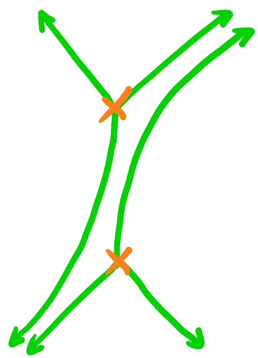
Example:

$$\begin{array}{l} \Sigma = \{ y^2 = x^2 - m \} \\ \big\downarrow {\scriptstyle 2:1} \\ C = \mathbb{C} \end{array}$$



real 1-dim'l trajectories defined by $\lambda \cdot v \in e^{i\vartheta}\mathbb{R}$:

generic ϑ:

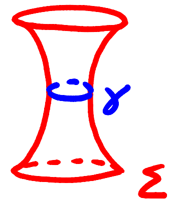


Example:

$$\Sigma = \{ y^2 = x^2 - m \}$$

$$\downarrow 2:1$$

$$C = \mathbb{C}$$



real 1-dim'l trajectories defined by $\lambda \cdot v \in e^{i\vartheta}\mathbb{R}$:

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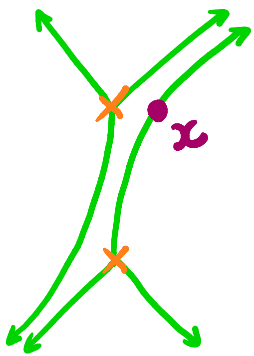
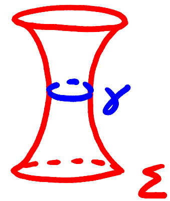


insert surface defect $\mathbb{T}_x$

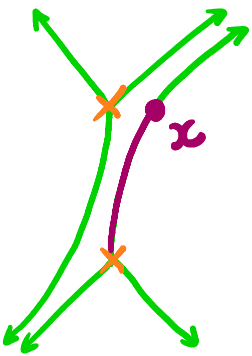
Example: $\Sigma = \{ y^2 = x^2 - m \}$

$\downarrow$ 2:1

$C = \mathbb{C}$

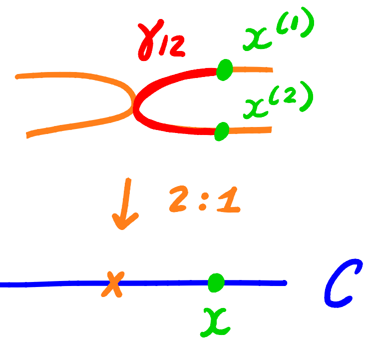


real 1-dim'l trajectories defined by $\lambda \cdot v \in e^{i\vartheta}\mathbb{R}$:



consider open trajectory ending on x

↳ lifts to Σ



say $\vartheta = \vartheta_c^{2d}$

2d BPS state in T_x

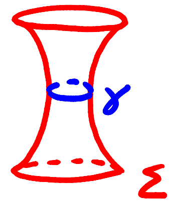
with $\arg(Z_{2d}) = \vartheta_c^{2d}$

Example:

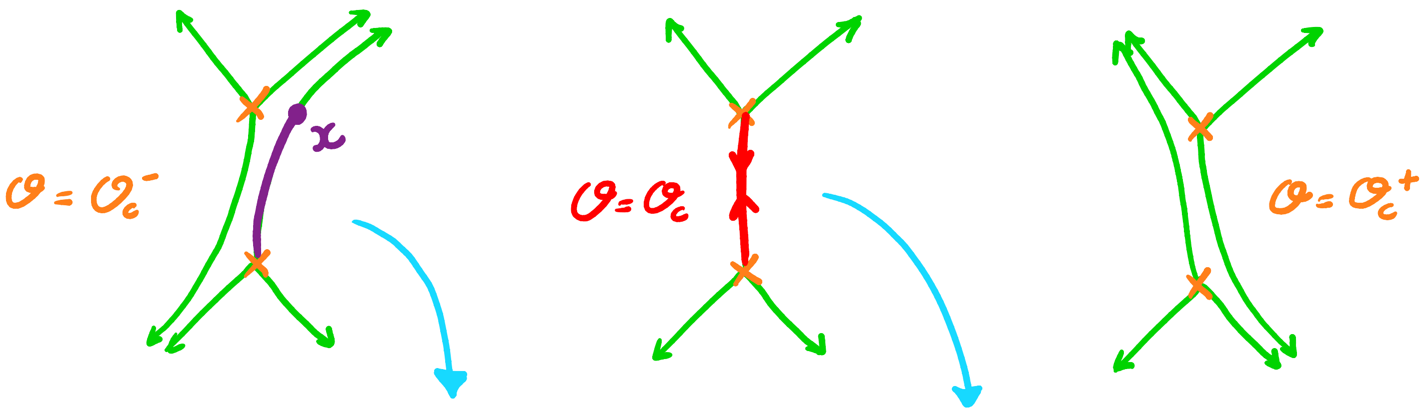
$$\Sigma = \{ y^2 = x^2 - m \}$$

$$\downarrow 2:1$$

$$C = \mathbb{C}$$



real 1-dim'l trajectories defined by $\lambda \cdot v \in e^{i\vartheta}\mathbb{R}$:

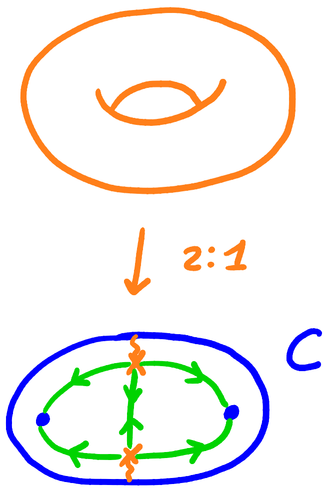


2d BPS state in T_x
with $\arg(Z_{2d}) = \vartheta_c^-$

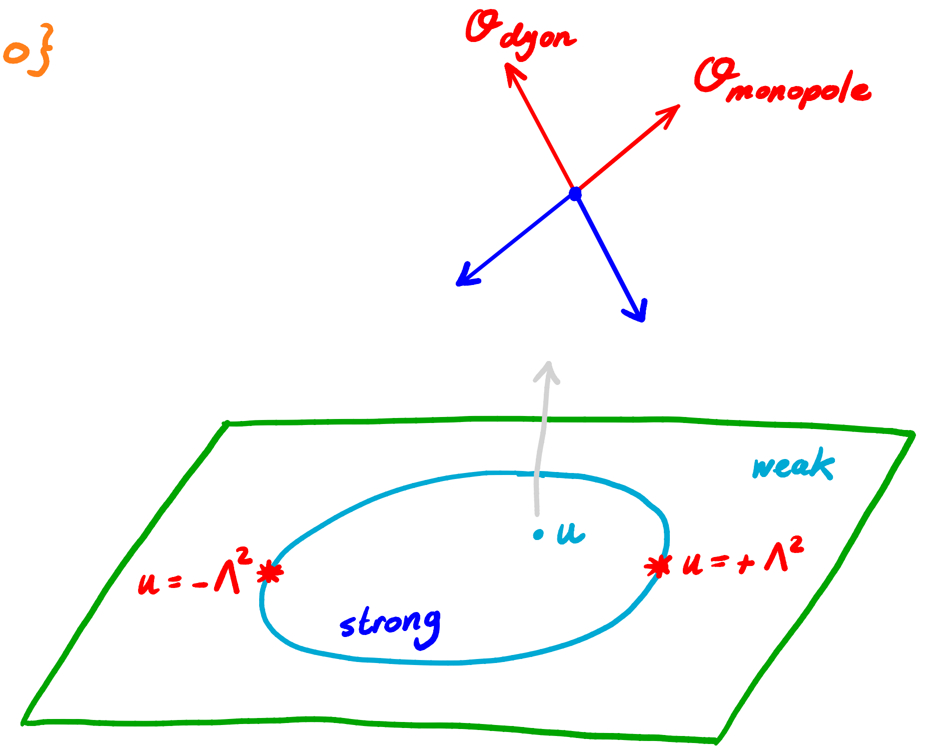
4d BPS state in T_C
with $\arg(Z_{4d}) = \vartheta_c$

Ex: 4d BPS states in pure SU(2) SYM

$\Sigma_u : \{ w^2 - \varphi_2(z) = 0 \}$



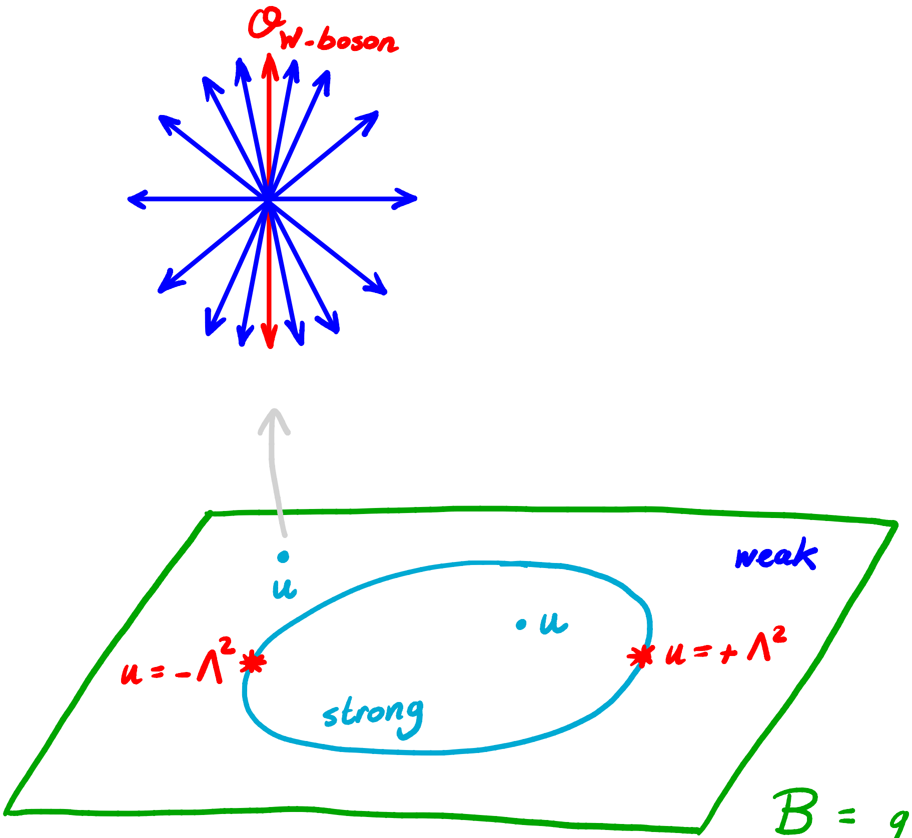
$\vartheta = \vartheta_{monopole}$

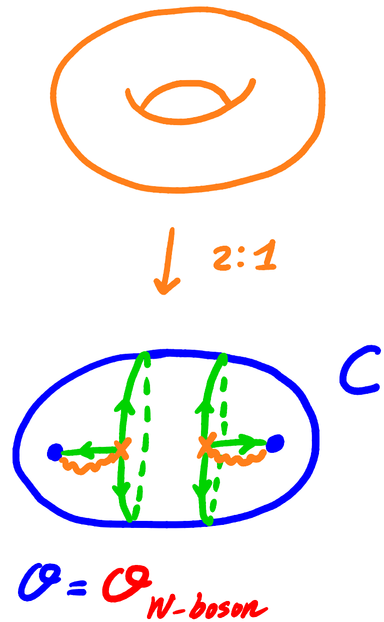


B = quantum Coulomb branch

Ex: 4d BPS states in pure SU(2) SYM

$\Sigma_u : \{ w^2 - \varphi_2(z) = 0 \}$



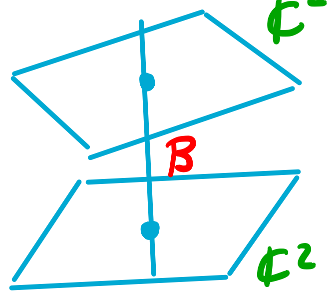


B = quantum Coulomb branch

$\frac{1}{2}\,\Omega$-background and quantization

Ω - background [Nekrasov, Nekrasov-Okounkov]

5d $\mathcal{N}=1$ SYM



on $\mathbb{C}^2$-bundle over S^1_β:

$$(z_1, z_2, x_5) \to (z_1 e^{i\beta\epsilon_1}, z_2 e^{i\beta\epsilon_2}, x_5+\beta)$$

also need $SU(2)_I$ rotation

$$\begin{pmatrix} e^{i\beta(\epsilon_1+\epsilon_2)/2} & 0 \\ 0 & e^{-i\beta(\epsilon_1+\epsilon_2)/2} \end{pmatrix}$$

to preserve susy

$\beta \to 0 \quad \downarrow$

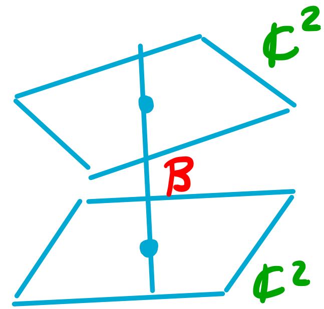
and gauge rotation $e^{i\beta \underline{a}} \in G$

4d $\mathcal{N}=2$ SYM

in the Ω-background

$\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$

Partition function in Ω- background



In 5d compute index

$$Z^{Nek}_{5d}(\epsilon_{1,2}, \tau, \underline{a}) = \sum (-)^F e^{-\beta H} g$$

$$\Big\downarrow \ \beta \to 0$$

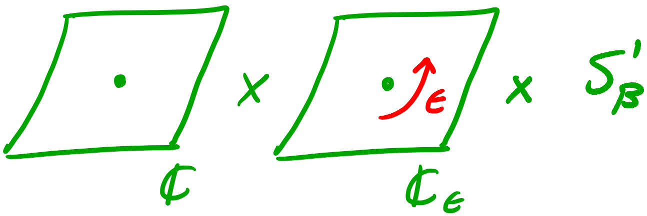
Reduces in 4d to instanton partition function:

$$\log Z^{Nek}(\epsilon_{1,2}, \tau, \underline{a}) = \frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_0(\tau, \underline{a}) + \text{terms less singular in } \epsilon_{1,2}$$

SW prepotential

[Nekrasov], [Nekrasov-Okounkov]

Today we are interested in $\frac{1}{2}\Omega$-background :



i.e. $\mathbb{C}^2$-bundle over S^1_β

aka $\mathbb{C} \times (\mathbb{C} \times_q S^1_\beta)$

The $\frac{1}{2}\Omega$-background effectively defines

a 3d $\mathcal{N}=2$ theory on $\mathbb{C} \times S_\beta$

$\beta \to 0$

a 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{C}$

At low energies $E \ll |\epsilon|$, this theory is described by an effective twisted superpotential

$$\widetilde{W}^{\text{eff}}(\epsilon, \tau, \underline{a}) = \lim_{\epsilon_2 \to 0} \epsilon_2 \, Z^{\text{Nek}}(\epsilon_{1,2}, \tau, \underline{a})$$

[Nekrasov-Shatashvili]

$\frac{1}{2}\,\Omega$-background quantizes the Seiberg-Witten geometry:
$(\hbar = \epsilon)$

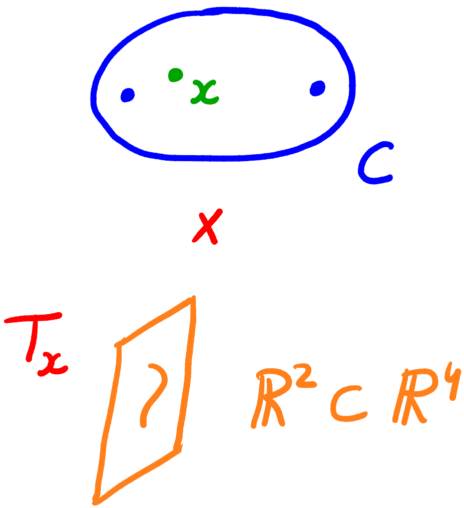
SW-curve Σ $\underset{4d}{\rightsquigarrow}$ differential operator d_ϵ
(= oper = quantum curve)

pure SU(2) SYM: $w^2 - \varphi_2(z) = 0$ $\rightsquigarrow$

Mathieu diff'l eqn $d_\epsilon = \epsilon^2 \partial_z^2 - \left(\varphi_2(z) + \frac{\epsilon^2}{4z^2}\right)$

$\underset{5d}{\rightsquigarrow}$ difference operator D_ϵ
(= q-difference oper)

pure U(1) SYM: $y = \frac{1-QX}{1-X}$ $\rightsquigarrow$ $D_\epsilon = (1-X)\, e^{\frac{\epsilon}{2\pi}\partial_X} - (1-QX)$



Solutions to the diff'l eqn

$$d_\epsilon \psi^{(i)}(x) = 0$$

characterize the vacuum expectation value of the surface defect T_x in the vacuum $x^{(i)}$

The monodromies of $\psi^{(i)}(x)$ around cycles $\gamma \in H^1(\bar{\Sigma})$ define the quantum periods:

$$\left.\begin{aligned} Z_\epsilon(\alpha^I) &\equiv a^I_\epsilon \\ Z_\epsilon(\beta_I) &\equiv a^\epsilon_{D,I} \end{aligned}\right\} \qquad \frac{\partial \widetilde{W}^{eff}(a^I_\epsilon)}{\partial a^I_\epsilon} = a^\epsilon_{D,I}$$

[Mironov-Morozov]

Note: the quantum periods a^I_ϵ and $a^\epsilon_{D,I}$ are perturbative in ϵ

What do we actually expect non-perturbatively in ϵ?

Is there a single (exact in ϵ) solution to $d_\epsilon \psi(x) = 0$?

[Maldacena, Moore, Shih, Seiberg]

Exact WKB analysis and non-pert partition functions

Something natural to consider: [Grassi, Marino, Gu, Schiappa, ..]

take the Borel sum of $\psi^{(i)}(x)$ and $\widetilde{W}^{\mathrm{eff}}(a^I)$

in fact, because both have Gevrey-1 asymptotics,

this is the unique thing one can do

[Nikolaev]

Questions:

- physics interpretation?
- how do we compute in particular $\widetilde{W}^{\mathrm{eff}}$ exact in ϵ?

Remember, the Borel sum $\mathcal{B}_\theta f(\epsilon)$ of a formal, divergent series $f(\epsilon)$ is defined as:

divergent $f(\epsilon) = \sum_{n=0}^{\infty} c_n \epsilon^n$ where $c_n \sim n!$ ← Gevrey-1 series

$\rightsquigarrow$ Borel transform $\mathcal{T}f(s) = \sum_{n=0}^{\infty} \frac{c_n}{n!} s^n$

$\rightsquigarrow$ Borel sum $\mathcal{B}_\theta f(\epsilon) = \int_0^{\infty e^{i\theta}} e^{-s}\, \mathcal{T}f(\epsilon s)\, ds$

Might think of Borel sum as "transseries":

zero modes from inst + anti-inst

$$\mathcal{B}_\theta f(\epsilon) = \sum\sum c_{k,\ell,n}\, \epsilon^n e^{-kc/\epsilon} \ln\left(\pm\frac{1}{\epsilon}\right)^\ell$$

non-pert instantons

Borel sums are particularly natural in context of diff'l eqns

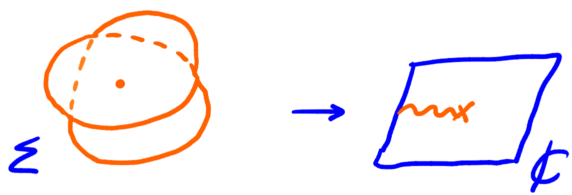
e.g. Schrödinger eqns, more generally opers

$\rightsquigarrow$ exact WKB analysis [Voros]

example: diff'l eqn on $\mathbb{C}$ with irreg sing at ∞

say Airy diff'l eqn $d_\epsilon = \epsilon^2 \partial_z^2 - z$, $\epsilon \in \mathbb{C}$

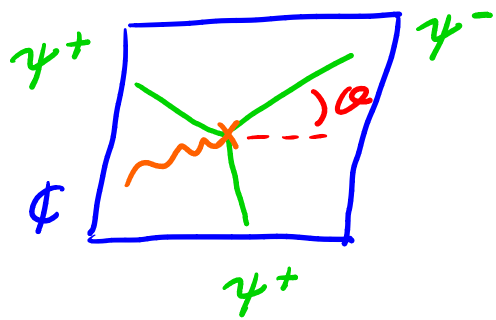
$\rightsquigarrow$ spectral curve $\Sigma = \{y^2 = x\} \xrightarrow[2:1]{} C = \mathbb{C}$



Airy diff'l eqn $d_\epsilon = \epsilon^2 \partial_z^2 - z$ $\quad \arg(\epsilon) = \vartheta$

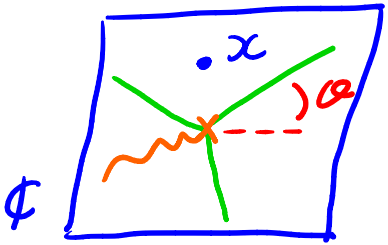
consider asymptotic solutions $\psi^\pm(x,\epsilon) \sim e^{\pm 2x^{3/2}/3\epsilon}$

and Stokes rays where these solutions decay fastest:



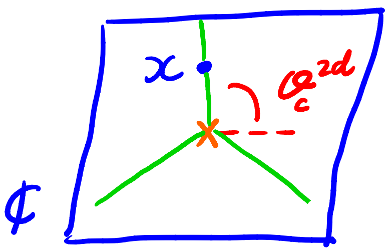
Note that these rays are precisely ϑ-trajectories of the spectral network defined by $\lambda = \sqrt{x}\,dx$

Pick $x \in \mathbb{C}$ and consider Borel sum $\mathcal{B}_\vartheta \psi_\pm(x,\epsilon)$



Remember that $\mathcal{B}_\vartheta \psi_\pm$ is analytic in ϵ and locally constant in ϵ

In particular, $\mathcal{B}_\vartheta(x)$ is not defined when we rotate ϑ such that the Stokes ray intersects with x:



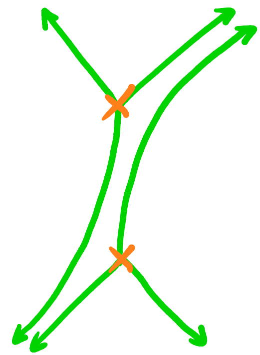
But this precisely happens when the 2d theory T_x supports a 2d BPS state with $\arg(Z) = \vartheta_c^{2d}$

More generally, in exact WKB one considers any (C, d_ϵ)

and ansatz $\psi^{(i)}(x) = \exp\left(\sum_{k=-1}^{\infty} \int^x \epsilon^k S_k^{(i)}(x')\, dx' \right)$

The set of Stokes lines

becomes a Stokes graph:



And, basically since $S_{-1}^{(i)} = \lambda^{(i)}$

the Stokes graph for $\arg(\epsilon) = \vartheta$

is equivalent to a spectral network $\mathcal{W}_\vartheta$

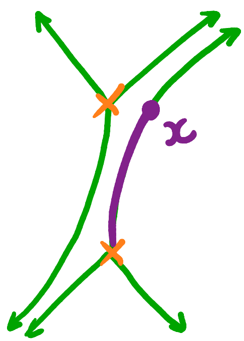
[LH, Neitzke]

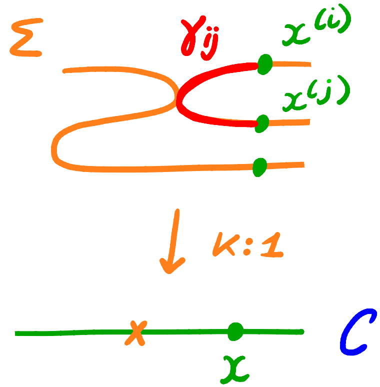
[LH, Kidwai]

The Borel sum $\mathcal{B}_\vartheta \psi^{(i)}$ is not defined along some critical directions ϑ_c

These critical directions ϑ_c correspond precisely to the phases $\vartheta_c^{2d} = \arg Z_\gamma^{2d}$ of the 2d BPS particles in the theory T_x

$\vartheta = \vartheta_c^{2d}$





Hence the jumps of $\mathcal{B}_\vartheta \psi^{(i)}(x,\epsilon)$ encode the 2d BPS particle spectrum!

[Grassi, Hao, Neitzke], [Alim, LH, Tulli]

Now define exact quantum periods $\mathcal{X}_\gamma^\vartheta$ as the monodromies of $\mathcal{B}_\vartheta \psi^{(i)}$ around cycles $\gamma \in H^1(\bar{\Sigma})$

new object: [LH, Rüter, Szabo]

$$\left.\begin{aligned} \log \mathcal{X}_{\alpha^I}^\vartheta &\equiv a_\vartheta^I \\ \log \mathcal{X}_{\beta_I}^\vartheta &\equiv a_{D,I}^\vartheta \end{aligned}\right\} \qquad \frac{\partial \widetilde{W}^{\text{eff}}_{u,\vartheta}(a_\vartheta^I)}{\partial a_\vartheta^I} = a_{D,I}^\vartheta$$

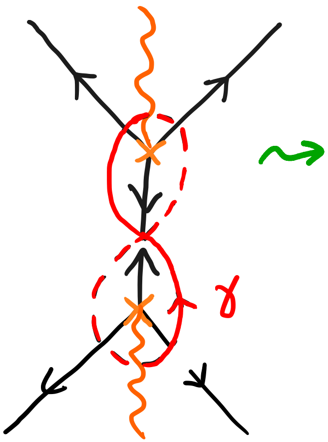
The exact quantum periods $\mathcal{X}_\gamma^\vartheta$ are also known as:

Voros periods (exact WKB) or

spectral coordinates ($\mathcal{W}$-abelianization)

$\hookrightarrow$ geometric interpretation exact WKB on $\mathcal{M}_{\text{flat}}(C, SL_K)$ [LH, Neitzke]

The non-perturbative superpotential $\widetilde{W}^{eff}_{u,\vartheta}$ is again piece-wise constant in ϑ. It jumps whenever there is a saddle trajectory in the spectral network $\mathcal{W}_{u,\vartheta}$:



4d BPS particle with $Z_\gamma = \oint_\gamma \lambda$

mass BPS particle

$$\sum_{k=1}^{\infty} \frac{e^{\pi i k x/\epsilon}}{k^2}$$

$$\parallel$$

$$\Delta \widetilde{W}^{eff}_{u,\vartheta} = \frac{\epsilon}{2\pi i} \, Li_2\left(e^{\pi i x/\epsilon}\right)$$

Hence $\widetilde{W}^{eff}_{u,\vartheta}$ encodes the 4d BPS particle spectrum!

[LH, Rüter, Szabo]

- Whereas the Borel sum of $\mathcal{W}^{\text{eff}}$ can only be computed with a weakly coupled description, the definition of $\mathcal{W}_{u,\varpi}(\epsilon,\tau,\underline{a})$ in terms of quantum periods may be extended to all of $\mathcal{B}$ (and used for theories without a weakly coupled description)

- The resulting $\mathcal{W}_{u,\varpi}(\epsilon,\tau,\underline{a})$ is the solution to a Riemann-Hilbert problem specified by the corresponding BPS structure.

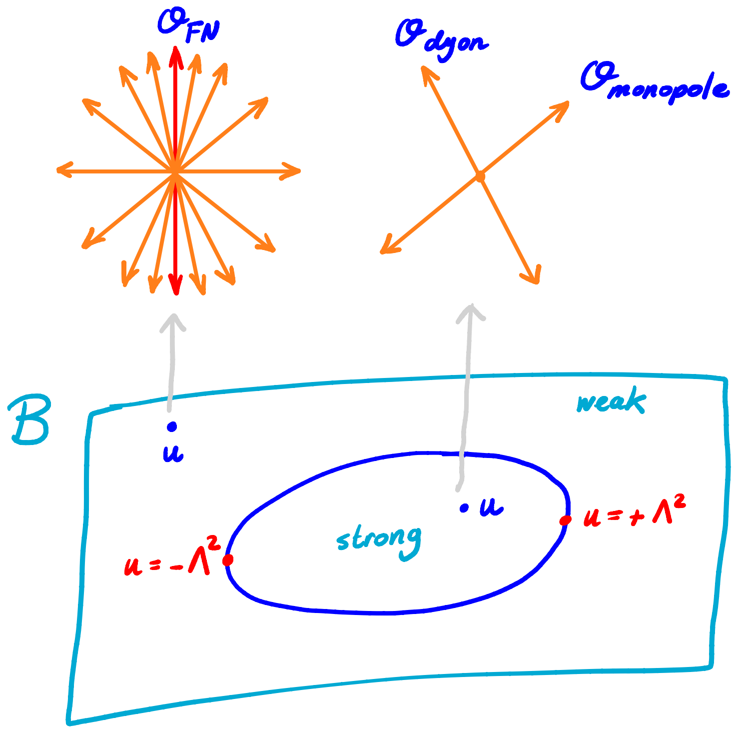
[Bridgeland], inspired by [Gaiotto, Moore, Neitzke]

similar to [Alim, Saha, Tulli, Teschner], [Alim, LH, Tulli] in 5d

- Geometrically, $\exp W_{u,\varphi}(\epsilon, \tau, \underline{a})$ transforms as a section of a "classical Chern-Simons" line bundle over $\mathcal{M}_{flat} \times \mathbb{C}^*_\epsilon$

↑ defined by gluings via generalized cluster charts

this line bundle is introduced / described in [Alexandrov, Persson, Pioline] [Neitzke], [Neitzke, Freed]



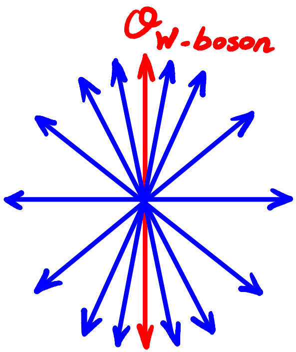
$$\rightsquigarrow \; W_{u,\varphi}(\epsilon, \tau, \underline{a}) \in \mathcal{L}$$
$$\downarrow$$
$$\mathcal{M}_{flat}(C) \times \mathbb{C}^*_\epsilon$$

it features in similar set-ups in:
[Alim-Saha-Tulli-Teschner] (5d, 5-d)
[Coman-Longhi-Teschner] (4d, 5-d)

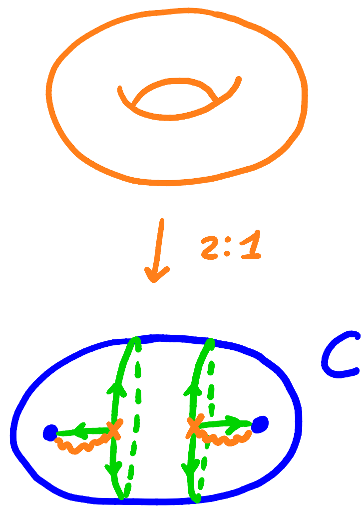
Example of a computation: pure SU(2) SYM

[LH-Rüter-Szabo]

weakly coupled



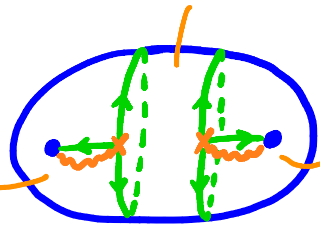
$$\Sigma_u : \{ w^2 - \varphi_2(z) = 0 \}$$



$$\mathcal{O} = \mathcal{O}_{W\text{-boson}} \equiv \mathcal{O}_{FN}$$

One can write $\mathcal{X}_\alpha^{FN}$ and $\mathcal{X}_\beta^{FN}$ as Wronskians of special solutions (using $\mathcal{W}^{FN}$- abelianization or exact WKB)

[LH, Neitzke]



$\mathcal{X}_\alpha^{FN}$ = eigenvalue of s_2'' around pants cycle

$$\mathcal{X}_\beta^{FN} = \frac{s_1' \wedge s_1''}{s_1 \wedge s_1''} \; \frac{s_1 \wedge s_2''}{s_1' \wedge s_2''}$$

(lateral summation)

Now we need to know s_i and s_i' in terms of s_i''

i.e. we need to compute connection coeff for the Mathieu eqn

$$s_i = \frac{r_i}{\Lambda} s_2'' + \frac{t_i}{\Lambda} s_1''$$

computed perturbatively in Λ in [LH-Küber-Szabo],

(proven by [Lisovyy, Naidiuk] for Heun eq)

$$\left.\begin{aligned} \log \mathcal{X}_\alpha^{FN} &\equiv a \\ \log \mathcal{X}_B^{FN} &\equiv a_D \equiv \frac{\partial \widetilde{W}^{\text{eff}}_{FN}(a)}{\partial a} \end{aligned}\right\} \Rightarrow \quad \widetilde{W}^{\text{eff}}_{FN} = \widetilde{W}^{\text{eff}}_{NS} \; !$$

This connects exact WKB with the NRS proposal that $\widetilde{W}^{\text{eff}}(\epsilon, \tau, \underline{a})$ is the generating function of opers in Fenchel-Nielsen coordinates

idea [Nekrasov-Rosly-Shatashvili]

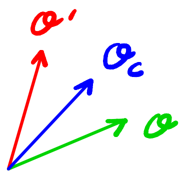
geom comp [LH-Kidwai]

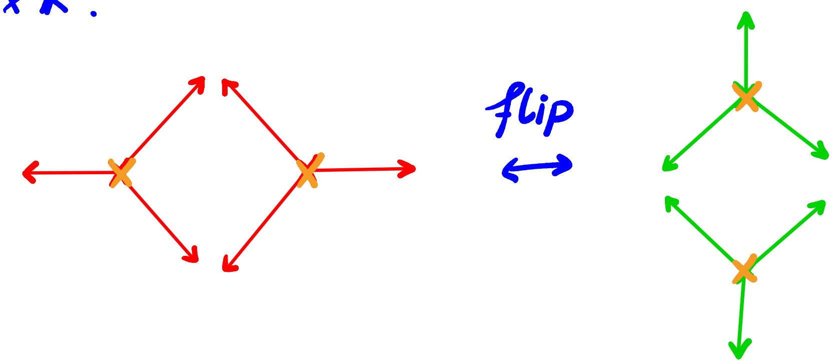
gauge thy derivation [Nekrasov-Jeong]

any other class S example [LH-Küber-Szabo]

- In terms of field theory, change of phase ϑ corresponds to coupling a 3d $\mathcal{N}=2$ thy to the boundary of $\mathbb{R}^2_\epsilon \times \mathbb{R}^2$.

simplest example





$\sim$ coupling 3d $T(\triangle)$ to boundary

$$a^{\vartheta'} = a^{\vartheta}$$

$$a_D^{\vartheta'} = a_D^{\vartheta} + \log\left(1 - e^{\pi i a^{\vartheta}/\epsilon}\right) \quad \rightsquigarrow \quad \Delta W = -\frac{\epsilon}{2\pi i} \mathrm{Li}_2\left(e^{\pi i a^{\vartheta}/\epsilon}\right)$$

[LH, Rüter, Szabo], [Dimofte, Gaiotto, Veen]

Summary

- We have defined a non-perturbative superpotential $\widetilde{W}^{\text{eff}}_{u,\vartheta}(\epsilon, \tau, \underline{a})$ which depends on $u \in \mathcal{B}$ and $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$.

- This superpotential may be computed as a generating function of opers in terms of certain generalized cluster coord.

- It may also be defined through exact WKB analysis in terms of a Borel sum of quantum periods.

- $\widetilde{W}^{\text{eff}}_{u,\vartheta}$ encodes the BPS particle spectrum in its jumps.

To do:

- extension to full $Z^{Nek}(t_1, t_2)$ of this picture
  relation to isomonodromy / Painleve
  as studied in [Bonelli, Tanzini, ...], [Gavrylenko, Grassi, ...]

- one may be able to derive 4d TS/ST through this picture

- lift to 5d : resolved conifold example [Grassi, Hao, Neitzke]
  [Alim, LH, Tulli]

  dream to understand local $\mathbb{P}^2$ / $\mathbb{P}^1 \times \mathbb{P}^1$

  as recently studied by [Gu, Kashani-Poor, Klemm, Marino]
  and relatedly split attractor flow in [Bousseau, Descombes, Le Floch, Pioline]

Application: Spectral problems

$d_\epsilon \psi(x) = 0$ defines a spectral problem if we choose:

some reality condition on ϵ & $\psi \in \mathcal{H}$ Hilbert space

s.t. $d_\epsilon \psi = 0$ with $\psi \in \mathcal{H}$ has discrete set of solutions

Ex: quantum harmonic oscillator

$$\hbar^2 \psi''(x) + (x^2 - E)\psi(x) = 0 \quad \text{with} \quad \psi(x) \in L^2(\mathbb{R})$$

Claim: perspective of spectral networks / exact WKB analysis for all ϑ is helpful to define and analyze new and old spectral problems

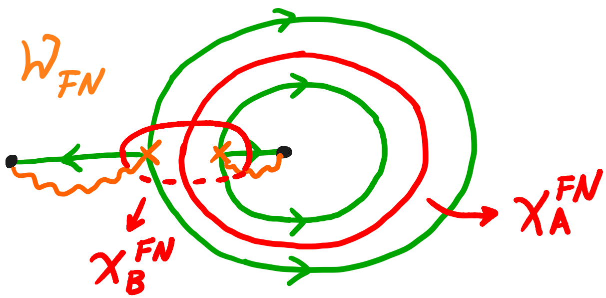
$\rightsquigarrow$ new spectral problems: T3-equation [LH, Neitzke]

resolved conifold [Alim, LH, Tulli]

Mathieu diff'l eqn: $d_\epsilon = \epsilon^2 \partial_x^2 + \left(\frac{1}{x^3} + \frac{2E + \frac{1}{4}\epsilon^2}{x^2} + \frac{1}{x} \right)$

$\epsilon = \hbar > 0$ and $x = e^{ix}$ with $x \in \mathbb{R}$ $\rightsquigarrow$

Mathieu spectral problem $\hbar^2 \psi''(x) - (2\cos(x) + 2E)\psi(x) = 0$



with $\psi(x+2\pi) = \psi(x)$, $E<0$

$\rightsquigarrow$ bound states $\sim$ $\mathcal{X}_A^{FN} = 1$

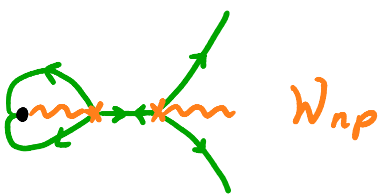
needs to be written in terms of $\mathcal{X}_{A,B}^{np}$

to analyze using WKB

[LH, Neitzke]

$x = ix' + \pi$ and $x' \in \mathbb{R}$ $\rightsquigarrow$

modified Mathieu spectral problem $\hbar^2 \psi''(x') - (\cosh x' - 2E)\psi(x') = 0$



with $\psi(x') \in L^2(\mathbb{R})$, $E>0$

$\rightsquigarrow$ exact quantization condition $\mathcal{X}_B^{FN} = 1$

can be analyzed using WKB since $E>0$