Some Applications of Free Supersymmetric Fields

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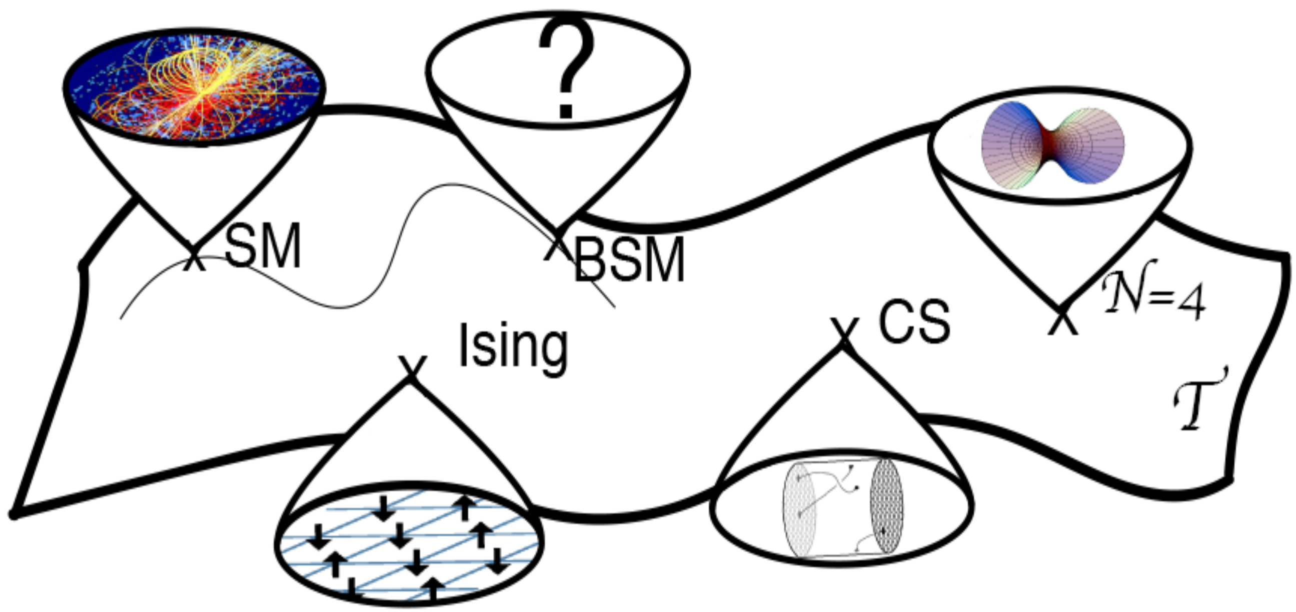
Introduction and Motivation

• In different ways, we are all interested in the space of quantum field theories, $\mathcal{T}$.



Introduction and Motivation (cont...)

• Given a particular QFT, we have a sense of what $\mathcal{T}$ should look like locally. But even here it is complicated: infinitely many (locally irrelevant) directions.

• Global questions—like what is the topology of $\mathcal{T}$?— seem even further out of reach.

• To make sense of such questions, we might specialize to spaces of theories that we can solve completely and that have sufficiently "few" degrees of freedom: Virasoro minimal models **[Vafa]**

• Even here, while the c-theorem and Morse theory give hints, they leave open many possibilities: $\mathbb{C}P^\infty$, loop space of $SU(2)$, $\cdots$.

Introduction and Motivation (cont...)

• But there are some reasons for optimism. For example, supersymmetry gives some analytical control on non-trivial deformations.

• As an elementary example, consider a free chiral superfield, Φ, in 3d $\mathcal{N} = 2$. We can study the interesting deformation

$$W = \lambda \Phi^3 \ , \tag{1}$$

• RG flow takes us from Φ_{UV} with dimension 1/2 to Φ_{IR} with dimension 2/3. In language of superconformal representations

$$A_2\bar{B}_1[0]^{(1/2)}_{1/2} \xrightarrow{\text{RG}} L\bar{B}_1[0]^{2/3}_{2/3} \ . \tag{2}$$

• Change in quantum #'s is of order original quantum #'s.

Introduction and Motivation (cont...)

• SUSY even allows us to "change coordinates" in the space of theories without trouble and gain analytical insight.

• For example, 4d $SU(N_c)$ $\mathcal{N}=1$ SQCD with N_f flavors satisfying $N_c+1 < N_f \leq 3N_c/2$ is famously described in the UV via

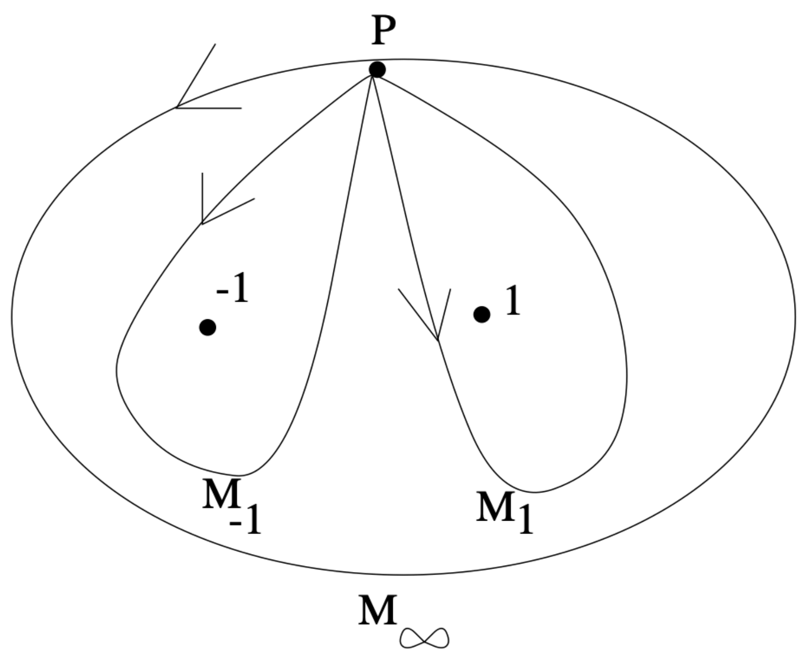
$$W_\alpha \ , \quad Q^i_a \ , \quad \tilde{Q}^a_i \ , \tag{3}$$

and in the IR via an $SU(N_f-N_c)$ gauge theory with **[Seiberg]**

$$W'_\alpha \ , \quad q^{i'}_{a'} \ , \quad \tilde{q}^{a'}_{i'} \ , \quad M^{i'}_{j'} \ , \quad W = Mq\tilde{q} \ . \tag{4}$$

Introduction and Motivation (cont...)

• An even more relevant example to us is the Seiberg-Witten solution to the IR of $SU(2)$ $\mathcal{N} = 2$ SYM



Introduction and Motivation (cont...)

• All of these examples involve crucial uses of free (chiral) fields and symmetries.

• Since symmetries are topological in nature, we expect them to have something important to say about the topology of the space of QFTs.*

• The ubiquity of free fields in these cases (and all others we have constructed to date using SUSY!) motivates us to ask: are free QFTs connected to all other QFTs in $\mathcal{T}$ via continuous deformations? See also **[Douglas],...**.

*We also expect them to have something to say about the categorical nature of this space

Introduction and Motivation (cont...)

• Since the above free fields are local, it is natural to specialize to local theories (i.e., those with EM tensors). This is of course still a very hard question.

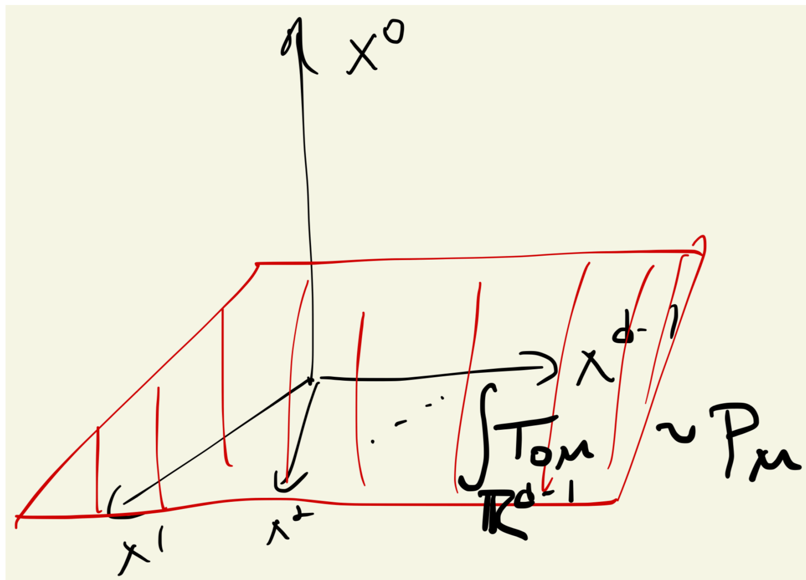
• To make things more concrete, we will focus on the representations of symmetries and ask the following specific question:

Question [M.B., Jiang]: When is it possible to construct arbitrary unitary multiplets of the superconformal algebra (SCA) with eight Poincaré supercharges that are compatible with locality from (continuous deformations of) representations in free field theories?

• I'll mainly discuss the most interesting case of $d = 4$, but we also studied $d = 2, 3, 5$ as I will briefly mention.

Unitarity and Locality

• To impose unitarity we only act with charges on states



• For example, imposing positivity of the norm on $P^2|\phi\rangle$ leads to the condition

$$\Delta \geq (d-2)/2 \ . \tag{5}$$

Unitarity and Locality (cont...)

• There are various similar computations involving other charges in the SCA.

• **Note:** in principle, these need not have corresponding Noether currents (i.e., they may be non-locally realized)

• For example, the long-range Ising model

$$H = -J \sum_{i,j} \frac{1}{r_{ij}^{d_s+\sigma}} s_i s_j \ . \tag{6}$$

has no EM tensor in the continuum (for $0 \leq \sigma < \sigma_*$), but it does have a conformal point.

Unitarity and Locality (cont...)

• Similar comments apply to generalized free fields.

• And even more generally, to QFTs living on defects and boundaries that are coupled to a bulk QFT.

• Imposing locality for us means that we will impose the existence of an EM tensor (and a corresponding SUSY multiplet).

• This implies more than the existence of a multiplet. It implies Ward identities on correlators like

$$\langle \mathcal{O}(x) T_{\mu\nu}(y) \mathcal{O}^\dagger(z) \rangle \ , \cdots \ . \tag{7}$$

Unitarity and Locality (cont...)

• It also implies things like the ANEC **[Faulkner et. al.], [Hartman et. al.]**

$$\int_{-\infty}^{\infty} d\alpha \langle\psi|T_{\mu\nu}|\psi\rangle u^{\mu}u^{\nu} \geq 0 \ . \tag{8}$$

and various generalizations **[Melzer],....**

• In principle, locality can rule out many unitary reps of the SCA.

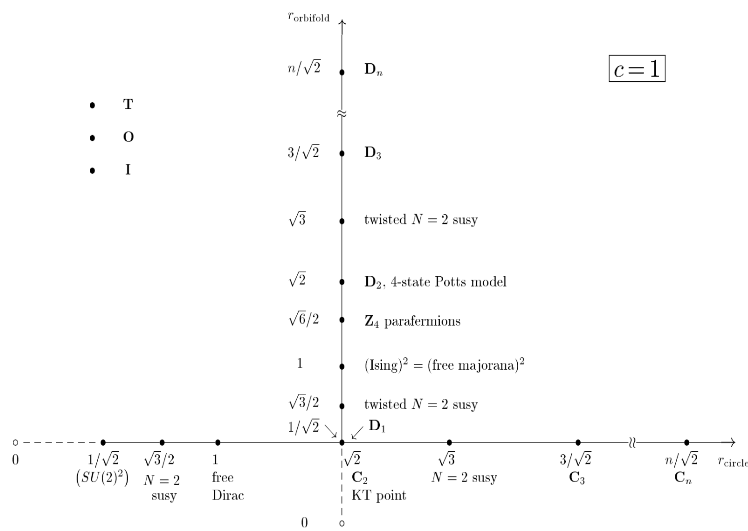
• We will see that this is not the case in low dimensions ($d < 4$), but it plays a crucial role in $d = 4$.

Spacetime Dimension

- Roughly speaking, we expect the relevant deformations of Lagrangians to be richer the lower in dimension we get.

$$\mathcal{L} = \int d^d x \partial^\mu \phi \partial_\mu \phi + \cdots \ , \tag{9}$$

- For example, in 2d we start from



Spacetime Dimension (cont...)

• In the 2d compact free boson, relevant deformations take us to the infinitely many Virasoro minimal models. Moreover, the compact boson theory space is very rich.

• In addition, in 2d, we can often construct not just abstract representations from free fields but also operator algebras.

• In line with this intuition, we gave a constructive proof involving $\mathcal{N} = (4,4)$ T^4 sigma models to show that:

Theorem [M.B. and Jiang]: Any representation of the "small" 2d $(4,4)$ SCA can be realized by free fields.

Spacetime Dimension (cont...)

• Similarly, in 3d $\mathcal{N} = 4$, we have lots of power for free fields through mirror symmetry. This fact is reflected in the statement that many "non-Lagrangian" 4d $\mathcal{N} = 2$ theories have simple 3d $\mathcal{N} = 4$ Lagrangians.

Theorem [M.B. and Jiang]: Any short representation of the 3d $\mathcal{N} = 4$ SCA can be realized by free fields. Up to continuous deformations, the same is true for any long multiplet.

• On the other hand, in $d > 4$ we have a sense that Lagrangians are less powerful. Indeed, it is possible to show that the $\mathcal{A}_4[0,0]_4^{(0)}$ 5d $\mathcal{N} = 1$ SCA irrep is not constructible via free fields (or continuous deformations), but it does exist in the E_2 interacting SCFT **[M.B., Hayling, Papageorgakis]**.

$d = 4$

• 4d is between the less powerful Lagrangians of higher dimensions and the more powerful Lagrangians of lower dimensions.

• Also, $\mathcal{N} = 2$ has an abelian quantum number, $U(1)_R \subset U(1)_R \times SU(2)_R$ that makes it a more subtle case.

• Our central conjecture is:

Conjecture: It is always possible to construct arbitrary multiplets of the 4d $\mathcal{N} = 2$ SCA from (continuous deformations of) representations in free field theories.

• We proved some aspects of this conjecture and found a web of supporting evidence and mutually consistent conjectures more generally.

$d = 4$ (cont...)

• This conjecture interfaces with intuition from studying 4d $\mathcal{N} = 2$ SCFTs. It also **(1)** rules out many SCA representations in local unitary theories and **(2)** constrains the way allowed representations transform under symmetries lying outside the SCA.

• Roughly speaking, there are two kinds of short SCA irreps in this case. Those that contain chiral operators*

$$\bar{\mathcal{E}}_{r(j,0)} \oplus \hat{\mathcal{B}}_R \oplus \bar{\mathcal{D}}_{R(j,0)} \oplus \bar{\mathcal{B}}_{R,r(j,0)} \ , \tag{10}$$

and those that do not†

$$\hat{\mathcal{C}}_{R(j,\bar{j})} \oplus \bar{\mathcal{C}}_{R,r(j,\bar{j})} \ . \tag{11}$$

*Conjugate multiplets contain anti-chiral operators.

†There is also a conjugate $\mathcal{C}$ multiplet.

$d = 4$ (cont...)

• Heuristically, we expect the following picture to hold:

• In many ways, chiral operators behave like free chiral fields. Indeed, consider chiral operators, $\mathcal{O}_{1,2}$, with quantum numbers

$$\mathcal{O}_i \leftrightarrow (\Delta_i, R_i, r_i, j_i)\ . \tag{12}$$

• Up to null relations that don't follow from statistics, these operators have a multiplication with quantum numbers that add

$$\mathcal{O}_1\mathcal{O}_2 \leftrightarrow (\Delta_1 + \Delta_2, R_1 + R_2, r_1 + r_2, j_1 + j_2)\ . \tag{13}$$

• Therefore, it is reasonable to imagine that all chiral representations of the SCA can be realized by free chiral fields (up to continuous deformations that account for anomalous dimensions).

$d = 4$ (cont...)

• In the case of the non-chiral multiplets, it is useful to develop some intuition for what happens when we have free fields.

• First, from locality, we have a stress tensor multiplet, $\widehat{\mathcal{C}}_{0(0,0)}$.

• More generally, we have an infinite set of closely-related higher-spin currents, $\widehat{\mathcal{C}}_{0(j,\bar{j})}$. In particular, they have $R = 0$. Roughly speaking these multiplets look as follows

$$\phi_i \partial^n \bar{\phi}_i + q_a \partial^n \bar{q}_a + \cdots \quad . \tag{14}$$

• A careful analysis of such multiplets reveals they are of type

$$\widehat{\mathcal{C}}_{0(j,j)} \oplus \widehat{\mathcal{C}}_{0(k,k-1)} \oplus \widehat{\mathcal{C}}_{0(k-1,k)} \oplus \widehat{\mathcal{C}}_{0(j,j-1/2)} \oplus \widehat{\mathcal{C}}_{0(j-1/2,j)} \ , \tag{15}$$

where $j \in \mathbb{N}/2$ and $k \in \mathbb{N}$.

$d = 4$ (cont...)

• More generally, we have $\widehat{\mathcal{C}}_{R(j,\bar{j})}$ and $\bar{\mathcal{C}}_{R,r(j,\bar{j})}$. However, for $j + \bar{j} \gg R$, CFT spectrum takes on properties of a free theory (e.g., additivity of twists).

• Moreover, in the large spin limit, we can, in some sense "forget" R and find effective higher spin multiplets. This suggests that, for large spin, we get the constraints of free fields.

• For $j + \bar{j} \lesssim R$ we can construct any multiplet from free fields.

• This follows from the fact that hypermultiplet primaries are fundamentals of $SU(2)_R$ and vector multiplet primaries are only charged under $U(1)_r$.

$d = 4$ (cont...)

• Let us now consider each representation more carefully and see how locality enters.

• First consider $\hat{\mathcal{B}}_R$ multiplets. They parameterize Higgs branches. Since the hypermultiplet is a fundamental of $SU(2)_R$ we have

$$\hat{\mathcal{B}}_R \leftrightarrow q^{2R} \ . \tag{16}$$

• This representation theory fact is compatible with various non-renormalization theorems.

• These multiplets are also related to states in a 2d VOA via the correspondence of **[Beem et. al.]**. We'll return to this fact.

$d = 4$ (cont...)

• Next, consider the $\bar{\mathcal{E}}_{r(j,0)}$ multiplets. For $j = 0$ these are the multiplets of Seiberg-Witten theory (i.e., they paramterize the Coulomb branch). In all known examples, they are connected to free irreps via RG flow.

• The primaries are annihilated by all the anti-chiral supercharges

$$\bar{Q}^i_{\dot{\alpha}} \mathcal{O}_{\alpha_1 \cdots \alpha_{2j}} = 0 \ . \tag{17}$$

• In a free theory, the only fields we can use to construct such a primary are the vector multiplet primaries, ϕ_i (derivatives lead to descendants in the chiral ring). So, we have

$$\mathcal{O} \leftrightarrow \prod_i \phi_i \ . \tag{18}$$

$d = 4$ **(cont...)**

- In particular, we are forced to have $j = 0$.

- More generally, Ward identities of the correlators

$$\langle \bar{\mathcal{E}}_{r,(j,0)} \widehat{\mathcal{C}}_{0(0,0)} \mathcal{E}_{-r(0,j)} \rangle \ , \tag{19}$$

are satisfied iff $j = 0$ **[Manenti]**. Therefore, $j > 0$ multiplets are absent in local unitary 4d $\mathcal{N} = 2$ SCFTs. This is in line with our heuristic intuition: up to deformations of r, all allowed multiplets realized in free theories.

- Next, let us consider $\bar{\mathcal{D}}_{R(j,0)}$ multiplets. Examples include

$$\phi_i q_a \in \bar{\mathcal{D}}_{1/2(0,0)} \ . \tag{20}$$

They contain extra supercurrents and appear on some mixed branches. They also appear in associated 2d VOAs.

$d = 4$ (cont...)

• In free theories, it is easy to convince oneself that $j \leq R$.

• More generally, the ANEC implies that $j \leq R$ as well **[Manenti, Stergiou, Vichi]**. So $\bar{\mathcal{D}}_{R(j,0)}$ with $j > R$ is forbidden in a local unitary theory. This is again compatible with our heuristic understanding.

• Finally, consider the remaining chiral multiplets $\bar{\mathcal{B}}_{R,r(j,0)}$. Examples include

$$\phi^n q_a \in \bar{\mathcal{B}}_{1/2,n(0,0)} \ , \quad n > 1 \ . \tag{21}$$

These appear for various reasons including mixed branches, local unitary $\mathcal{N} > 2$ SCFTs, higher-rank Coulomb branches, and certain 't Hooft anomalies **[M.B., Banerjee]**.

$d = 4$ **(cont...)**

• In free theories, we again have $j \leq R$.

• It is easy to extend the ANEC analysis of **[Manenti, Stergiou, Vichi]** to show that such multiplets satisfy $j \leq R$ for $r < j + 2$.*

• More generally, all examples we are aware of have $\bar{\mathcal{B}}$ multiplets coming from OPEs of the following type

$$\bar{\mathcal{B}} \in \hat{\mathcal{B}} \times \bar{\mathcal{E}}\ ,\ \bar{\mathcal{D}} \times \bar{\mathcal{D}}\ ,\ \bar{\mathcal{E}} \times \bar{\mathcal{E}}\ , \cdots\ . \tag{22}$$

*Using a more general $\mathcal{N} = 2$ superspace analysis, we suspect this bound holds for any r compatible with unitarity.

$d = 4$ **(cont...)**

• Or they can arise via gauging*

$$\mathrm{Tr}(\phi^n \prod_i \mathcal{O}_i) \ , \tag{23}$$

with $\mathcal{O}_i \in \hat{\mathcal{B}}, \bar{\mathcal{E}}, \bar{\mathcal{D}}$ or of the above $\bar{\mathcal{B}}$ type.

• In any of these cases, we can argue our bounds are satisfied.

• Next, consider the non-chiral $\hat{\mathcal{C}}_{R(j,\bar{j})}$ multiplets. Like the $\hat{\mathcal{B}}$ and $\bar{\mathcal{D}}$ multiplets, they form part of the "Schur" sector of operators that correspond to 2d VOA states.

*There are also non-chiral production channels, but they do not seem to yield qualitatively new multiplets.

$d = 4$ (cont...)

• Most general possible bound compatible with CPT and linearity is

$$a_r|r| + a_+(j + \bar{j}) + a_R R + a \leq 0 \ . \tag{24}$$

• Locality implies that $a \leq 0$. The spectrum of possible higher spin multiplets implies that $a_+ = 0$. Finally, the existence of

$$q^{2R} \cdot q\bar{q} \in \hat{\mathcal{B}}_R \times \hat{\mathcal{C}}_{0(0,0)} \in \hat{\mathcal{C}}_{R(0,0)} \ , \tag{25}$$

for any R implies that $a_R \leq 0$. Then to get a constraint, WLOG, we can take $a_r = 1$.

• Combined with the fact that $\hat{\mathcal{C}}_{R(j,-1/2)} \cong \bar{\mathcal{D}}_{R+1/2(j,0)}$ we have

$$|r| = |j - \bar{j}| \leq R + 1 \ . \tag{26}$$

$d = 4$ (cont...)

• This is also the bound satisfied by free fields.

• It is possible to check compatibility of these results with known conjectures on the generation of the associated 2d VOA by $\hat{\mathcal{B}}_R$ states in the case of a theory with a (pure) Higgs branch.

• Also, it is easy to check compatibility of this conjecture with class $\mathcal{S}$ theories with regular punctures built from gauging "trinion" theories.

• Finally, the above results combine together to lead to the following conjecture for $\bar{\mathcal{C}}_{R,r(j,\bar{j})}$ multiplets **[M.B., Jiang]**

$$j \leq \bar{j} + R + 1 \ , \tag{27}$$

which is also satisfied by free fields.

$d = 4$ (cont...)

• There is additional evidence for the above bound in a spirit similar to that for the $\bar{\mathcal{B}}$ multiplets.

• What additional implications does this line of thinking lead to?

• For one, it implies there is another generally well-defined limit of the superconformal index

$$\begin{aligned} \mathcal{I}(q,u,t) &:= \mathrm{Tr}(-1)^F q^{2j} u^{\bar{j}-j-r} t^{R+\bar{j}-j} \\ &\xrightarrow{t\to 0,q,u \text{ fixed}} \mathcal{I}_{GC}(q,u) = \mathrm{Tr}(-1)^F q^{2j} u^{\bar{j}-j-r} \ , \end{aligned} \tag{28}$$

where states satisfying $\Delta - 2R - 2\bar{j} - r = 0$ contribute; the final trace is restricted to states satisfying $R + \bar{j} - j = 0$.

$d = 4$ (cont...)

• In the case of a free theory of n_V abelian vectors and n_H hypers, this index only receives contributions from free vector fields ϕ_i, $\lambda^1_{i,+}$, $\partial_{+\dot{+}}$ **[Gadde, Rastelli, Razamat]**

$$\mathcal{I}_{GC}(q,u) = \text{P.E.}\left(\frac{u-q}{1-q}\right) \ . \tag{29}$$

• In a Lagrangian theory with freely generated Coulomb branch we have

$$\mathcal{I}(q,u) = \prod_r \text{P.E.}\left(\frac{(u-q)u^{r-1}}{1-q}\right) \ . \tag{30}$$

The multiplets in the arguments are the $\bar{\mathcal{E}}_r$ generators of the Coulomb branch chiral ring.

$d = 4$ (cont...)

• More generally, we analysed theories that don't have $\mathcal{N} = 2$ Lagrangians and found that **[M.B., Jiang], [Bhargava, M.B., Jiang]**

$$\mathcal{I}_{GC}(q,u) = \prod_r \text{P.E.}\left(\frac{(u-q)u^{r-1}}{1-q}\right) \ . \tag{31}$$

• Therefore, we conjecture **[M.B., Jiang]**: all local unitary $\mathcal{N} = 2$ SCFTs w/ freely generated Coulomb branches satisfy above.

• Since the above indices are generated by $\bar{\mathcal{E}}_r$ multiplets, they are independent of flavor (this follows from general OPE arguments) and so

$$\partial_a \mathcal{I}_{GC} = 0 \ . \tag{32}$$

$d = 4$ **(cont...)**

• It is then natural to conjecture that any multiplet contributing to $\mathcal{I}_{GC}$ are flavor neutral. This leads to the conclusion that **[M.B., Jiang]**

$$\bar{\mathcal{C}}_{j-\bar{j}-1,r(j,\bar{j})} \oplus \bar{\mathcal{B}}_{j,r(j,0)} \ , \tag{33}$$

are neutral under any flavor symmetry in any irreducible local unitary 4d $\mathcal{N} = 2$ SCFT with a freely generated Coulomb branch.

• Note that we can also take $u \to 0$. This leads to a "reduced" Macdonald limit of the index. To get a non-trivial index, we clearly need generators with $r = 1$. So it is natural to conjecture that such an index is non-trivial iff the theory has free abelian vector multiplets **[M.B., Jiang]**.

$d = 4$ (cont...)

• Therefore, by looking at the most general multiplets that can contribute in this limit, we conjecture that they can only appear in theories with free vectors

$$\hat{\mathcal{C}}_{j-\bar{j}-1(j,\bar{j})} \oplus \bar{\mathcal{D}}_{j(j,0)} \ . \tag{34}$$

• The case $j = \bar{j} + 1$ is associated with higher-spin symmetries and hence free theories **[Maldacena, Zhiboedov]**. Here we see a much more general set of multiplets that are associated with free theories. This should lead to implications for the bootstrap.

• There are also interesting implications for a vertex algebra, "$\sqrt{\text{VOA}}$," associated with the VOAs of **[Beem et. al.]**

Conclusions and Further Directions

• We used representation theory to learn new things about the space of SCFTs with 8 Poincaré supercharges.

• Would be interesting to connect our results further with the topology of space of QFTs and also with more categorical constructions.

• When particular SCA irreps can't be realized by free fields but correspond to deformation of an irrep realized by free fields, is it possible to associate a "closest" free irrep? Interpretation in terms of RG interfaces or more general conformal interfaces?

• Spin plays an important role in our bounds. What if rotational invariance is relaxed, can we say interesting things for such theories?