Elastic scattering of three ultracold bosons

EFB24, 6 September 2019

arXiv:1905.07205

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Outline

Three-body scattering hypervolume D of bosons with finite-range interactions

- Effects on BECs
- Methods
- Results for D from weak to strong interactions
- Conclusion and outlook

P. Mestrom, V. Colussi, T. Secker, S. Kokkelmans, arXiv:1905.07205 (2019)

TU/e

Ultracold atomic Bose gases

Bose-Einstein condensation (BEC)

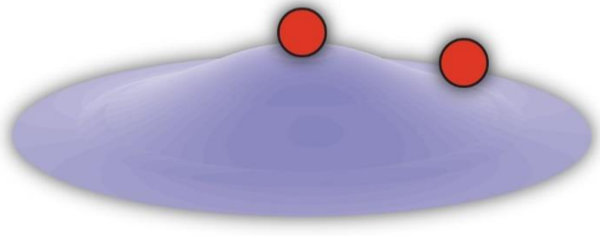
$$n\,\lambda_{\mathrm{db}}^3 \gtrsim 1$$

n = density

λ_{db} = de Broglie wave length $\propto T^{-1/2}$

T = temperature

$$n\,\lambda_{\mathrm{db}}^3 \gg 1$$



Ultracold atomic Bose gases

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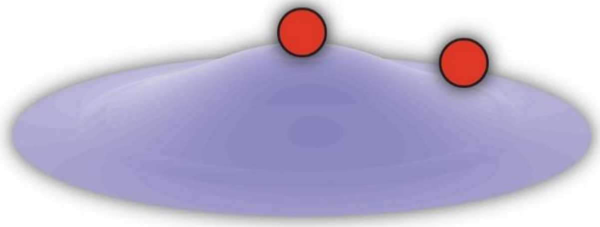

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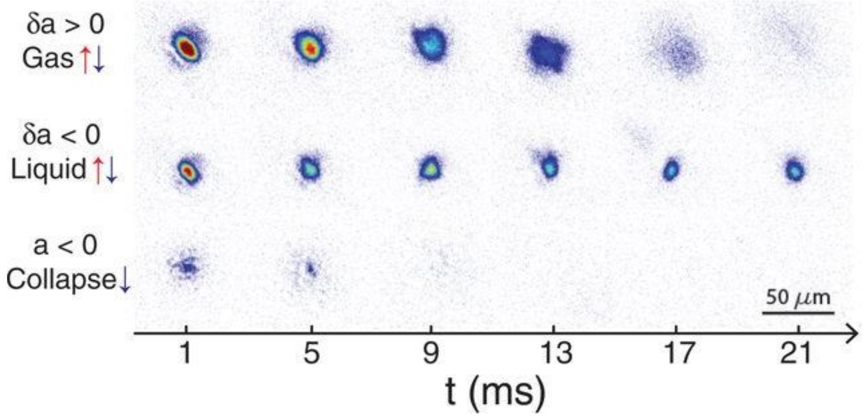


Quantum droplets (self-bound, dilute)

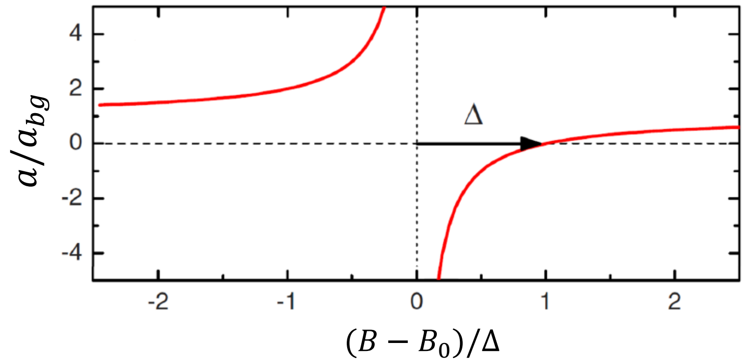
- Observed in
  - Mixtures of BECs
  - Dipolar BECs
- Stabilized by repulsive beyond-mean-field forces

C. R. Cabrera et al., Phys. Rev. Lett. 120, 135301 (2018)

Ultracold atomic Bose gases

High experimental control

Scattering length a → effective two-body interaction strength



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Ultracold atomic Bose gases

High experimental control

Scattering length a → effective two-body interaction strength

Ground-state energy density of a BEC:

$$\mathcal{E}(n) = \frac{2\pi\hbar^2 n^2 a}{m}\left\{1 + \frac{128}{15\sqrt{\pi}}\sqrt{na^3} + \ldots\right\}$$

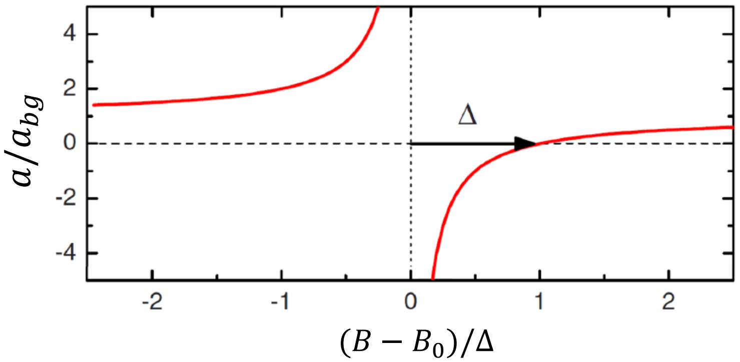
n = density

m = mass of boson

$a > 0$

$na^3 \ll 1$

T. D. Lee, K. Huang, C. N. Yang, Phys. Rev., 106, 1119 (1957)



Ultracold atomic Bose gases

Is there a three-body analog of the two-body scattering length?

Ultracold atomic Bose gases

Is there a three-body analog of the two-body scattering length?

Scattering length a → effective two-body interaction strength

Scattering hypervolume D → effective three-body interaction strength

$$\mathcal{E}(n) = \frac{2\pi\hbar^2 n^2 a}{m}\left\{1 + \frac{128}{15\sqrt{\pi}}\sqrt{na^3} + \left[\frac{8(4\pi - 3\sqrt{3})}{3}\ln(na^3) + \frac{D}{12\pi a^4} + \pi r_s/a + 118.5\right] na^3 + \ldots\right\}$$

n = density

m = mass of boson

r_s = effective range

$a > 0$

$na^3 \ll 1$

S. Tan, Phys. Rev. A, 78, 013636 (2008)

T. T. Wu, Phys. Rev., 115, 1390 (1959)

T. D. Lee, K. Huang, C. N. Yang, Phys. Rev., 106, 1119 (1957)

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Three-body scattering hypervolume

- Beyond LHY correction

$$\mathcal{E}(\rho) = \frac{1}{6}\frac{\hbar^2 n^3}{m} D + \ldots \qquad (a = 0)$$

- Quantum droplets
  - Stabilized by effective three-body interactions

A. Bulgac, Phys. Rev. Lett., 89, 050402 (2002)

Three-body scattering hypervolume

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$$\mathcal{E}(\rho) = \frac{1}{6}\frac{\hbar^2 n^3}{m} D + \ldots \qquad (a = 0)$$

- Quantum droplets
  - Stabilized by effective three-body interactions

    A. Bulgac, Phys. Rev. Lett., 89, 050402 (2002)

Pairwise interactions with a finite range R tuned from weak ($|a| \lesssim R$) to strong ($|a| \gg R$)

Alt-Grassberger-Sandhas (AGS) equations

Two-body transition operator T → a

Three-body transition operators U_{fi} → D

$$\begin{cases} U_{00}(z) = \sum_{\alpha=1}^{3} T_\alpha(z) G_0(z) U_{\alpha 0}(z) \\ U_{\alpha 0}(z) = G_0^{-1}(z) + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{3} T_\beta(z) G_0(z) U_{\beta 0}(z) \\ \qquad \text{for } \alpha = 1, 2, 3, \end{cases}$$ [1]

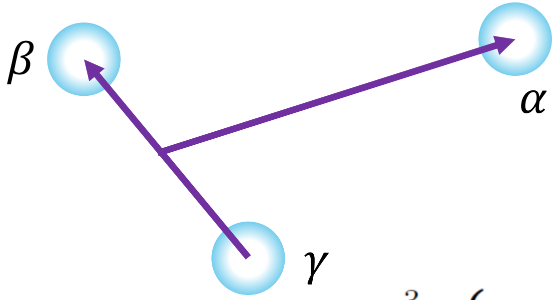
α	**configuration**
0	A + B + C
1	A + BC
2	B + CA
3	C + AB

- $G_0(E) = (E - H_0)^{-1}$ where H_0 is the three-body kinetic energy operator
- T_α is the transition operator of a two-particle subsystem

  E.g. $T_1(E) = V_{23} + V_{23} G_0(E) T_1(E)$

[1] E. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B **2**, 167 (1967)

Elastic scattering amplitude



$$\boldsymbol{p}_\alpha = \frac{1}{2}(\boldsymbol{P}_\beta - \boldsymbol{P}_\gamma) \qquad (\alpha\beta\gamma = 123, 231, 312)$$

$$\boldsymbol{q}_\alpha = \frac{2}{3}\left(\boldsymbol{P}_\alpha - \frac{1}{2}(\boldsymbol{P}_\beta + \boldsymbol{P}_\gamma)\right)$$

$$\langle \mathbf{p}, \mathbf{q} | U_{00}(0) | \mathbf{0}, \mathbf{0} \rangle = \sum_{\alpha=1}^{3} \left\{ \delta(\mathbf{q}_\alpha) \langle \mathbf{p}_\alpha | T(0) | \mathbf{0} \rangle + A_1 \frac{a^2}{q_\alpha^2} + A_2 \frac{a^3}{q_\alpha} + A_3\, a^4 \ln\left(\frac{q_\alpha |a|}{\hbar}\right) \right.$$
$$\left. + A_4 \frac{p_\alpha^2 + \frac{3}{4} q_\alpha^2}{q_\alpha^2} a + \frac{1}{3} \frac{1}{(2\pi)^6} \frac{D}{m\hbar^4} + O\left(q_\alpha \ln\left(\frac{q_\alpha |a|}{\hbar}\right), \frac{p_\alpha^2}{q_\alpha}\right) \right\}$$

where $\boldsymbol{p} \equiv \boldsymbol{p}_1$

$\boldsymbol{q} \equiv \boldsymbol{q}_1$

Elastic scattering amplitude

$$
\begin{aligned}
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\end{aligned}
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Elastic scattering amplitude

two-body scattering

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Elastic scattering amplitude

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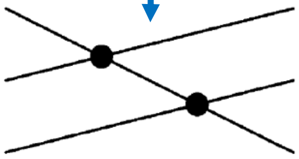
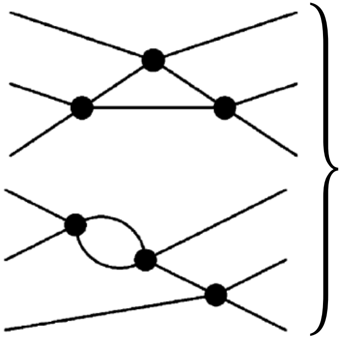


A_1 to A_4 are constants

Elastic scattering amplitude



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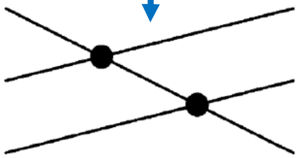

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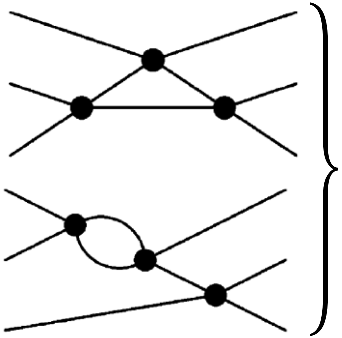

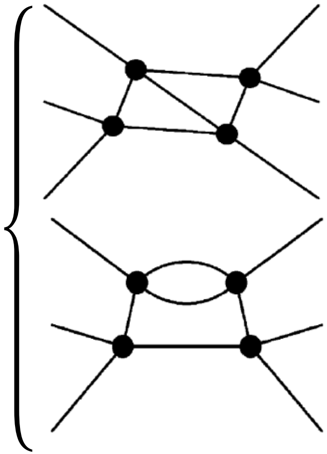

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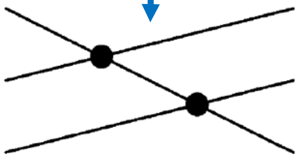

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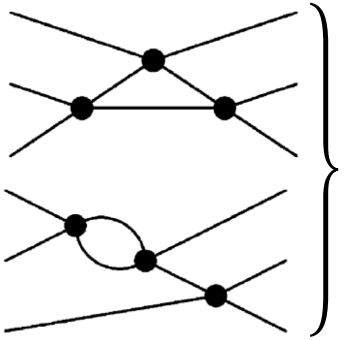

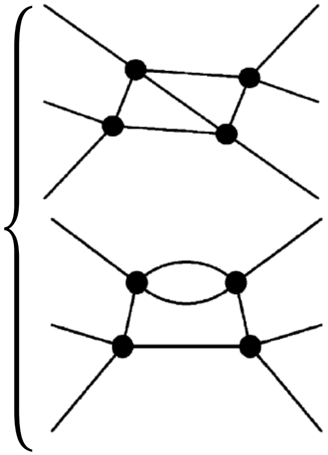

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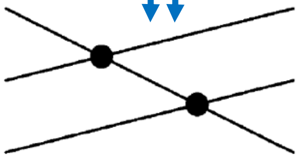

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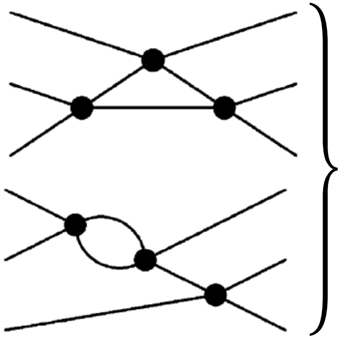

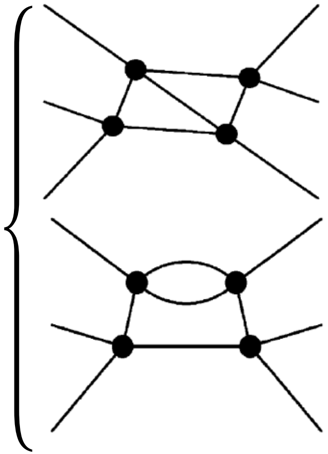

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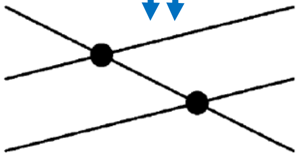

A_1 to A_4 are constants

D = three-body scattering hypervolume

dimension: (length)^4

$\mathrm{Im}(D) \propto$ three-body recombination rate

Numerical method

- Write out AGS equations in momentum space
- Subtract the singular terms from $\langle \boldsymbol{p}, \boldsymbol{q} | U_{00}(0) | \mathbf{0}, \mathbf{0} \rangle$

$$\langle \mathbf{p}, \mathbf{q} | U_{00}(0) | \mathbf{0}, \mathbf{0} \rangle = \sum_{\alpha=1}^{3} \Bigg\{ \boxed{\delta(\mathbf{q}_\alpha) \langle \mathbf{p}_\alpha | T(0) | \mathbf{0} \rangle} + \boxed{A_1 \frac{a^2}{q_\alpha^2}} + \boxed{A_2 \frac{a^3}{q_\alpha}} + \boxed{A_3\, a^4 \ln\left(\frac{q_\alpha |a|}{\hbar}\right)}$$
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- Perform a partial-wave expansion and Weinberg expansion

$$T(z) = \sum_{n=1}^{\infty} |g_n(z)\rangle \tau_n(z) \langle g_n(z)|$$

S. Weinberg, Phys. Rev. 131, 440 (1963)

P. M. A. Mestrom et al., Phys. Rev. A 99, 012702 (2019)

- Solve the resulting integral equation as a matrix equation

Finite-range interaction

We consider three identical bosons interacting via pairwise square-well potentials

$$V(r) = \begin{cases} -V_0, & 0 \le r < R \\ 0, & r \ge R \end{cases}$$

Dimensionless interaction strength: $\bar{V}_0 = mV_0R^2/\hbar^2$

Repulsive: $-V_0 > 0$

Attractive: $-V_0 < 0$

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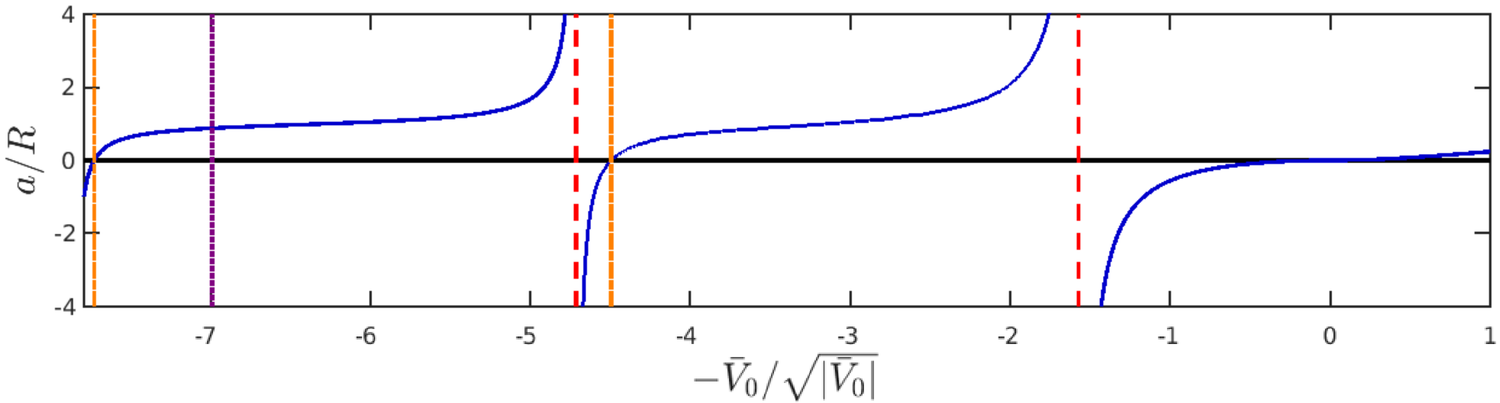
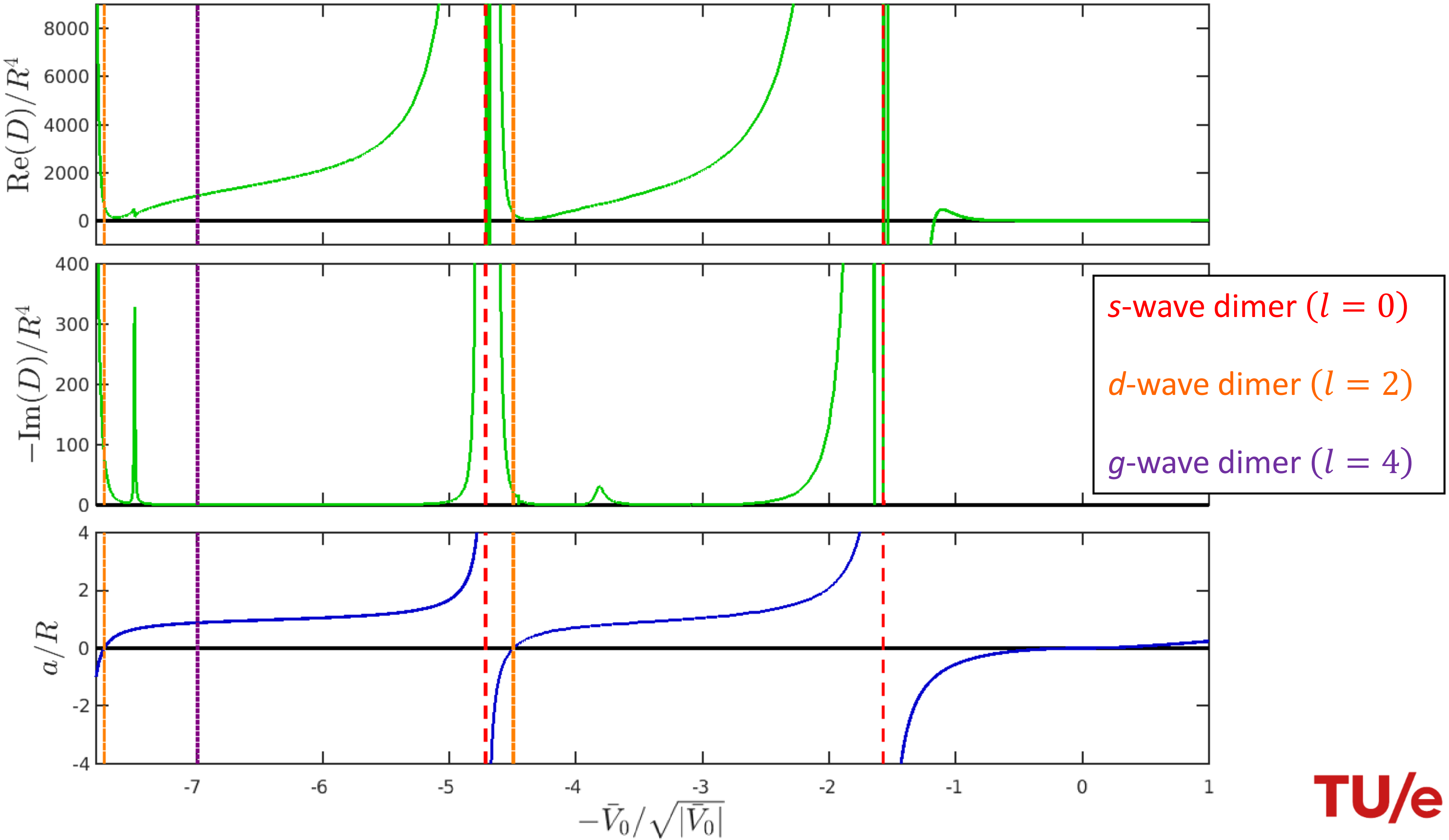
s-wave dimer $(l = 0)$

d-wave dimer $(l = 2)$

g-wave dimer $(l = 4)$





8000
6000
4000
2000
0
$\mathrm{Re}(D)/R^4$
400
300
200
100
0
$-\mathrm{Im}(D)/R^4$
4
2
0
-2
-4
a/R
-7
-6
-5
-4
-3
-2
-1
0
1
$-\bar{V}_0/\sqrt{|\bar{V}_0|}$
s-wave dimer ($l = 0$)
d-wave dimer ($l = 2$)
g-wave dimer ($l = 4$)

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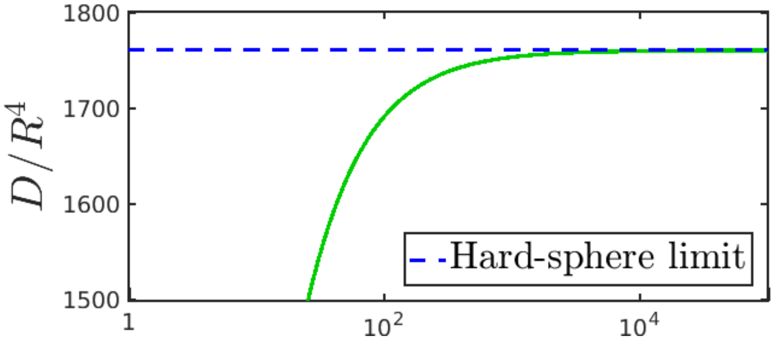
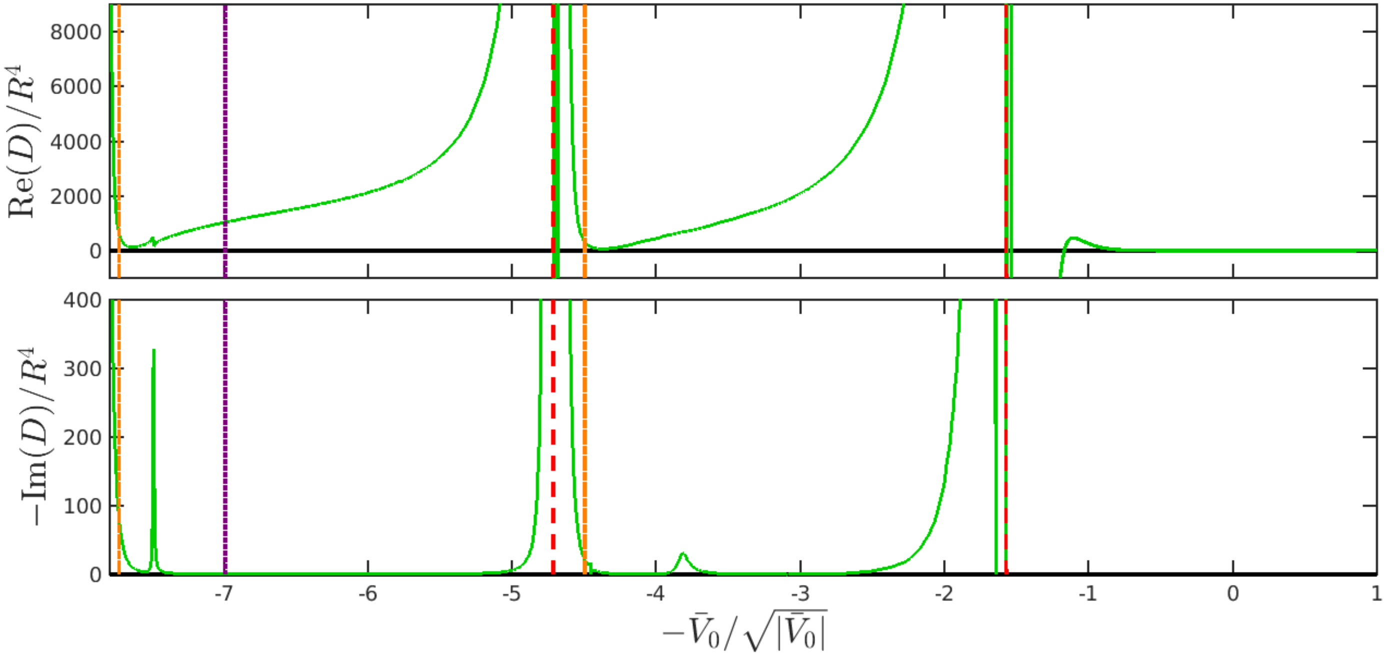
Hard-sphere limit

$$D/a^4 \underset{\bar{V}_0 \to -\infty}{=} 1761 \pm 1$$

S. Tan, Phys. Rev. A, 78, 013636 (2008)





s-wave dimer ($l = 0$)

d-wave dimer ($l = 2$)

g-wave dimer ($l = 4$)

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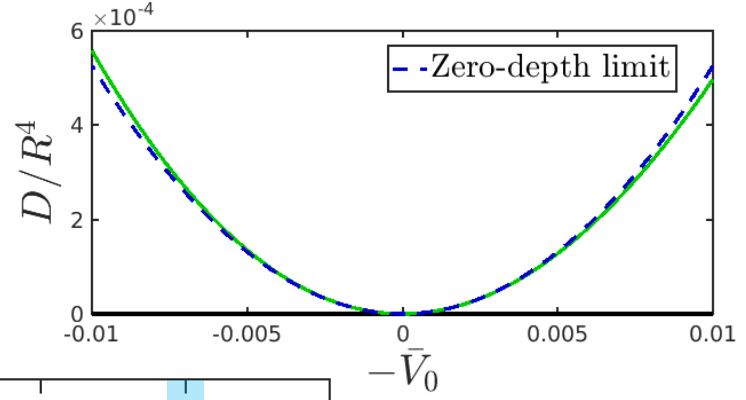
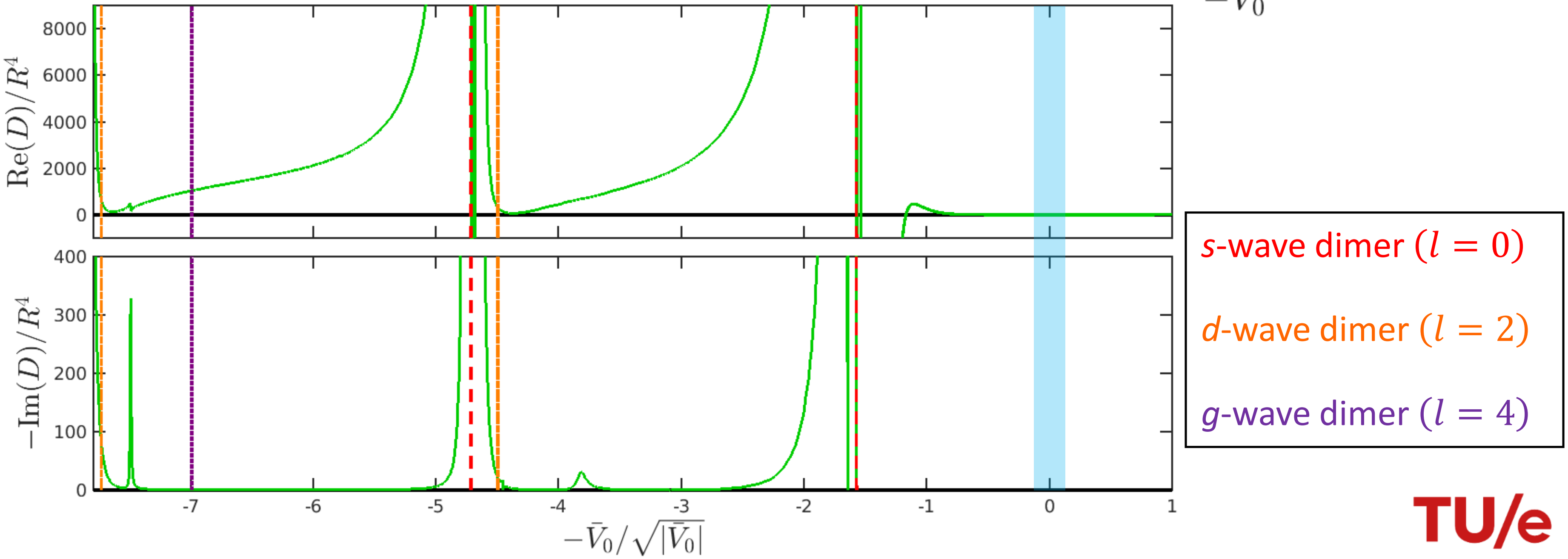
Zero-depth limit

$$D = -96\pi^6 m^2 \hbar^4 \langle \mathbf{0}|V|\mathbf{0}\rangle \left.\frac{\partial^2 \langle \mathbf{p}|V|\mathbf{0}\rangle}{\partial p^2}\right|_{p=0} + O(\bar{V}_0^3)$$

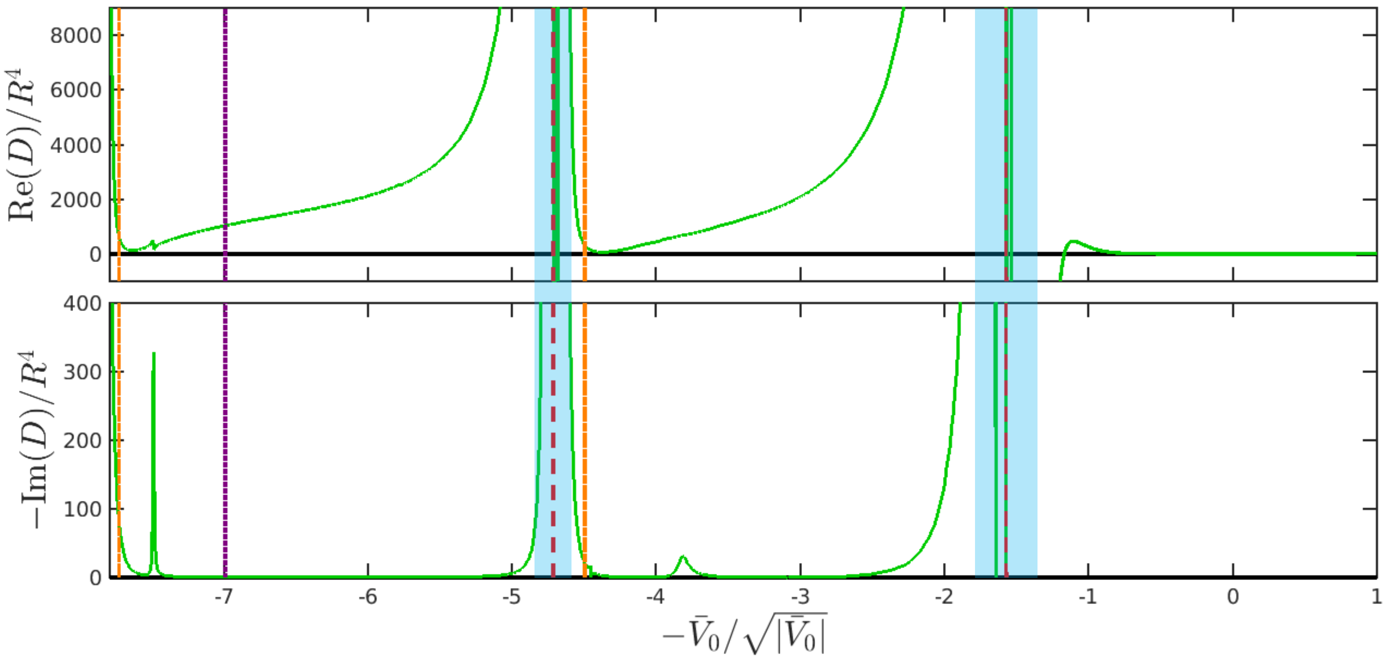
$$= \frac{8}{15}\pi^2 \bar{V}_0^2 R^4 + O(\bar{V}_0^3)$$





Strongly-interacting regime

$$|a|/R \gg 1$$
$$D \propto a^4$$



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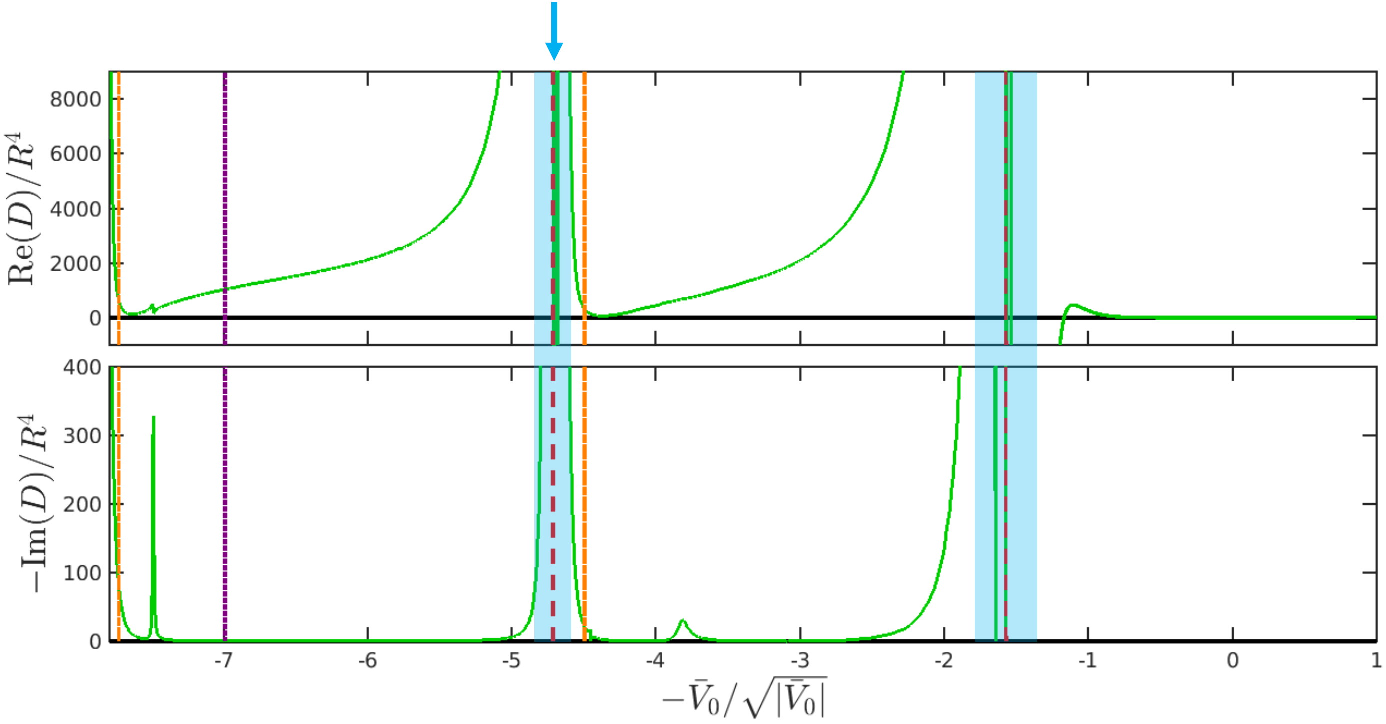

s-wave dimer ($l = 0$)

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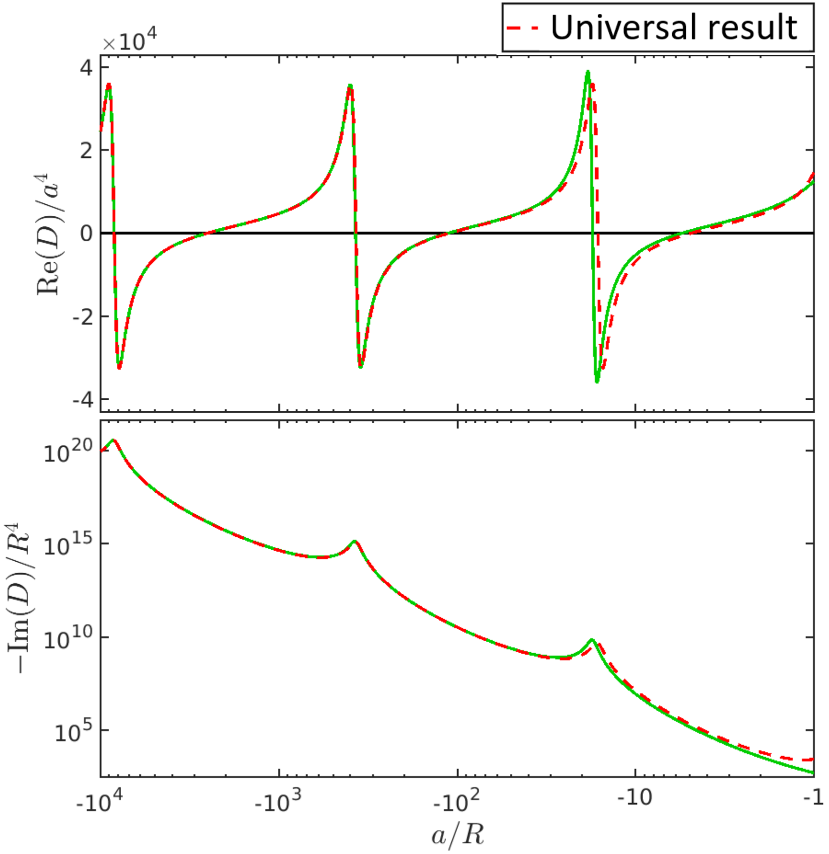
Second potential resonance $a < 0$



Efimov physics:

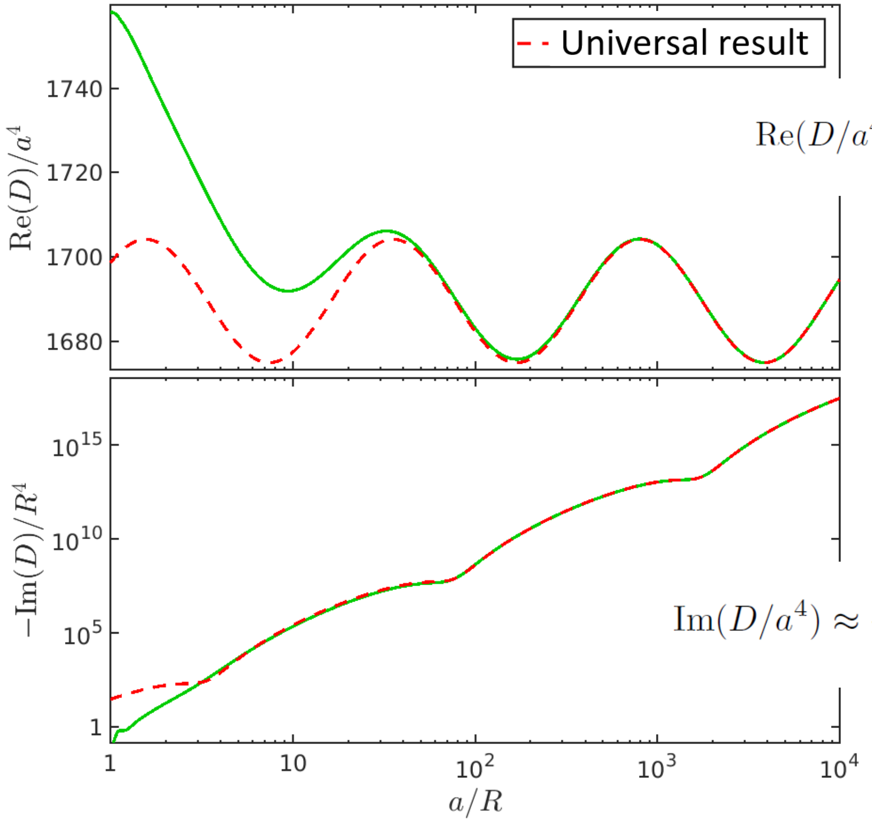
$$\mathrm{Re}(D/a^4) \approx C\left(c_- + \frac{1}{2}b_- \frac{\sin\left(2s_0\ln(a/a_-)\right)}{\sin^2\left(s_0\ln(a/a_-)\right) + \sinh^2(\eta)}\right)$$

	Our result	Result of Ref. [1]
b_-	3.153(5)	3.16
c_-	1.140(2)	1.14
η	0.068(1)	-

$$\mathrm{Im}(D/a^4) \approx -\frac{1}{2}C_- \frac{\sinh(2\eta)}{\sin^2\left(s_0\ln(a/a_-)\right) + \sinh^2(\eta)}$$

[1] E. Braaten, H.-W. Hammer, T. Mehen, Phys. Rev. Lett., 88, 040401 (2002)

Second potential resonance $a > 0$



Efimov physics:

$$\mathrm{Re}(D/a^4) \approx C\left(c_+ + \frac{1}{2}b_+(1-e^{-2\eta}) + b_+ e^{-2\eta}\sin^2\left(s_0 \ln(a/a_+) - \pi/4\right)\right)$$

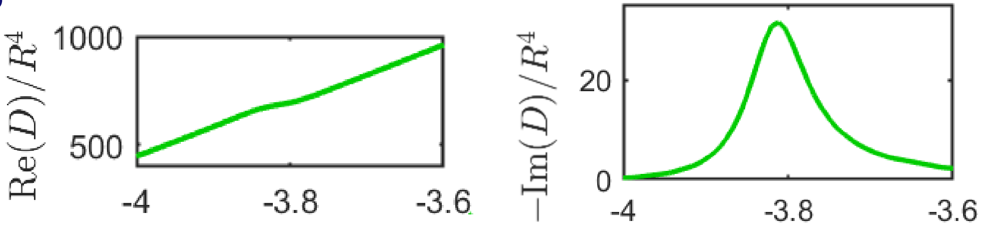
	Our result	**Result of Ref. [1]**
b_+	0.0226(5)	0.021
c_+	1.1288(5)	1.13
η	0.068(1)	-

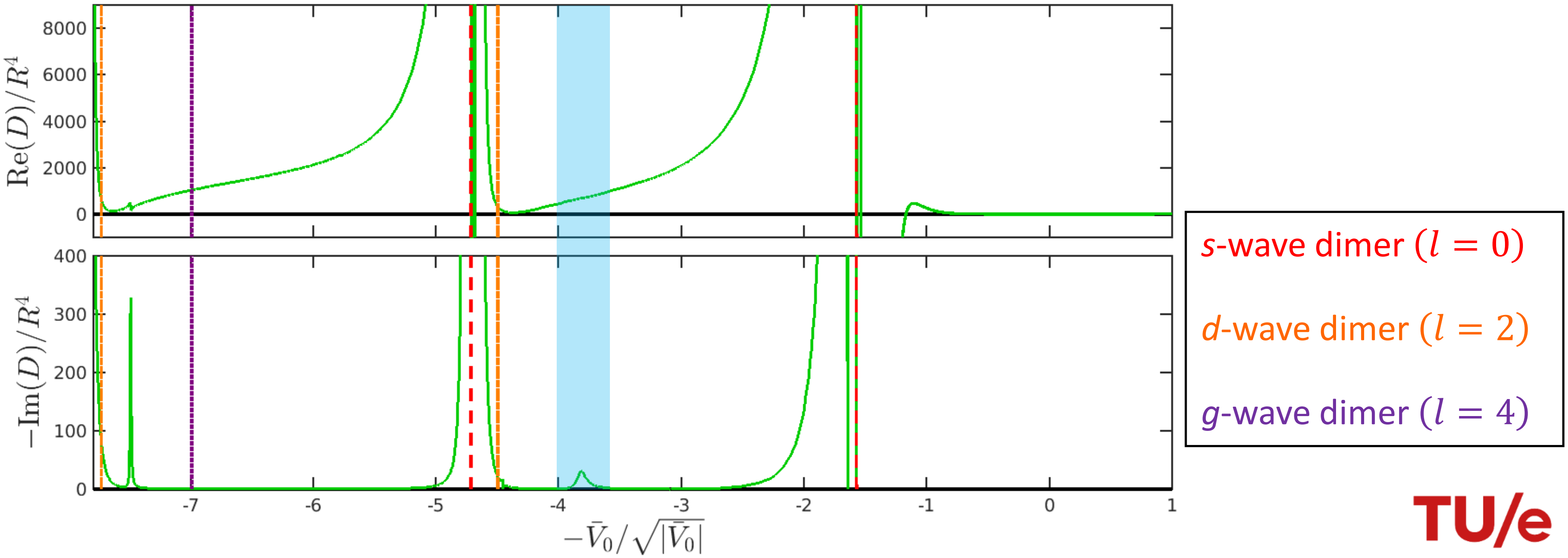
$$\mathrm{Im}(D/a^4) \approx -\frac{1}{2}C_+\left(\frac{1}{4}\left(1-e^{-4\eta}\right) + e^{-2\eta}\left(\sin^2\left(s_0 \ln(a/a_+)\right) + \sinh^2(\eta)\right)\right)$$

[1] E. Braaten, H.-W. Hammer, T. Mehen, Phys. Rev. Lett., 88, 040401 (2002)

Weakly-interacting regime

Three-body quasibound states
with strong *d*-wave character

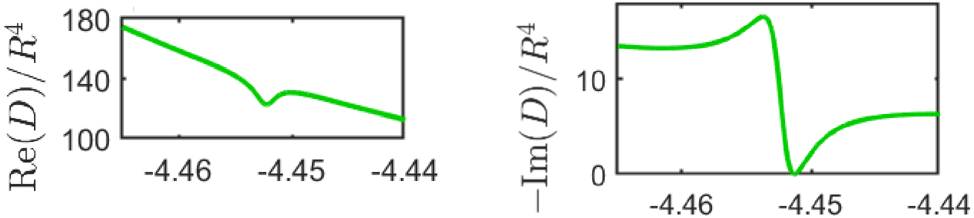


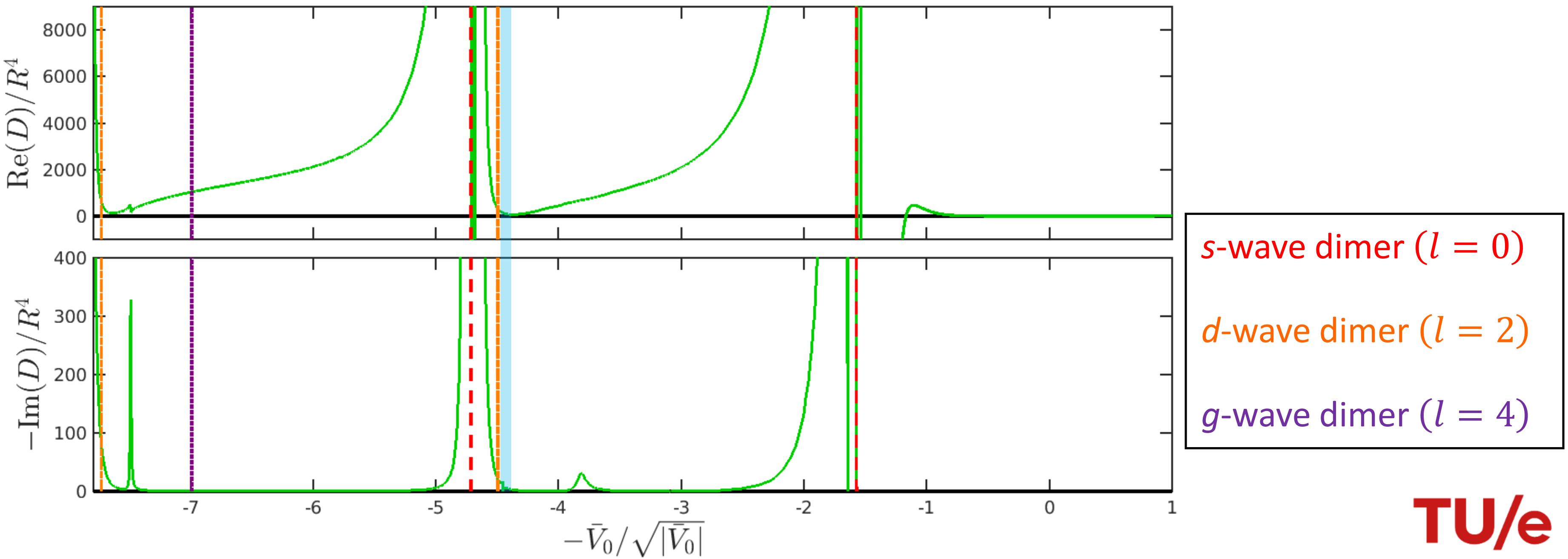


TU/e

Weakly-interacting regime

Three-body quasibound states
with strong *d*-wave character





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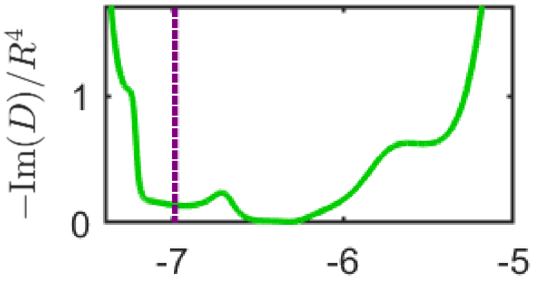
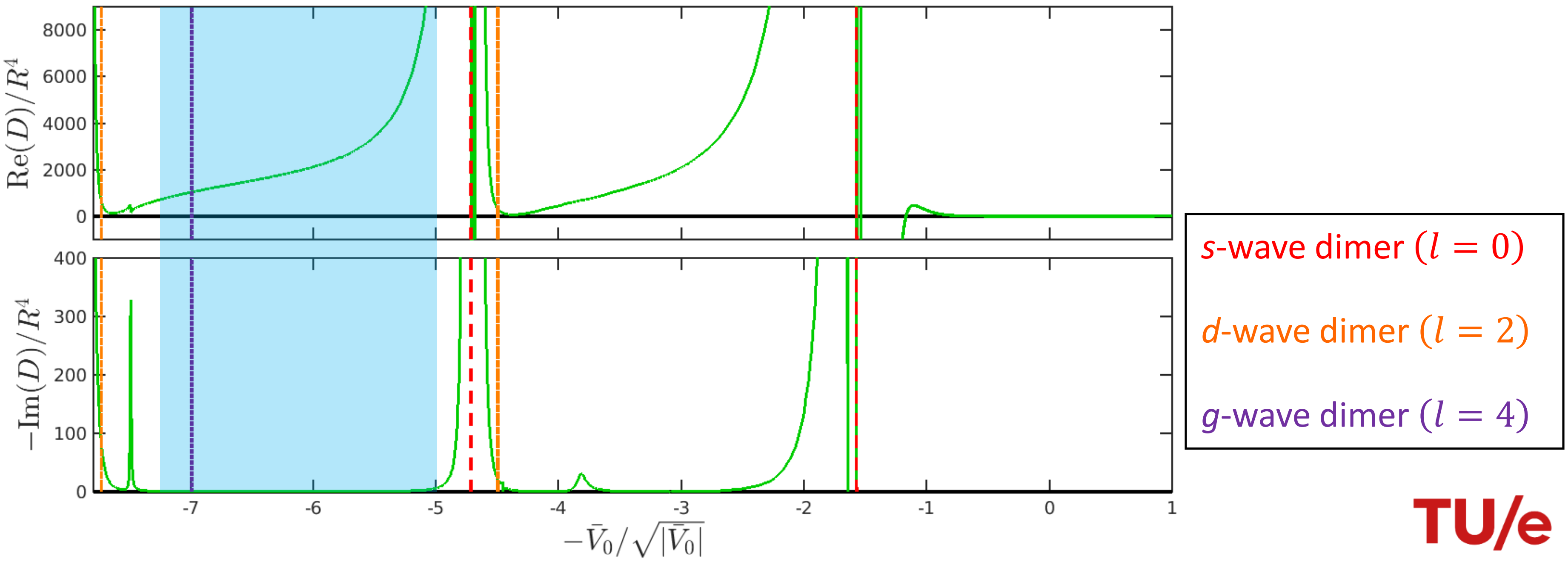
Weakly-interacting regime

Three-body quasibound states
with strong *g*-wave character





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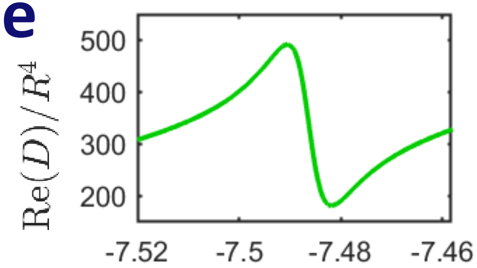
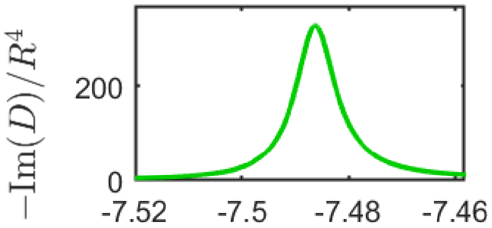
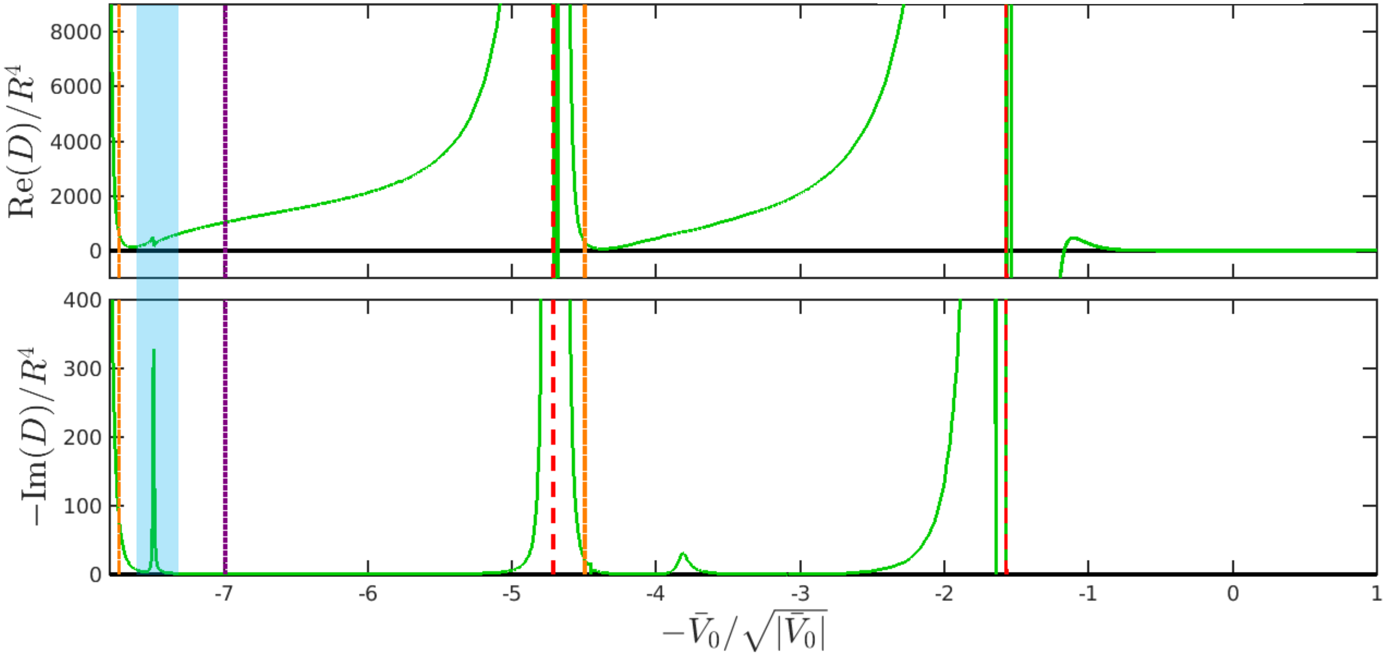
Weakly-interacting regime

Three-body quasibound states with strong *d*-wave and *g*-wave character







s-wave dimer ($l = 0$)

d-wave dimer ($l = 2$)

g-wave dimer ($l = 4$)

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Conclusion

Three-body scattering hypervolume

- three-body analog of the two-body scattering length
- important for studying ultracold Bose gases beyond the LHY correction

Our results for finite-range interactions:

- agreement with theoretical predictions (hard-sphere limit, zero-depth limit, Efimov effect)
- additional trimer resonances with strong *d*-wave and/or *g*-wave character

Outlook

- van der Waals potentials → quantum droplets?

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