Tracing a few-fermion system inside the unitary window

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Preliminaries

- definition of the unitary window:
  A region in the (a, E) plane in which some physical points could lie
  a is the two-body scattering length verifying $a >> r_0$
  r_0 the interaction range.
  We are dealing with systems having a large scattering length
- When a is large a shallow bound (virtual) state appears
  $E \approx \hbar^2/ma^2$
- In this region the low-energy scattering parameters and the shalow state are related

$$k \cot \delta_0 = -1/a + \frac{1}{2} r_{eff} k^2$$

with r_{eff} the effective range.

$$\kappa = 1/a + \frac{1}{2} r_{eff} \kappa^2$$

with $E = \hbar^2 \kappa^2/m = \hbar^2/ma^2 \left(1 + \frac{r_{eff}}{a} + \ldots\right)$

Properties inside the unitary window

Scale invariance

- $E \approx \hbar^2/ma^2$
- $< r^2 > \approx a^2/2$
- $D \approx \sqrt{2/a}$

Limits

This two scales define two limits:

- scaling limit: $r_0 \to 0$ (Thomas collapse)
- unitary limit: $a \to \infty$ (Efimov states)
- in both cases the ratio $r_0/a \to 0$

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Universal behavior in few-body systems

Examples

- The helium dimer (as given by the TTY potential):
  $E_d = 1.309$ mk
  $a = 188.78$ a.u.
  $r_{eff} = 13.845$ a.u.
  $E(a, r_{eff}) = 1.311$ mk
- The deuteron:
  $E_d = 2.225$ MeV
  $a^1 = 5.419 \pm 0.007$ fm
  $r^1_{eff} = 1.753 \pm 0.008$ fm
  $E(a, r_{eff}) = 2.223$ fm

Gaussian potential model

$$V(r) = V_0 e^{-(r/r_0)^2}$$

with V_0 fixed to describe a or E_d and the role of r_0 to be discussed.

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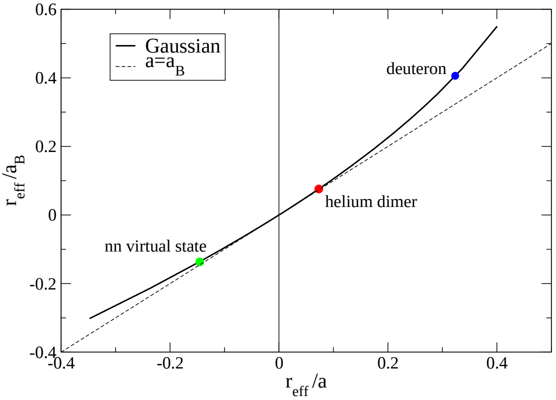
Universal behavior in few-body systems

- When a shallow state exists, a Gaussian potential gives a reasonable description of the low energy regime, bound and scattering states. Here $a_B = \sqrt{\hbar^2/mE_2}$.



The three-boson system

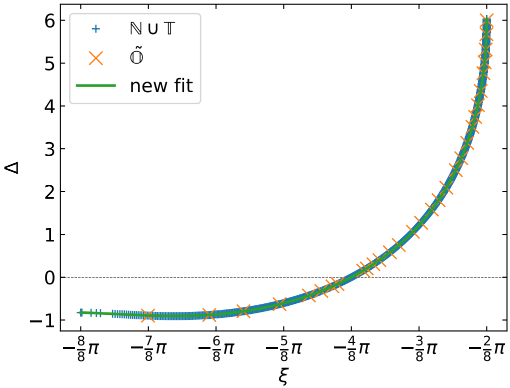
Zero-Range Equations: Efimov spectrum

$$E_3^n/(\hbar^2/ma^2) = \tan^2\xi$$

$$\kappa_* a = \mathrm{e}^{\pi(n-n_*)/s_0}\mathrm{e}^{-\Delta(\xi)/2s_0}/\cos\xi$$

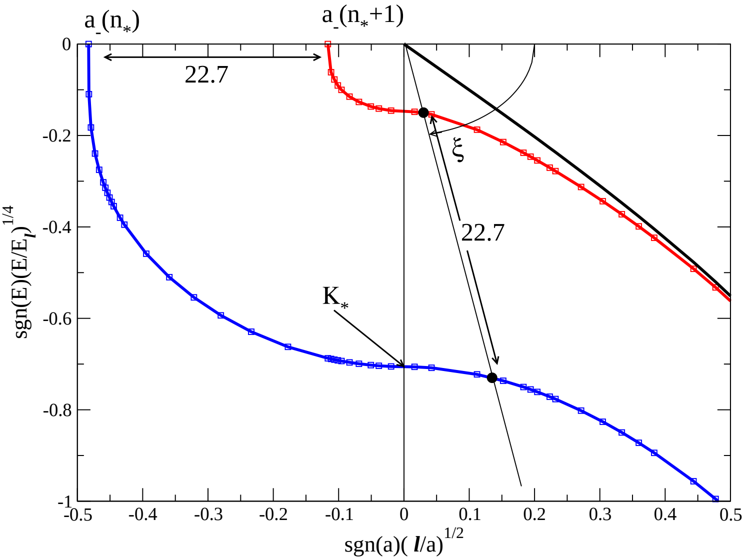
- The ratio E_3^n/E_2 defines the angle ξ
- The three-body parameter κ_* defines the energy of the system at the unitary limit $E_u = \hbar^2\kappa_*^2/m$
- The product $\kappa_* a$ is a function of ξ governed by the universal function $\Delta(\xi) = s_0 \log\left[\frac{E_3+E_2}{E_u}\right]$
- The universal function $\Delta(\xi)$ is obtained by solving the STM equations (Faddeev equation in the zero-range limit) and is the same for all levels n

The Universal function Δ



M. Gattobigio, M. Göbel, H.-W. Hammer, and A. Kievsky
Few-Body Systems **60**, 40 (2019)

Efimov plot



a_(n*)
a_(n*+1)
22.7
ξ
22.7
K*
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
sgn(E)(E/E_l)^1/4
sgn(a)(l/a)^1/2

The three-boson system

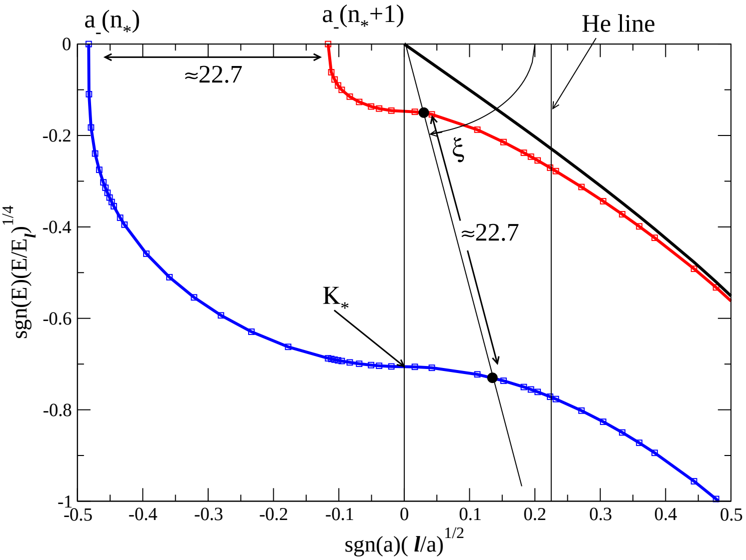
Finite-Range Equations: Gaussian spectrum

$$E_3^n/E_2 = \tan^2 \xi_n$$

$$\kappa_*^n a_B = \mathrm{e}^{-\widetilde{\Delta}_n(\xi_n)/2s_0} / \cos\xi$$

- The ratio E_3^n/E_2 defines the angle ξ_n
- The three-body parameter κ_*^n defines the energy of the *n*-level at the unitary limit $E_u = \hbar^2(\kappa_*^n)^2/m$
- The product $\kappa_*^n a_B$ is a function of ξ_n governed by the level function: $\widetilde{\Delta}(\xi_n) = s_0 \ln\left(\frac{E_3^n + E_2}{E_u^n}\right)$
- The level function $\widetilde{\Delta}(\xi)$ is obtained by solving the Schrödinger equation in the desired region.
- For $n > 1$ $\widetilde{\Delta}(\xi) \to \Delta(\xi)$

Three boson spectrum with a Gaussian potential



Scale invariance

The three-body parameter κ_* for the helium trimer

- The quantity $\kappa_* a_B = f(\xi)$ is a function of ξ, where $E_3/E_2 = \tan^2\xi$
- To determine ξ we use experimental results $E_3 = 126$mK and $E_2 = 1.3$mK. Accordingly $\tan^2\xi = 97.0$

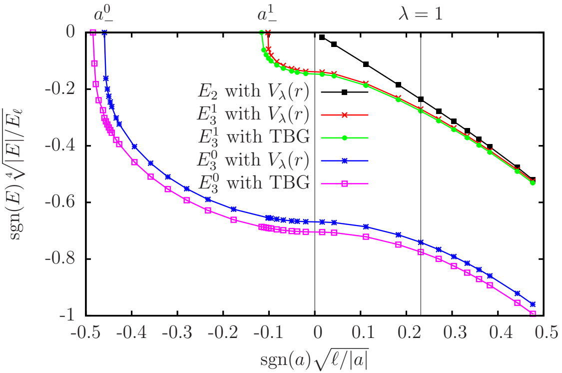
$$[\kappa_* a_B]^{He} = [\kappa_* a_B]^{gaussian} = [r_0\kappa_*(a_B/r_0)]^{gaussian} \approx 0.488/0.061 = 8$$

$$[\kappa_*]^{He} \approx 8/a_B \approx 0.044\, a.u.$$

$$\hbar^2\kappa_*^2/m = E_u = 87.3\ \mathrm{mK}$$

- Using the LM2M2 potential we obtain 87.3 mK
  Scale invariance is well verified!

Gaussian vs Realistic Efimov plot



$$V_\lambda(r) = \lambda V_{He}(r)$$
$$TBG = V_0\ e^{-r^2/r_0^2}$$

Characteristic of the Efimov plot

- The scale invariance constrains the energy spectrum
- At unitary there are infinite excited states
- At the physical point the numbers of excited states is finite
- For the He system (the ratio $\ell/a \approx 0.23$) the prediction is two excited states

- For nuclear physics the triton has not excited states
- The α particle has no excited states
- The doublet scattering length $n-d$ is very small (0.65 fm)
- The doublet scattering length and the triton binding energy are correlated (Phillips line)
- The triton α particle binding energies are correlated (Tjon line)

- Tracing the few-fermion system inside the Efimov window many of these characteristics can be explained

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Soft Two-Body Gaussian Potential

Effective low-energy soft potential

- boson case

$$V(r) = V_0 \, \mathrm{e}^{-r^2/r_0^2}$$

- nuclear case

$$V(r) = \sum_{ST} V_{ST} \, \mathrm{e}^{-r^2/r_{ST}^2} \mathcal{P}_{ST}$$

where ST are the spin snd isospin channels. The singlet and triplet scattering lengths correspond to $ST = 0, 1$ and $1, 0$

Soft Two-Body Gaussian Potential

S-wave interaction, $ST = 01, 10$

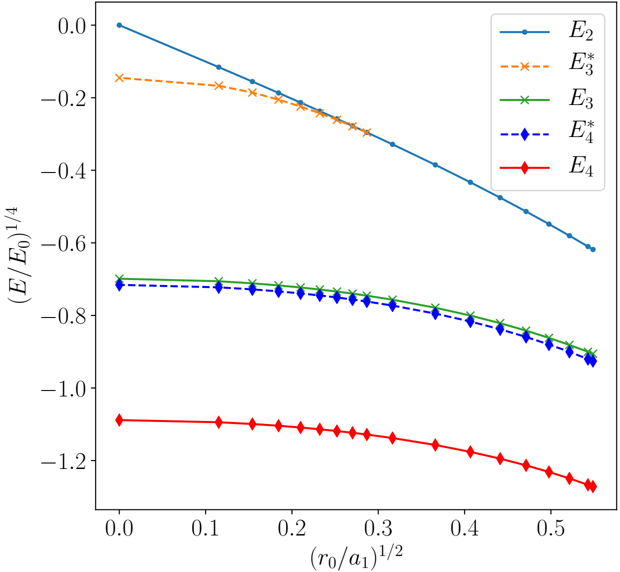
$$V(r) = V_{10}\, \mathrm{e}^{-r^2/r_0^2} \mathcal{P}_{10} + V_{01}\, \mathrm{e}^{-r^2/r_0^2} \mathcal{P}_{01}$$

as reference energy we use $E_0 = \hbar^2/mr_0^2$

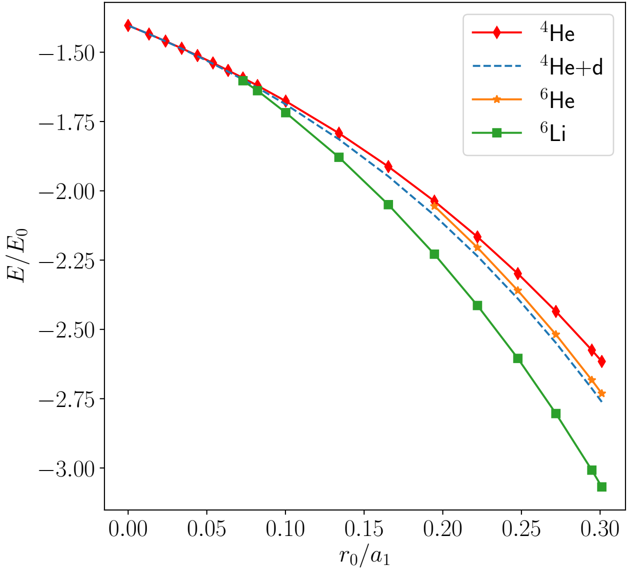
The physical point is located at the values:
$a^1 = 5.419 \pm 0.007$ fm
$r^1_{eff} = 1.753 \pm 0.008$ fm
$a^0 = -23.740 \pm 0.020$ fm
$r^0_{eff} = 2.77 \pm 0.05$ fm

The strength of the gaussian are related to verify $a^0/a^1 = \mathrm{constant}$

$A \leq 4$ Efimov plot



$A \leq 6$ Efimov plot



Comments on the $A \leq 6$ Efimov plot

- For $A = 3$ only one state survive at the physical point
- The excited state disappears at $r_0/a_1 \approx 0.09$
- The nuclear physics point is $r_0/a_1 \approx 0.3$
- Resembling the bosonic case, there is a shallow four-body excited state
- At the physical point $E_4/E_3 \approx 3.9$, very close to the experimental ratio $E(^4\mathrm{He})/E(^3\mathrm{He}) \approx 3.7$
- At unitary no five-body and six-body bound state appear.
- The ^{6}Li state appears quite close to the unitary limit
- The ^{6}He state appears a little bit further
- At the physical point they are very deep

- two ingredients are missing: the three-body force and the Coulomb interaction

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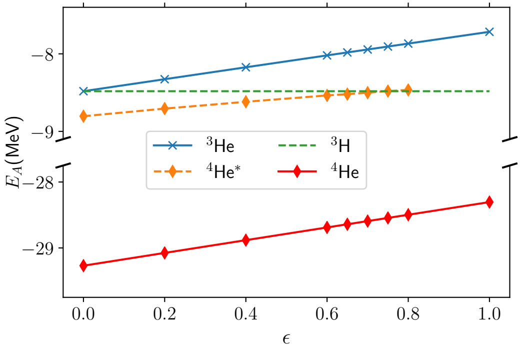
Three-body force

$$W(\rho) = W_0 \mathrm{e}^{-(r_{12}^2 + r_{13}^2 + r_{23}^2)/\rho_0^2}$$

$$\sum_{cyc} Z_0 \mathrm{e}^{-(r_{12}^2 + r_{13}^2)/R_0^2}$$

Including Coulomb at the physical point



Three-body force included to describe the triton and $V_c(r) = \epsilon\, e^2/r$

To be noticed that at the physical point with $\epsilon = 1$ the six-body states are not bound any more!

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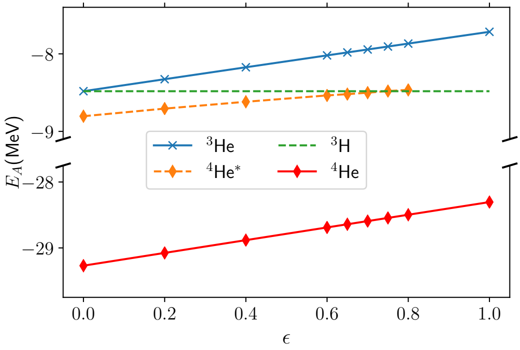

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Role of P-waves

• The study shown up to now has been done with potential

$$V(r) = \sum_{ST} V_{ST}\, \mathrm{e}^{-r^2/r^2_{ST}} \mathcal{P}_{ST}$$

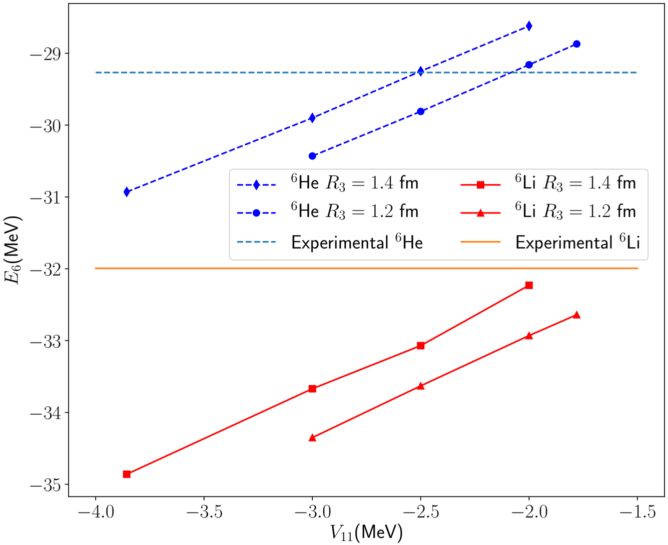
where V_{01} and V_{10} fixed to reproduce ${}^0a$ and ${}^1a$. The other two terms of the interaction V_{11} and V_{00} were set to zero.

• These potentials can be used to describe the (uncoupled) P-wave phase-shifts through the relation

$$S_k = k^3 \cot {}^{2S+1}P_J = \frac{-1}{{}^{2S+1}a_J} + \frac{1}{2}\, {}^{2S+1}r_J k^2$$

• V_{00} and V_{11} have been fixed from the ${}^1P_1$ and ${}^3P_0$ scattering parameters. V_{00} is repulsive whereas V_{11} is sligthly attractive.

$A = 6$ spectrum


^{6}He $R_3 = 1.4$ fm
^{6}He $R_3 = 1.2$ fm
Experimental ^{6}He
^{6}Li $R_3 = 1.4$ fm
^{6}Li $R_3 = 1.2$ fm
Experimental ^{6}Li
E_6(MeV)
-29
-30
-31
-32
-33
-34
-35
-4.0
-3.5
-3.0
-2.5
-2.0
-1.5
V_{11}(MeV)

Conclusions form the Efimov window

- Nuclear systems are well inside the Efimov window
- The light nuclei spectrum emerge continuously from the unitary limit as the singlet and triplet scattering lengths take their physical values
- Gaussian potentials naturally take into account range corrections.
- Quantitative values result from delicate cancellations
- To trace the excited state of ^{4}He as it enters the continuum is a very difficult but interesting task
- The study clarifies some correlations as the Phillips or Tjon lines
- Although some ratios are well described, the ^{6}Li-^{6}He mass difference is predicted too large. Probably the $L \cdot S$ operator could help to set this difference.