Mesons with charm and bottom quarks in a covariant quark model

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EFB24 University of Surrey, Guildford, September 2, 2019

Motivation for studying meson structure

- Meson properties are measured at many experimental facilities:
  LHC, *BABAR*, Belle, BES III, GlueX (JLab); in the future PANDA (GSI)
- Trying to find exotic mesons (hybrids, glueballs, … but maybe $q\bar{q}$ in disguise?)
- We also need to understand "conventional" $q\bar{q}$ -mesons in more detail
- Study production mechanisms, transition form factors
  (e.g., important for hadronic contributions to light-by-light scattering)

Theory: a very large amount of work has already been done on meson structure

Many different approaches:
- Lattice QCD
- Bethe-Salpeter/Dyson-Schwinger Equations
- Relativistic Quantum Mechanics (point form, front form, instant form)
- BLFQ (Basis Light-Front Quantization)
- Chiral quark models
- Constrained dynamics two-body Dirac equation
- Relativized Schrödinger equation, …

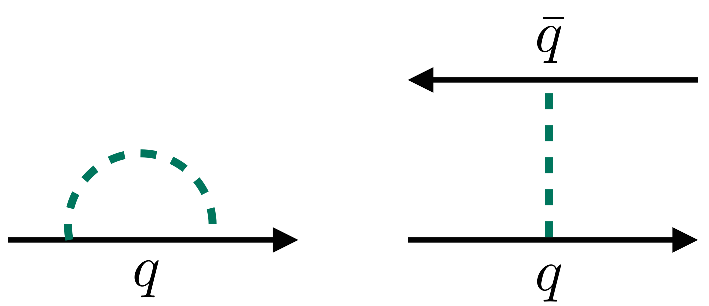
Our approach

CST - Covariant Spectator Theory

Main goals and features of our approach:

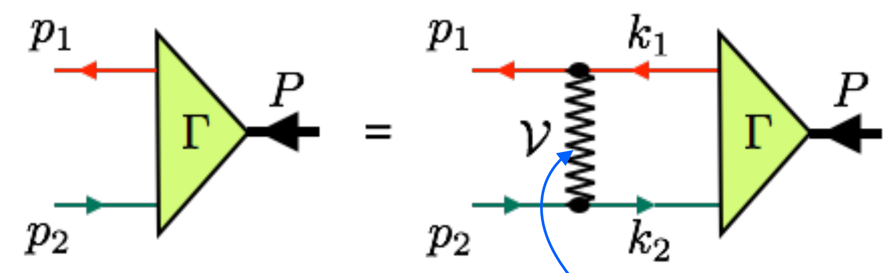
- Find $q\bar{q}$ interaction that can be used in all mesons (unified model)
- Relativistic covariance (work in Minkowski space)
- Confinement through a confining interaction kernel, which should reduce to linear+Coulomb in the nonrelativistic limit
- Learn about the Lorentz structure of the confining interaction
- Quark masses are dynamic: self-interaction should be consistent with $q\bar{q}$ interaction
- Chiral symmetry: massless pion in chiral limit, satisfy the axialvector Ward-Takahashi identity

Huge mass variation:
from pions (~0.14 GeV)
to bottomonium (> 10 GeV)

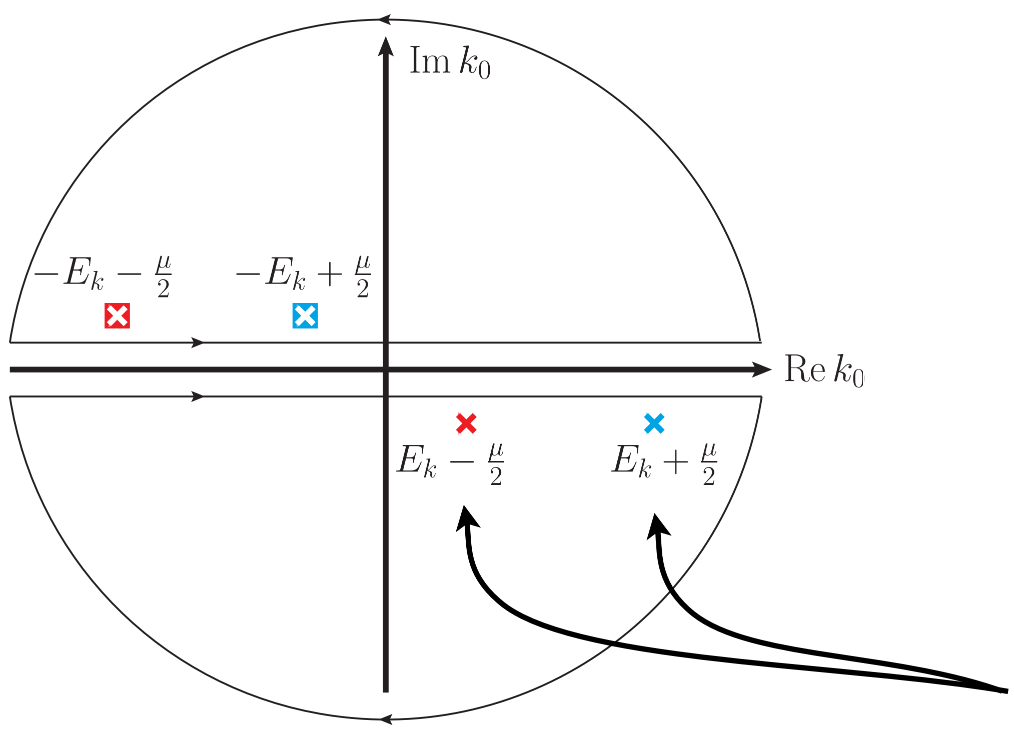


CST equation for two-body bound states

Bethe-Salpeter equation for $q\bar{q}$ bound-state with mass μ



2PI diagrams



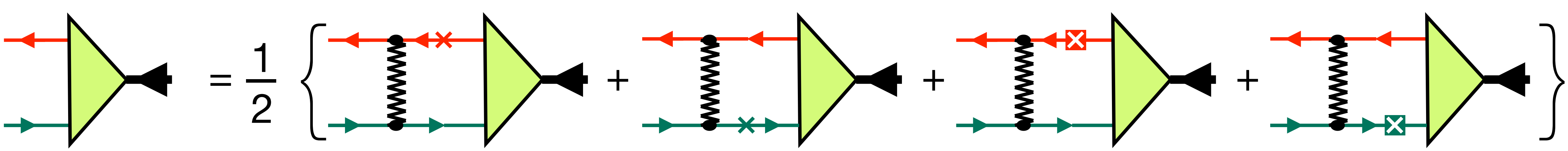
Integration over relative energy k_0:

- Keep only pole contributions from constituent particle propagators
- Poles from particle exchanges appear in higher-order kernels (usually neglected — tend to cancel)
- Reduction to 3D loop integrations, but covariant
- Correct one-body limit

If bound-state mass μ is small:
both poles are close together (both important)

Symmetrize pole contributions from both half planes: charge conjugation symmetry

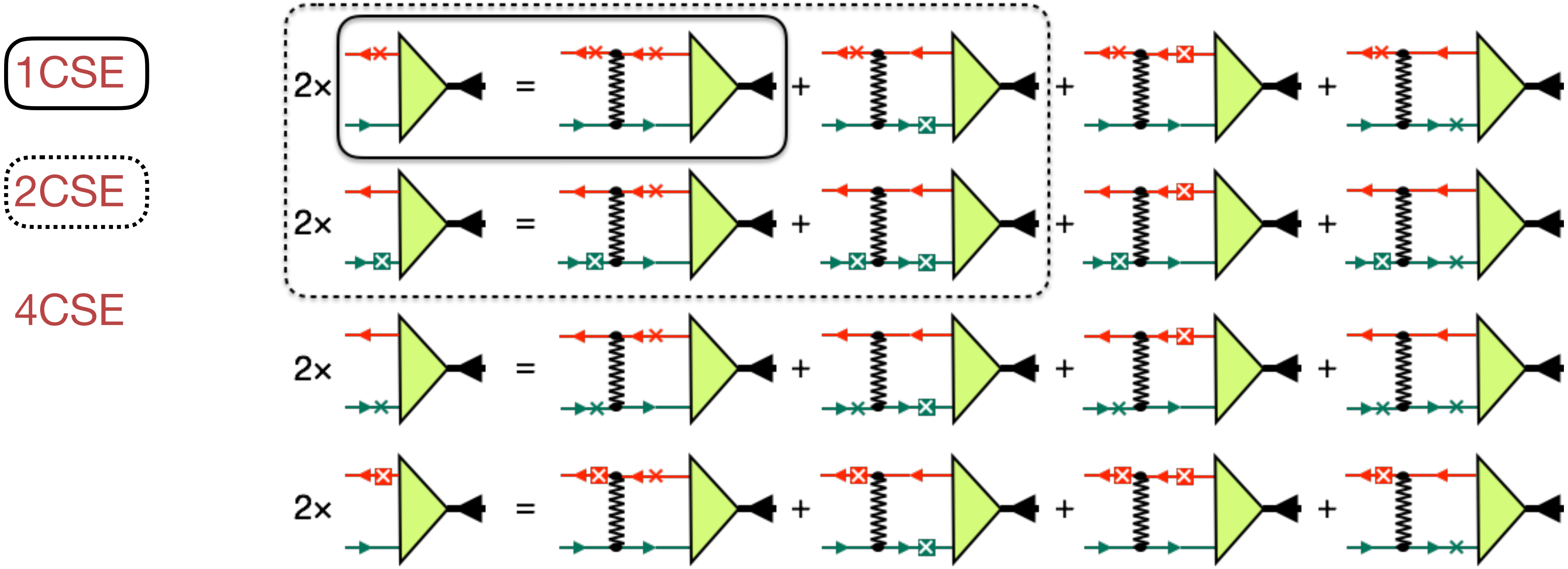
BS vertex (approx.) CST vertices



Once the four CST vertices (with one quark on-shell) are all known, one can use this equation to get the vertex function for arbitrary four-momenta.

CST equations

Closed set of equations when external legs are systematically placed on-shell



Solutions: bound state masses μ and corresponding vertex functions Γ

One-channel spectator equation (1CSE):

- Particularly appropriate for unequal masses
- Numerical solutions easier (fewer singularities)
- But not charge-conjugation symmetric

Two-channel spectator equation (2CSE):

- Restores charge-conjugation symmetry
- Additional singularities in the kernel

Four-channel spectator equation (4CSE):

- Necessary for light bound states (pion!)

All have smooth one-body limit (Dirac equation) and nonrelativistic limit (Schrödinger equation).

The covariant kernel

Our kernel:

$$\mathcal{V}(p,k;P) = \frac{3}{4}\mathbf{F}_1 \cdot \mathbf{F}_2 \sum_K V_K(p,k;P)\,\Theta_1^{K(\mu)} \otimes \Theta_{2(\mu)}^K$$

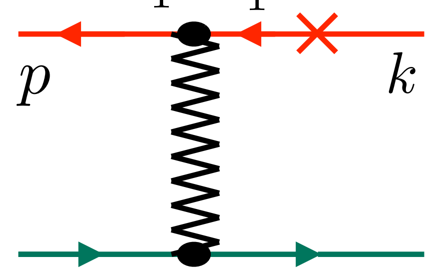
$F_a = \frac{1}{2}\lambda_a$ color SU(3) generators

1 for $q\bar{q}$ color singlets

momentum dependence

Lorentz structure

$$\Theta_i^{K(\mu)} = \mathbf{1}_i, \gamma_i^5, \gamma_i^\mu$$



- Confining interaction: Lorentz (scalar + pseudoscalar) mixed with vector
  Coupling strength σ, mixing parameter y

$y=0$	pure S+PS
$y=1$	pure V

for correct nonrelativistic limit

$$\mathcal{V}_{\mathrm{L}}(p,k;P) = \left[(1-y)\left(\mathbf{1}_1 \otimes \mathbf{1}_2 + \gamma_1^5 \otimes \gamma_2^5\right) - y\,\gamma_1^\mu \otimes \gamma_{\mu 2}\right] V_{\mathrm{L}}(p,k;P)$$

→ E.P. Biernat et al., PRD **90**, 096008 (2014)

equal weight (constraint from chiral symmetry)

- One-gluon exchange with constant coupling strength α_s
  + Constant interaction (in r-space) with strength C
  } Lorentz vector

$$\mathcal{V}_{\mathrm{OGE}}(p,k;P) + \mathcal{V}_{\mathrm{C}}(p,k;P) = -\gamma_1^\mu \otimes \gamma_{2\mu}\left[V_{\mathrm{OGE}}(p,k;P) + V_{\mathrm{C}}(p,k;P)\right]$$

- Nonrelativistic limit: Cornell type potential

$$V(r) = \sigma r - C - \frac{\alpha_s}{r}$$

(for any value of y)

Covariant confining kernel in CST

- Nonrelativistic linear potential in momentum space: FT of $\tilde{V}_L(r) = \sigma r$

$$\langle V_L \phi\rangle(\mathbf{p}) = \int \frac{d^3k}{(2\pi)^3} V_L(\mathbf{p}-\mathbf{k})\phi(\mathbf{k}) = -8\pi\sigma \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\mathbf{k})-\phi(\mathbf{p})}{(\mathbf{p}-\mathbf{k})^4}$$

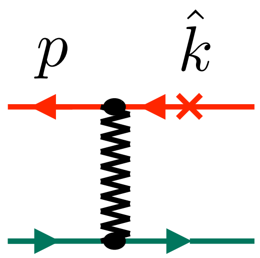
any regular function

highly singular

Cauchy principal value singularity

automatic subtraction

- Covariant generalization: $\mathbf{q}^2 \to -q^2$



initial state: either quark or antiquark onshell

This leads to a kernel that acts like

$$\langle V_L \phi\rangle(p) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E_k} V_L(p,\hat{k})\phi(\hat{k}) = -8\pi\sigma \int \frac{d^3k}{(2\pi)^3} \frac{m}{E_k} \frac{\phi(\hat{k})-\phi(\hat{k}_R)}{(p-\hat{k})^4}$$

$\hat{k} = (E_k, \mathbf{k})$ on mass shell

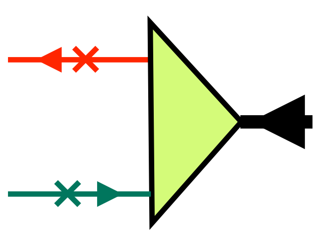
Complication: Singularity not only when $\mathbf{k} = \mathbf{p}$

$\hat{k}_R = (E_{k_R}, \mathbf{k}_R)$ $\qquad$ $\mathbf{k}_R = \mathbf{k}_R(p_0, \mathbf{p})$

value of $\mathbf{k}$ at which kernel becomes singular

- Does it still confine?

Yes: the vertex function vanishes if both quarks are on-shell!

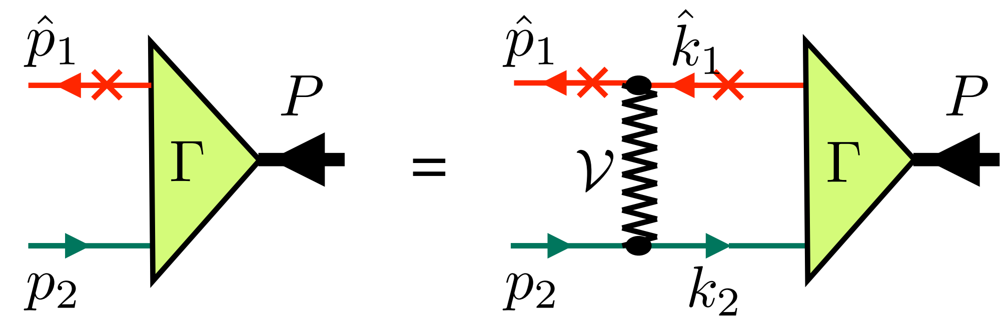


= 0

More details: Savkli, Gross, PRC **63**, 035208 (2001)

The One-Channel Spectator Equation (1CSE)

We solve the 1CSE for heavy and heavy-light systems



- Should work well for bound states with at least one heavy quark
- Much easier to solve numerically than 2CSE or 4CSE
- C-parity splitting small in heavy quarkonia
- For now with constant constituent quark masses (quark self-energies will be included later)

$$E_{ik} = \sqrt{m_i^2 + \mathbf{k}^2}$$

$$\Gamma(\hat{p}_1, p_2) = -\int \frac{d^3k}{(2\pi)^3} \frac{m_1}{E_{1k}} \sum_K V_K(\hat{p}_1, \hat{k}_1)\Theta_1^{K(\mu)} \frac{m_1 + \hat{\not{k}}_1}{2m_1} \Gamma(\hat{k}_1, k_2) \frac{m_2 + \not{k}_2}{m_2^2 - k_2^2 - i\epsilon} \Theta^K_{2(\mu)}$$

- Practical solution: solve equation for relativistic wave functions in a basis of eigenstates of total orbital angular momentum *L* and of total spin *S* (not necessary, but useful for spectroscopic identification of solutions)

Relativistic components

$$J^P = 0^\pm \qquad \int_0^\infty dp\, p^2 \left[\psi_S^2(p) + \psi_P^2(p)\right] = 1$$

$$J^P = 1^\pm \qquad \int_0^\infty dp\, p^2 \left[\psi_S^2(p) + \psi_D^2(p) + \psi_{P_s}^2(p) + \psi_{P_t}^2(p)\right] = 1$$

Normalization of radial wave functions → probabilities of partial waves

(No problem with parity: relativistic components also have opposite intrinsic parity factor!)

Data sets used in least-square fits of meson masses

	State	$J^{P(C)}$	Mass (MeV)	Data set S1	S2	S3
$b\bar{b}$	$\Upsilon(4S)$	1^{--}	10579.4±1.2		•	•
	$\chi_{b1}(3P)$	1^{++}	10512.1±2.3			•
	$\Upsilon(3S)$	1^{--}	10355.2±0.5		•	•
	$\eta_b(3S)$	0^{-+}	10337			
	$h_b(2P)$	1^{+-}	10259.8±1.2			•
	$\chi_{b1}(2P)$	1^{++}	10255.46±0.22±0.50			•
	$\chi_{b0}(2P)$	0^{++}	10232.5±0.4±0.5		•	•
	$\Upsilon(1D)$	1^{--}	10155			
	$\Upsilon(2S)$	1^{--}	10023.26±0.31		•	•
	$\eta_b(2S)$	0^{-+}	9999±4	•	•	•
	$h_b(1P)$	1^{+-}	9899.3±0.8			•
	$\chi_{b1}(1P)$	1^{++}	9892.78±0.26±0.31			•
	$\chi_{b0}(1P)$	0^{++}	9859.44±0.42±0.31		•	•
	$\Upsilon(1S)$	1^{--}	9460.30±0.26		•	•
	$\eta_b(1S)$	0^{-+}	9399.0±2.3	•	•	•
$b\bar{c}$	$B_c(2S)^{\pm}$	0^{-}	6842±6			•
	B_c^{+}	0^{-}	6275.1±1.0	•	•	•
$b\bar{s}$	$B_{s1}(5830)$	1^{+}	5828.63±0.27			•
$b\bar{q}$	$B_1(5721)^{+,0}$	1^{+}	5725.85±1.3			•
$b\bar{s}$	B_s^{*}	1^{-}	5415.8±1.5		•	•
	B_s^{0}	0^{-}	5366.82±0.22	•	•	•
$b\bar{q}$	B^{*}	1^{-}	5324.65±0.25		•	•
	$B^{\pm,0}$	0^{-}	5279.45	•	•	•

	State	$J^{P(C)}$	Mass (MeV)	Data set S1	S2	S3
$c\bar{c}$	$X(3915)$	0^{++}	3918.4±1.9		•	•
	$\psi(3770)$	1^{--}	3773.13±0.35		•	•
	$\psi(2S)$	1^{--}	3686.097±0.010		•	•
	$\eta_c(2S)$	0^{-+}	3639.2±1.2	•	•	•
	$h_c(1P)$	1^{+-}	3525.38±0.11			•
	$\chi_{c1}(1P)$	1^{++}	3510.66±0.07			•
	$\chi_{c0}(1P)$	0^{++}	3414.75±0.31		•	•
	$J/\Psi(1S)$	1^{--}	3096.900±0.006		•	•
	$\eta_c(1S)$	0^{-+}	2983.4±0.5	•	•	•
$c\bar{s}$	$D_{s1}(2536)^{\pm}$	1^{+}	2535.10±0.06			•
	$D_{s1}(2460)^{\pm}$	1^{+}	2459.5±0.6			•
$c\bar{q}$	$D_1(2420)^{\pm,0}$	1^{+}	2421.4			•
	$D_0^{*}(2400)^0$	0^{+}	2318±29		•	•
$c\bar{s}$	$D_{s0}^{*}(2317)^{\pm}$	0^{+}	2317.7±0.6		•	•
	$D_s^{*\pm}$	1^{-}	2112.1±0.4		•	•
$c\bar{q}$	$D^{*}(2007)^0$	1^{-}	2008.62			•
$c\bar{s}$	$D_s^{\pm}$	0^{-}	1968.27±0.10	•	•	•
$c\bar{q}$	$D^{\pm,0}$	0^{-}	1867.23	•	•	•

S1: 9 PS mesons

S2: 25 PS+V+S mesons

S3: 39 PS+V+S+AV mesons

q represents a light quark (u or d)

We use $m_u = m_d \equiv m_q$

Global fits with fixed quark masses and y=0

S. Leitão, A. S., M. T. Peña, E. Biernat, Phys. Lett. B **764** (2017) 38

First step: we perform global fits to the heavy + heavy-light meson spectrum

Adjustable model parameters: σ α_s C

Model parameters not adjusted in the fits:

Constituent quark masses (in GeV) m_b=4.892, m_c=1.600, m_s=0.448, m_q=0.346

Scalar + pseudoscalar confinement $y = 0$

Linear and OGE kernels need to be regularized
We chose Pauli-Villars regularizations with parameter $\Lambda = 2m_1$

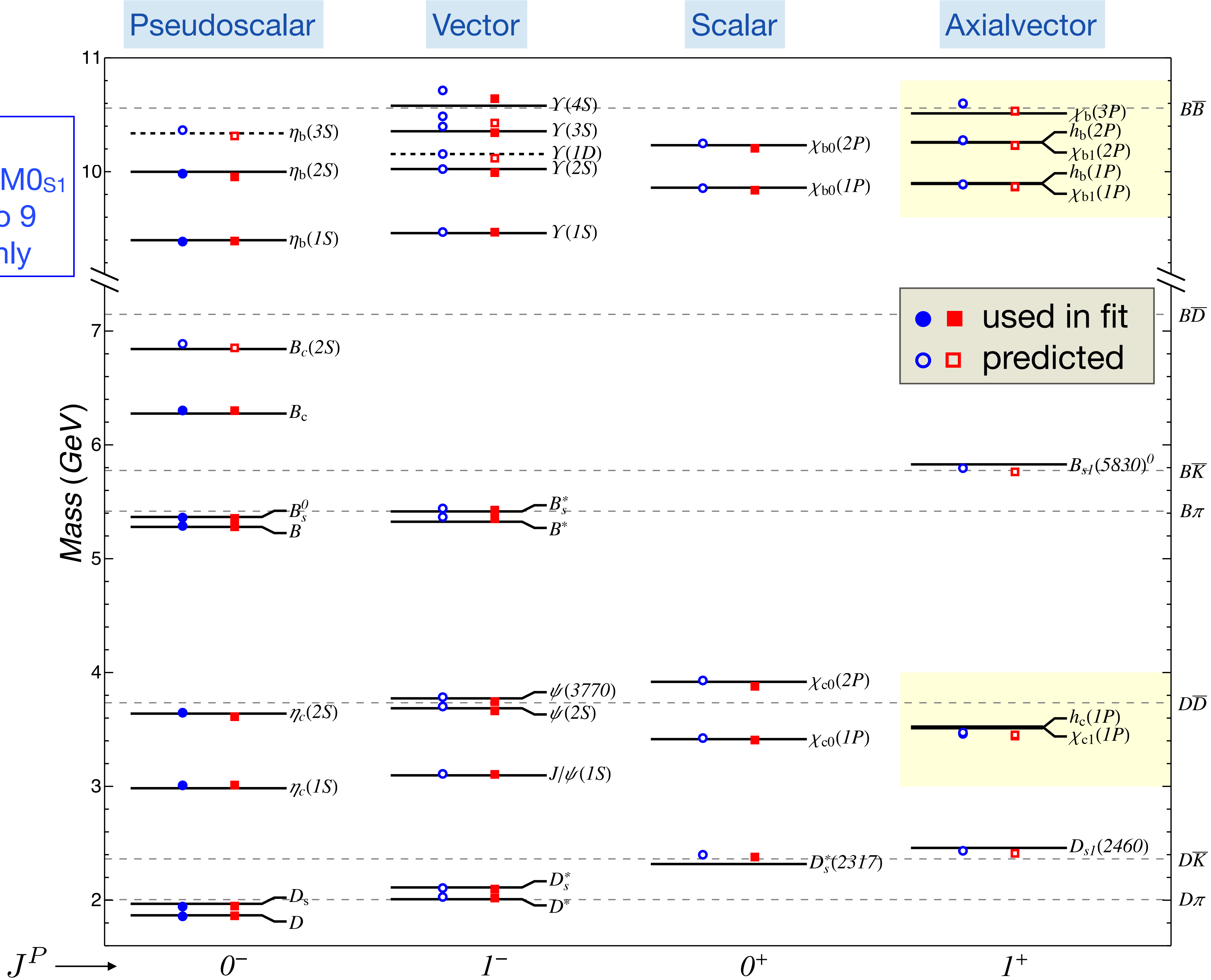
- Model $M0_{S1}$: fitted to 9 pseudoscalar meson masses only
- Model $M0_{S2}$: fitted to 25 pseudoscalar, vector, and scalar meson masses

(Previously called models P1 and PSV1)

Global fits with fixed quark masses and scalar confinement (y=0)

Blue: model M0$_{S1}$ fitted to 9 0^-(P) only

Red: model M0$_{S2}$ fitted to 25 P+S+V



Global fits with fixed quark masses and y=0

The results of the two fits are remarkably similar!

Model	σ [GeV^2]	α_s	C [GeV]
$M0_{S1}$	0.2493	0.3643	0.3491
$M0_{S2}$	0.2247	0.3614	0.3377

→

rms differences to experimental masses (set S3):

Model	Δ_{rms} [GeV]
$M0_{S1}$	0.037
$M0_{S2}$	0.036

- Kernel parameters are already well determined through pseudoscalar states ($J^P = 0^-$)

Almost 100% L=0, S=0
(S-wave, spin singlet)

$$\langle 0^- | \mathbf{L} \cdot \mathbf{S} | 0^- \rangle = 0$$ Spin-orbit force vanishes

$$\langle 0^- | S_{12} | 0^- \rangle = 0$$ Tensor force vanishes

$$\langle 0^- | \mathbf{S}_1 \cdot \mathbf{S}_2 | 0^- \rangle = -3/4$$ Spin-spin force acts in singlet only

- Good test for a covariant kernel:

Pseudoscalar states do not constrain spin-orbit and tensor forces, and cannot separate spin-spin from central force.

But they should be determined through covariance.

Model $M0_{S1}$ indeed predicts spin-dependent forces correctly!

Leitão, AS, Peña, Biernat, Phys. Lett. B **764** (2017) 38

Fits with variable quark masses and confinement (S+PS)-V mixing *y*

In a new series of fits we treat quark masses and mixing parameter *y* as adjustable parameters.

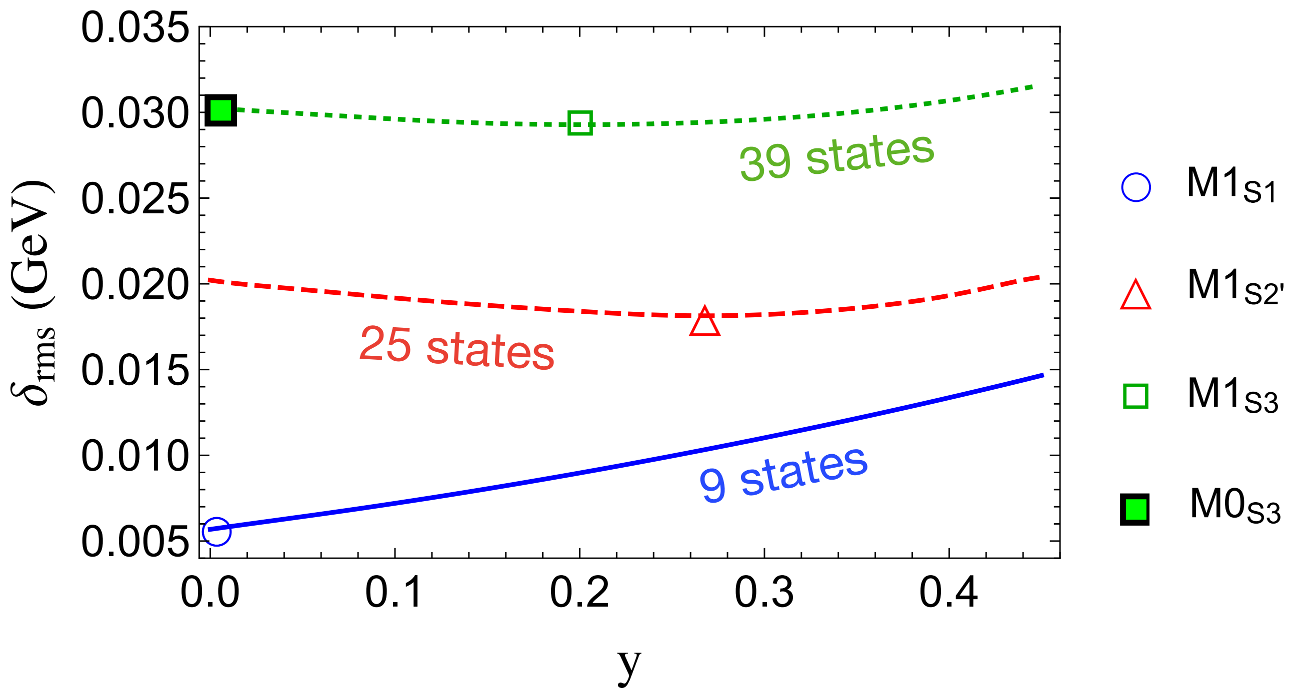
Model	Symbol	σ [GeV^2]	α_s	C [GeV]	y	m_b [GeV]	m_c [GeV]	m_s [GeV]	m_q [GeV]	N	$\delta_{\rm rms}$ [GeV]	$\Delta_{\rm rms}$ [GeV]
M0$_{S1}$		0.2493	0.3643	0.3491	**0.0000**	**4.892**	**1.600**	**0.4478**	**0.3455**	9	0.017	0.037
M1$_{S1}$	○	0.2235	0.3941	0.0591	0.0000	4.768	1.398	0.2547	0.1230	9	0.006	0.041
M0$_{S2}$		0.2247	0.3614	0.3377	**0.0000**	**4.892**	**1.600**	**0.4478**	**0.3455**	25	0.028	0.036
M1$_{S2}$		0.1893	0.4126	0.1085	0.2537	4.825	1.470	0.2349	0.1000	25	0.022	0.033
M1$_{S2'}$	△	0.2017	0.4013	0.1311	0.2677	4.822	1.464	0.2365	0.1000	24	0.018	0.033
M1$_{S3}$	□	0.2022	0.4129	0.2145	0.2002	4.875	1.553	0.3679	0.2493	39	0.030	0.030
M0$_{S3}$	■	0.2058	0.4172	0.2821	**0.0000**	4.917	1.624	0.4616	0.3514	39	0.031	0.031

include AV states in fit (M1$_{S3}$, M0$_{S3}$)

Parameters in **bold** were not varied during the fit

y held fixed, other parameters refitted

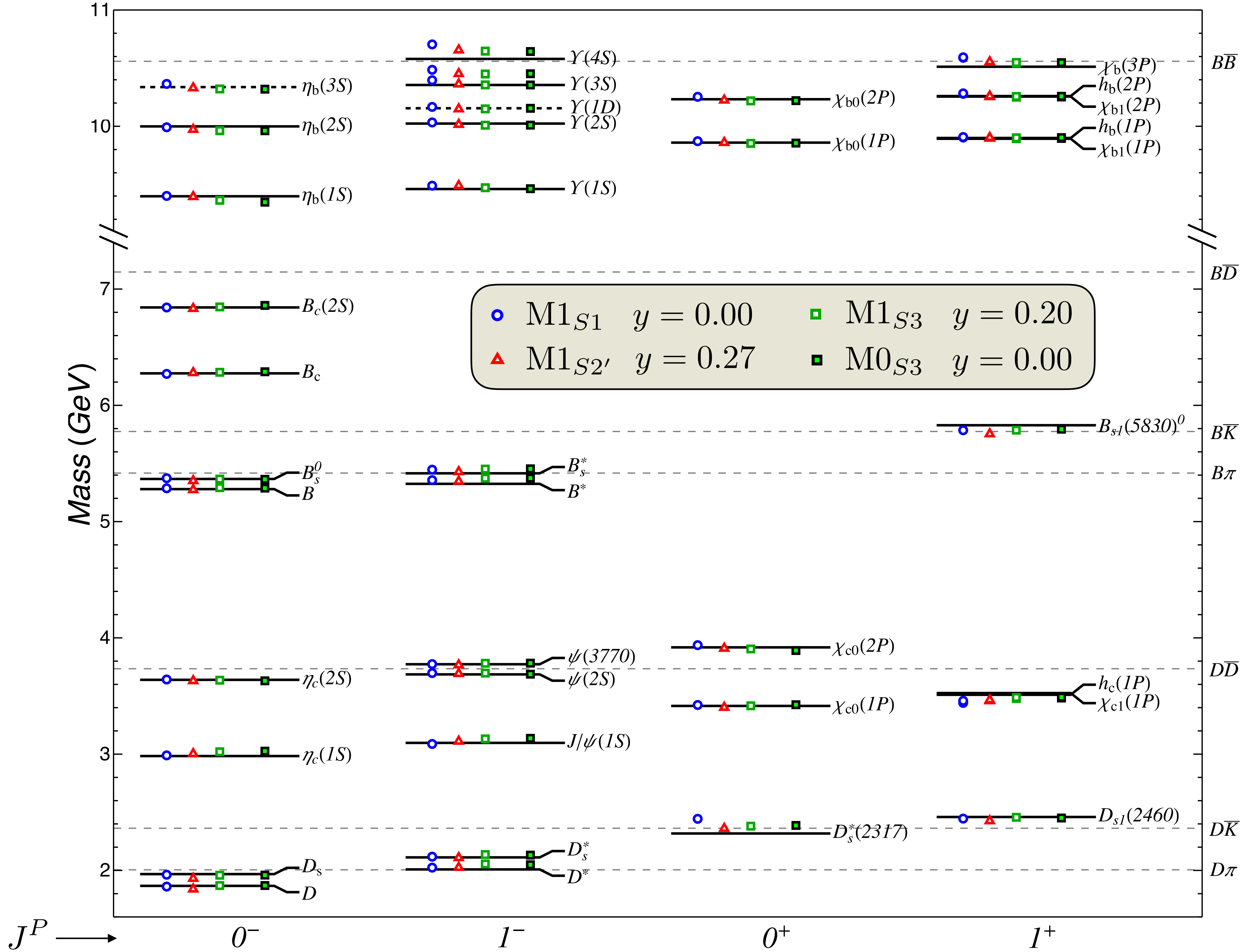
rms difference to fitted masses



- Quality of fits not much improved
- Best model M1$_{S3}$ has y=0.20, but minimum is very shallow

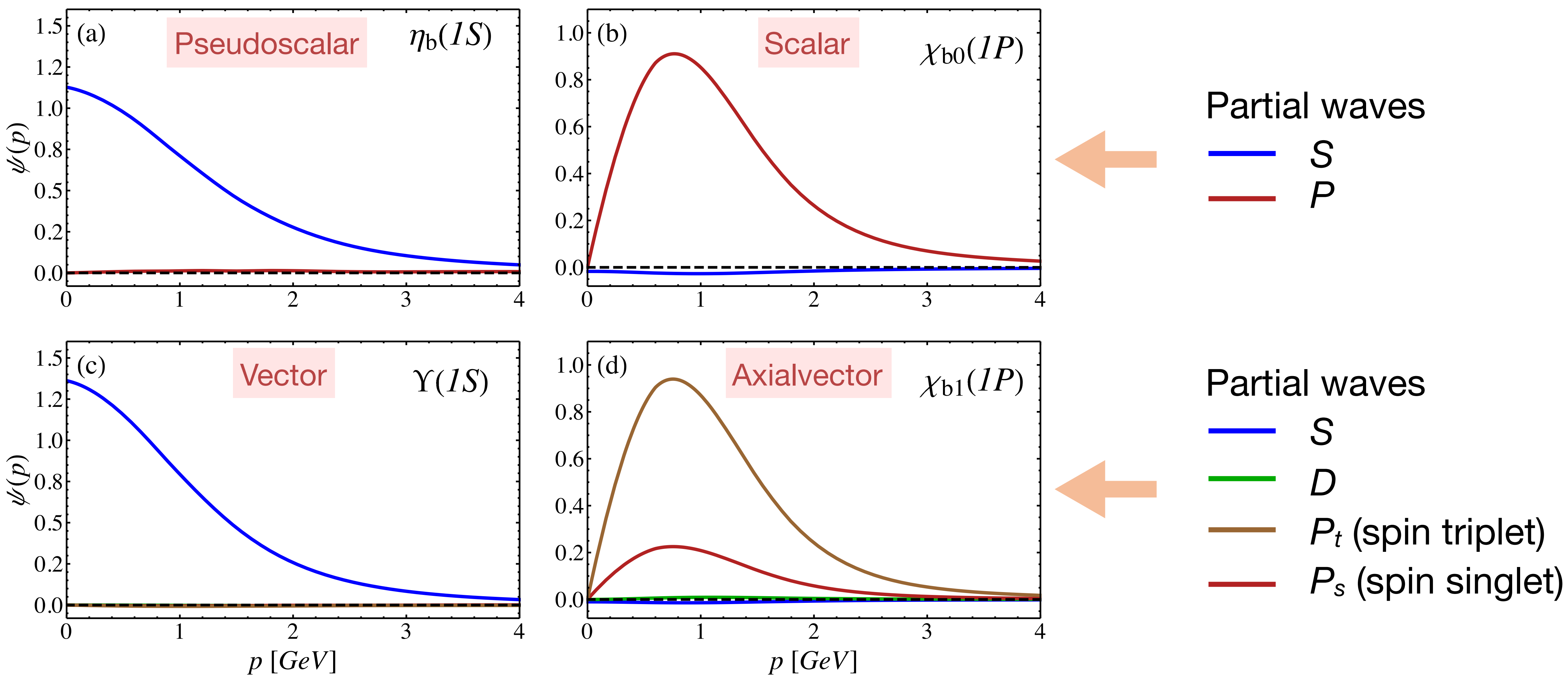
y and quark masses are not much constrained by the mass spectrum.

Mass spectra of heavy and heavy-light mesons



Bottomonium ground-state wave functions

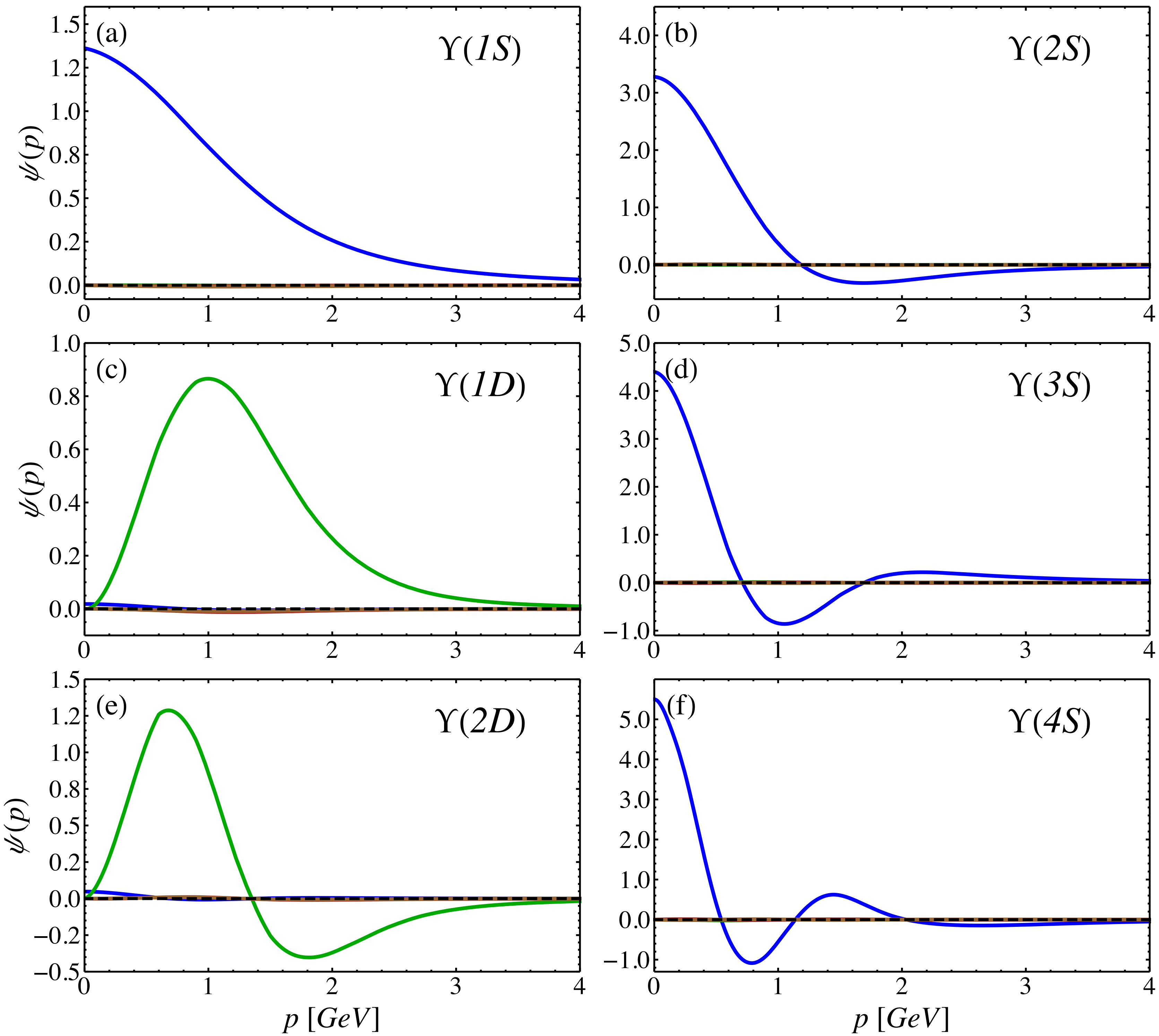
Calculated with model $M1_{S3}$



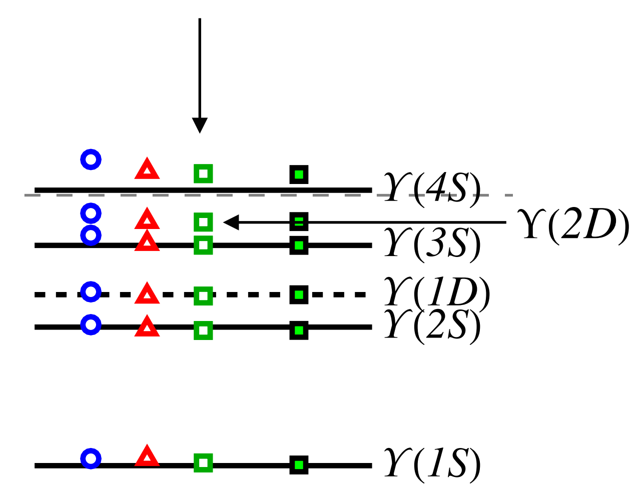
Relativistic wave function components are very small

Radial excitations in vector bottomonium

Wave functions of excited states look reasonable



Model M1$_{S3}$

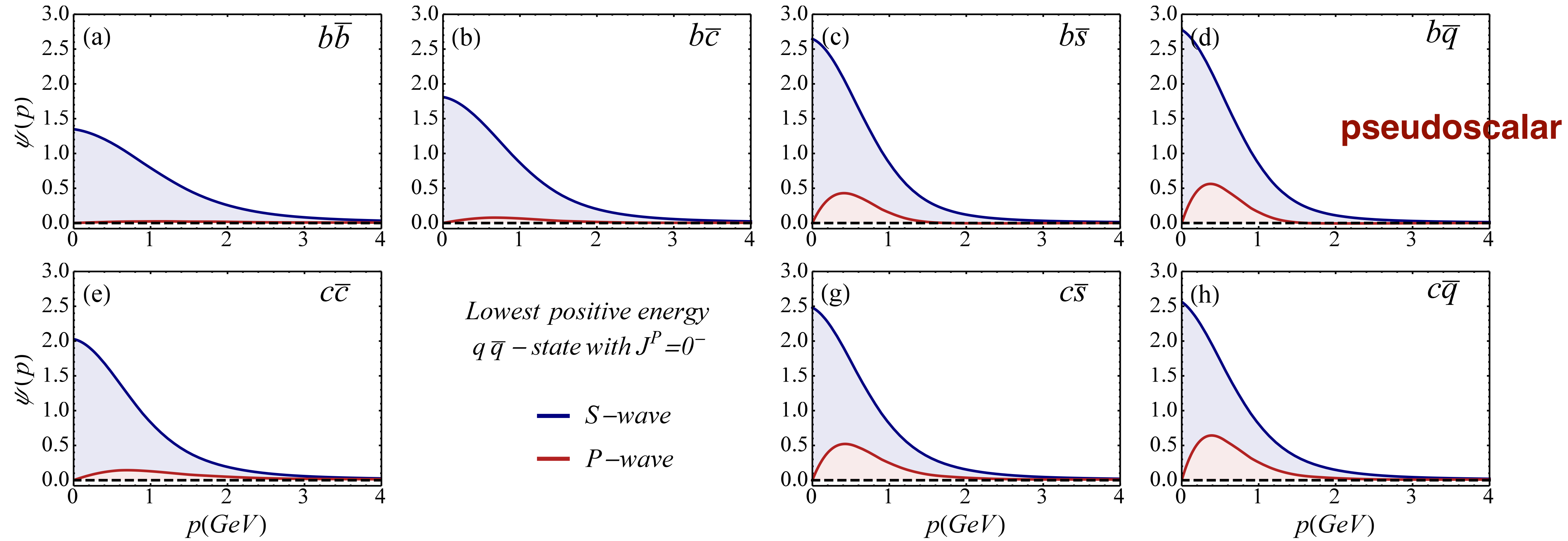


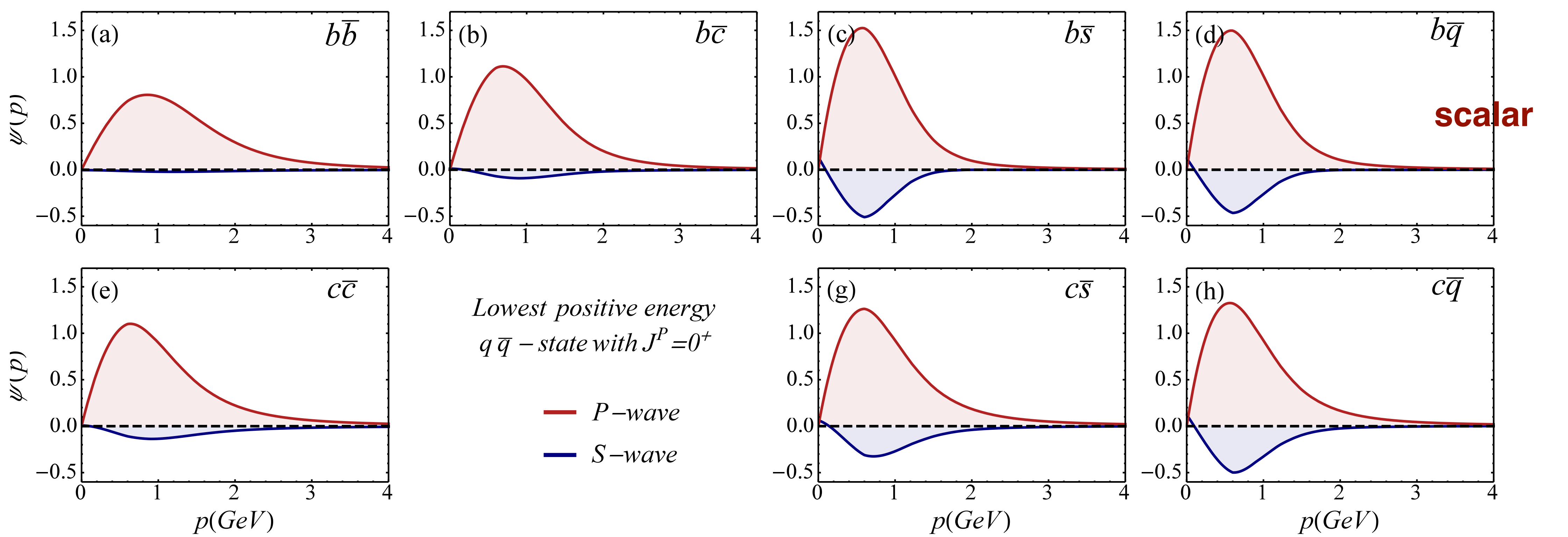
Partial waves

- S
- D
- P_t (spin triplet)
- P_s (spin singlet)

Importance of relativistic components

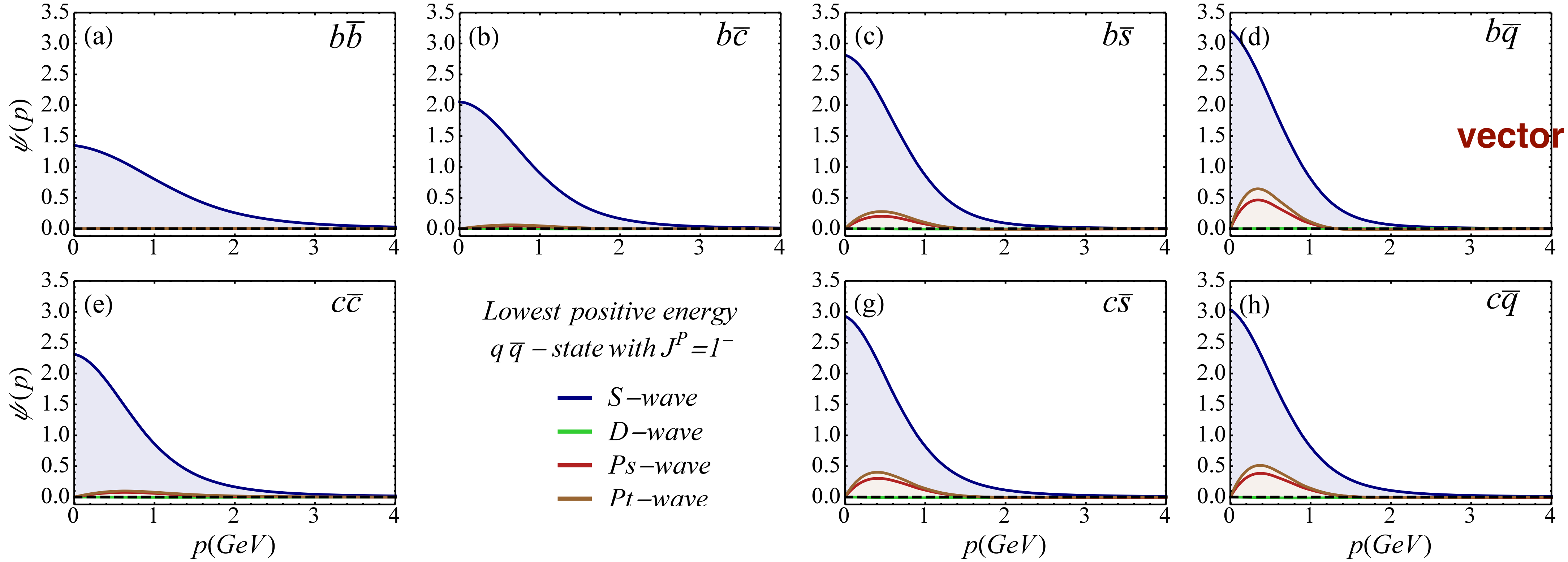
Ground-state wave functions of model M1$_{S3}$.

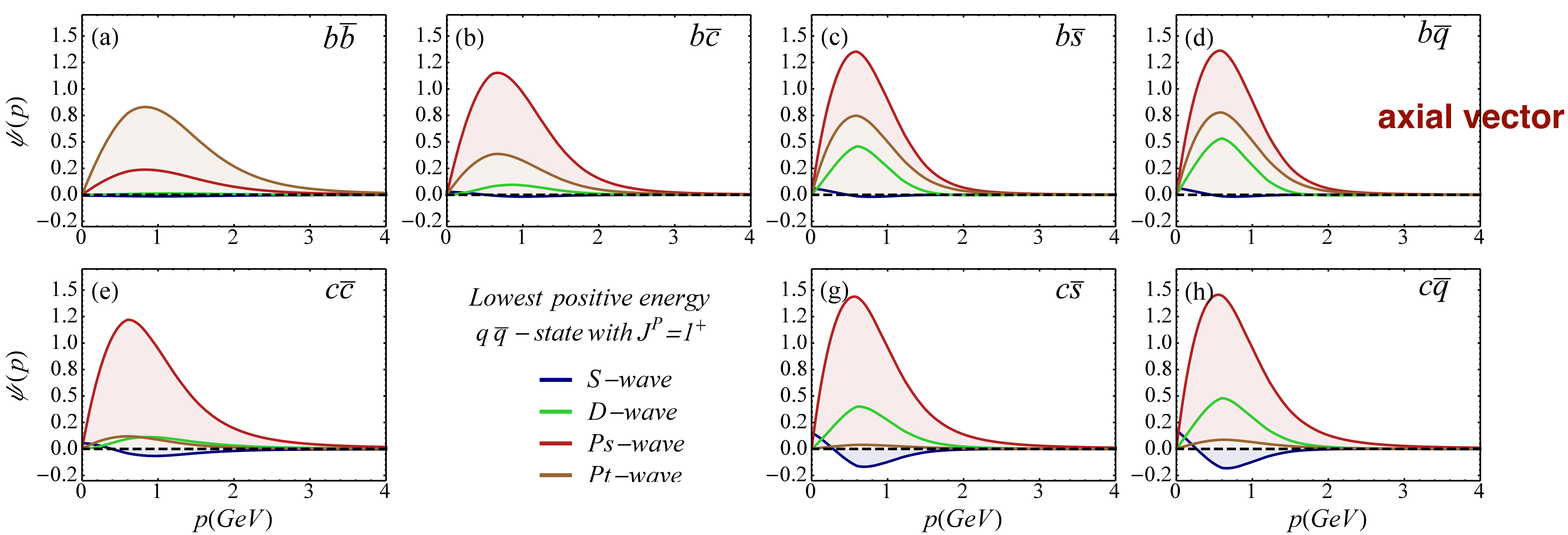




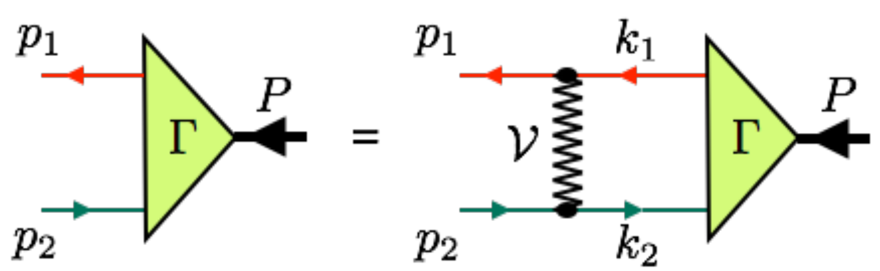
Importance of relativistic components

Ground-state wave functions of model $M1_{S3}$.





Regularization of the kernel



Loop integration needs to be regularized

- First models used Pauli-Villars regularization, with $\Lambda = 2m_1$ (for simplicity) (equivalent to rational form factor with $n_A = 1$ and $n_G = 1$)
- Generalize to other form factors (higher powers or exponential form) This is needed for convergence in calculations of decays constants

	Linear	OGE
Rational:	$V_{A\Lambda}(p,k) = -\frac{8\pi\sigma}{q^4}\left(\frac{\Lambda^4}{q^4+\Lambda^4}\right)^{n_A}$	$V_{G\Lambda}(p,k) = \frac{4\pi\alpha_s}{m_g^2 - q^2}\left(\frac{\Lambda^2 - m_g^2}{\Lambda^2 - q^2}\right)^{n_G}$
Exponential:	$V_{A\Lambda}(p,k) = -\frac{8\pi\sigma}{q^4}e^{-q^4/\Lambda^4}$	$V_{G\Lambda}(p,k) = \frac{4\pi\alpha_s}{m_g^2 - q^2}e^{-q^4/\Lambda^4}$

(we used mostly $m_g = 0$)

- We considered models with $(n_A, n_G) = (1,3)$ or $(2,5)$, or with exponential form
- We use global model parameters. But how to scale Λ? Is $\Lambda \propto m_1$ really the best choice?

Scaling of the cutoff Λ

Test different scaling rules for Λ with the quark masses in bottomonium and charmonium:

$\Lambda = \Lambda_s m_1$ $\Lambda = \Lambda_s m_1^{1/2}$ $\Lambda = \Lambda_s m_1^{1/3}$ $\Lambda = \Lambda_s m_1^{0}$ (independent of m_1)

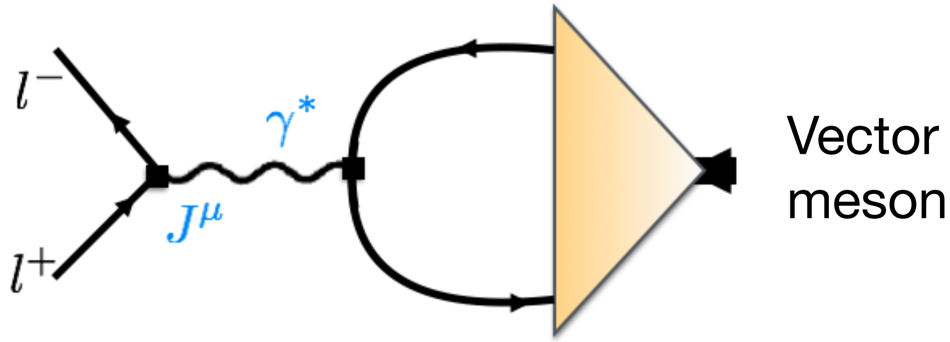
δ_{rms} (GeV) of fits to 22 experimental $b\bar{b}$ and $c\bar{c}$ states

Form factor	$\Lambda = \Lambda_s m_1$	$\Lambda = \Lambda_s m_1^{1/2}$	$\Lambda = \Lambda_s m_1^{1/3}$	$\Lambda = \Lambda_s m_1^{0}$
$n_A = 1, n_G = 3$	0.0287	0.0184	0.0148	0.0252
$n_A = 2, n_G = 5$	0.0347	0.0166	0.0148	0.0221
Exponential	0.0216	0.0146	0.0126	0.0197

For comparison: M1$_{\text{S3}}$ had $\delta_{\text{rms}} = 0.0247$ GeV

- Clearly, a linear scaling of Λ with m_1 is not the best choice to reproduce the spectrum
- A scaling $\Lambda \propto m_1^{1/3}$ works best
- This improved scaling leads to excellent combined fits to heavy quarkonia (essentially as good as fits to $b\bar{b}$ and $c\bar{c}$ separately)

Heavy quarkonium decay constants



Very precise measurements for some charmonium and bottomonium V and PS states (no data for S and AV)

Nonrelativistic: depend on $\Psi(r=0)$ (→ only S-waves contribute)

Relativistic: all partial waves can contribute

Pseudoscalar mesons

$$f_P = \frac{1}{\pi}\sqrt{\frac{N_c}{2\mu_P}}\int_0^\infty dk\, k^2 \sqrt{\left(1+\frac{m_1}{E_{1k}}\right)\left(1+\frac{m_2}{E_{2k}}\right)}\left[(1-\tilde{k}_1\tilde{k}_2)\psi_s(k) + (\tilde{k}_1+\tilde{k}_2)\psi_p(k)\right]$$

Vector mesons

$$f_V = \frac{1}{\pi}\sqrt{\frac{N_c}{2\mu_V}}\int_0^\infty dk\, k^2 \sqrt{\left(1+\frac{m_1}{E_{1k}}\right)\left(1+\frac{m_2}{E_{2k}}\right)}\left[(1+\frac{1}{3}\tilde{k}_1\tilde{k}_2)\psi_s(k) - \frac{2\sqrt{2}}{3}\tilde{k}_1\tilde{k}_2\psi_d(k) + \frac{1}{\sqrt{3}}(\tilde{k}_1+\tilde{k}_2)\psi_{p_s}(k) + \sqrt{\frac{2}{3}}(\tilde{k}_2-\tilde{k}_1)\psi_{p_t}(k)\right]$$

$$\tilde{k}_i \equiv \frac{|\mathbf{k}_i|}{E_{ik}+m_i}$$

Quarkonium decay constants (preliminary results)

Model $\mathrm{M}_{Q\bar{Q}}\Lambda_{\mathrm{OGE}}$: $n_A = 1, n_G = 3, \Lambda = 1.5 m_1$ $\delta_{\mathrm{rms}} = 0.048$ GeV

Very difficult to get a good fit! (especially in $c\bar{c}$)

CST fits indicate trade-off between descriptions of mass spectrum and decay constants.

CST

Quark content	n	Meson	$J^{P(C)}$	PDG	Lattice	DSE I	DSE II	BLFQ	$\mathrm{M}_{Q\bar{Q}}\Lambda_{\mathrm{OGE}}$ (this work)
	1	$\eta_b(1S)$	0^{-+}	–	667^{+6}_{-6}	773	756	589	795
	2	$\eta_b(2S)$	0^{-+}	–	–	419(8)	285	427	596
	3	$\eta_b(3S)$	0^{-+}	–	–	534(57)	333	331	536
	4	$\eta_b(4S)$	0^{-+}	–	–	–	40(15)	–	503
	1	$\Upsilon(1S)$	1^{--}	689^{+5}_{-5}	649^{+31}_{-31}	768	707	689	703
$b\bar{b}$	2	$\Upsilon(2S)$	1^{--}	479^{+4}_{-4}	481^{+39}_{-39}	467(17)	393	484	573
	3	1^3D_1	1^{--}	–	–	41(7)	371(2)	4.2	26
	4	$\Upsilon(3S)$	1^{--}	414^{+4}_{-4}	–	–	9(5)	366	536
	5	2^3D_1	1^{--}	–	–	–	165(50)	–	38
	6	$\Upsilon(4S)$	1^{--}	328^{+17}_{-18}	–	–	20(15)	–	518
	1	$\eta_c(1S)$	0^{-+}	330^{+13}_{-13}	393^{+9}_{-9}	401	378	368	547
	2	$\eta_c(2S)$	0^{-+}	211^{+35}_{-42}	–	244(12)	82	280	461
	3	$\eta_c(3S)$	0^{-+}	–	–	145(145)	206	–	417
$c\bar{c}$	4	$\eta_c(4S)$	0^{-+}	–	–	–	87	–	387
	1	J/ψ	1^{--}	407^{+5}_{-5}	405^{+6}_{-6}	450	411	404	525
	2	$\psi(2S)$	1^{--}	290^{+2}_{-2}	–	30(3)	155	290	531
	3	$\psi(3770)$	1^{--}	97.7^{+3}_{-3}	–	118(91)	45	0.9	98

Lattice: HPQCD Collaboration. Phys. Rev. D **86**, 074503 (2012); Phys. Rev. D **82**, 114504 (2010); Phys. Rev. D **86**, 094501 (2012); Phys. Rev. D **91**, 074514 (2015).

DSE: A. Krassnigg, M. Gomez-Rocha, and T. Hilger, Journal of Physics: Conference Series **742**, 012032 (2016).

BLFQ: Y. Li, P. Maris, and J. P. Vary, Phys. Rev. D **96**, 016022 (2017).

Spectrum of mesons with b-quarks

Quark content	Meson	$J^{P(C)}$	Exp. Mass	$\mathrm{M1_{S3}}$	$\mathrm{M_{S3}\Lambda_{OGE}}$	$\mathrm{M_{Q\bar{Q}}\Lambda_{OGE}}$
	$\eta_b(1S)$	0^{-+}	9399.0 ± 2.3	9363	9402	9474
	$\eta_b(2S)$	0^{-+}	9999 ± 4	9963	9991	9992
	$\eta_b(3S)$	0^{-+}	10337	10321	10356	10346
	$\Upsilon(1S)$	1^{--}	9460.30 ± 0.26	9472	9460	9505
	$\Upsilon(2S)$	1^{--}	10023.26 ± 0.31	10009	10015	10008
	$\Upsilon(1D)$?	1^{--}	10155	10150	10134	10096
	$\Upsilon(3S)$	1^{--}	10355.2 ± 0.5	10356	10373	10357
$b\bar{b}$	$\Upsilon(4S)$	1^{--}	10579.4 ± 1.2	10647	10668	10650
	$\chi_{b0}(1P)$	0^{++}	$9859.44 \pm 0.42 \pm 0.31$	9853	9838	9834
	$\chi_{b0}(2P)$	0^{++}	$10232.5 \pm 0.4 \pm 0.5$	10219	10227	10211
	$\chi_{b1}(1P)$	1^{++}	$9892.78 \pm 0.26 \pm 0.31$	9894	9871	9850
	$h_b(1P)$	1^{+-}	9899.3 ± 0.8	9901	9876	9852
	$\chi_{b1}(2P)$	1^{++}	$10255.46 \pm 0.22 \pm 0.50$	10250	10247	10221
	$h_b(2P)$	1^{+-}	10259.8 ± 1.2	10256	10251	10222
	$\chi_{b1}(3P)$	1^{++}	10512.1 ± 2.3	10543	10553	10527
$b\bar{c}$	B_c^+	0^-	6275.1 ± 1.0	6284	6264	–
	$B_c(2S)^\pm$	0^-	6842 ± 6	6846	6845	–
	B_s^0	0^-	5366.82 ± 0.22	5366	5344	–
$b\bar{s}$	B_s^*	1^-	5415.8 ± 1.5	5452	5423	–
	$B_{s1}(5830)$	1^+	5828.63 ± 0.27	5787	5767	–
	$B^{\pm,0}$	0^-	5279.45	5292	5274	–
$b\bar{q}$	B^*	1^-	5324.65 ± 0.25	5373	5352	–
	$B_1(5721)^{+,0}$	1^+	5725.85 ± 1.3	5703	5690	–

Model $\mathrm{M}_{Q\bar{Q}}\Lambda_{\mathrm{OGE}}$: $\delta_{\mathrm{rms}} = 0.048$ GeV

Spectrum of mesons with c-quarks

Quark content	Meson	$J^{P(C)}$	Exp. Mass	$\mathrm{M1_{S3}}$	$\mathrm{M_{S3}\Lambda_{OGE}}$	$\mathrm{M_{Q\bar{Q}}\Lambda_{OGE}}$
	$\eta_c(1S)$	0^{-+}	2983.4 ± 0.5	3021	3058	3079
	$\eta_c(2S)$	0^{-+}	3639.2 ± 1.2	3636	3670	3647
	$J/\Psi(1S)$	1^{--}	3096.900 ± 0.006	3132	3125	3131
	$\psi(2S)$	1^{--}	3686.097 ± 0.010	3697	3703	3701
$c\bar{c}$	$\psi(3770)$	1^{--}	3773.13 ± 0.35	3782	3719	3709
	$\chi_{c0}(1P)$	0^{++}	3414.75 ± 0.31	3416	3431	3436
	$X(3915)$	0^{++}	3918.4 ± 1.9	3905	3936	3941
	$\chi_{c1}(1P)$	1^{++}	3510.66 ± 0.07	3478	3443	3426
	$h_c(1P)$	1^{+-}	3525.38 ± 0.11	3488	3459	3447
	$D_s^{\pm}$	0^-	1968.27 ± 0.10	1959	1993	–
	$D_s^{*\pm}$	1^-	2112.1 ± 0.4	2137	2117	–
$c\bar{s}$	$D_{s0}^*(2317)^{\pm}$	0^+	2317.7 ± 0.6	2381	2411	–
	$D_{s1}(2460)^{\pm}$	1^+	2459.5 ± 0.6	2457	2425	–
	$D_{s1}(2536)^{\pm}$	1^+	2535.10 ± 0.06	2469	2447	–
	$D^{\pm,0}$	0^-	1867.23	1870	1905	–
	$D^*(2007)^0$	1^-	2008.62	2055	2038	–
$c\bar{q}$	$D_0^*(2400)^0$	0^+	2318 ± 29	2294	2329	–
	$D_1(2420)^{\pm,0}$	1^+	2421.4	2368	2342	–

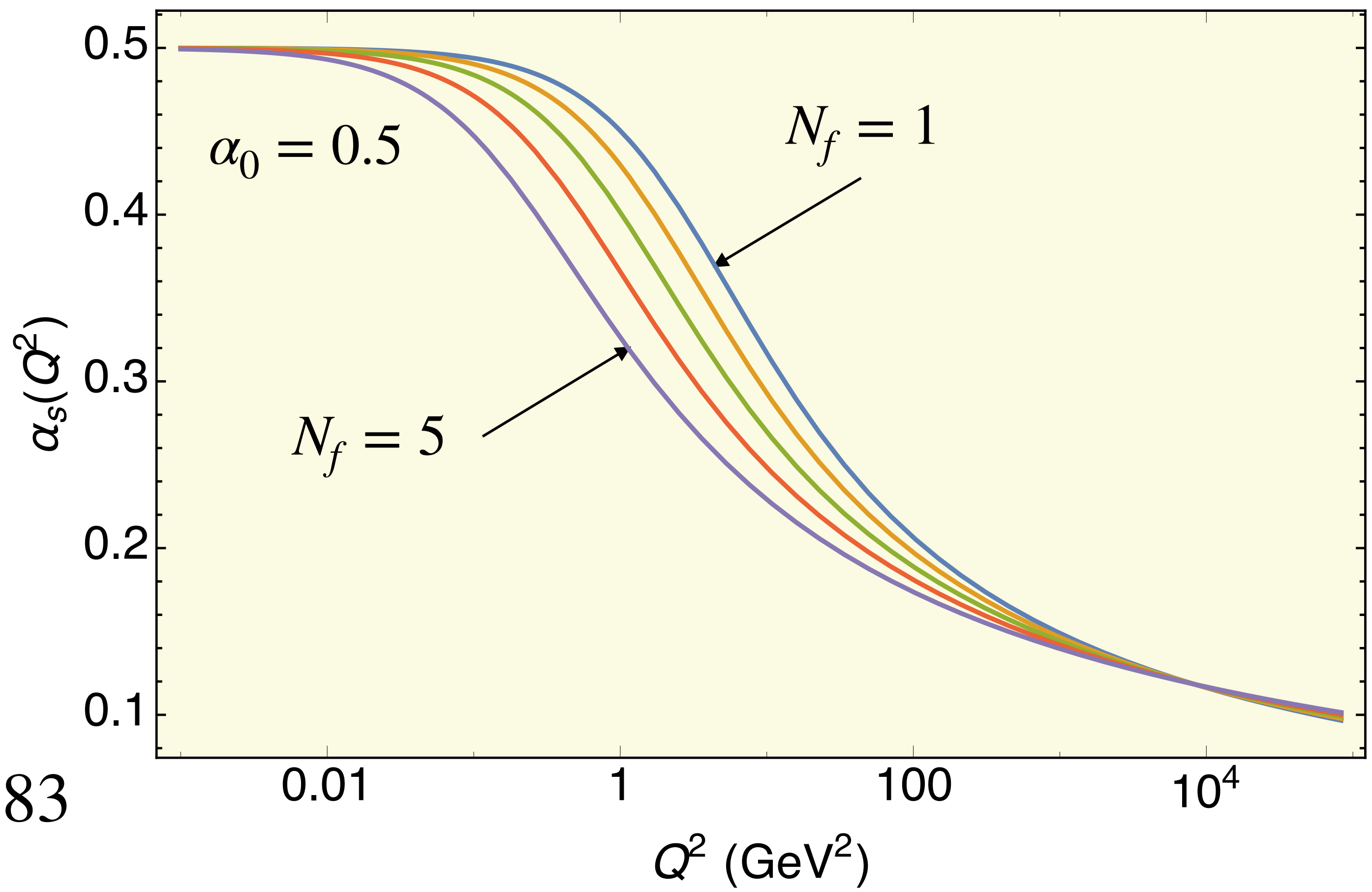
Model $\mathrm{M}_{Q\bar{Q}}\Lambda_{\mathrm{OGE}}$: $\delta_{\mathrm{rms}} = 0.048$ GeV

Running coupling $\alpha_s(Q^2)$ versus fixed coupling α_s

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \ln\left(\frac{Q^2}{\Lambda^2_{\mathrm{QCD}}} + \tau\right)}$$

$$\beta_0 = \frac{33 - 2N_f}{12\pi} \qquad \tau = \exp\left(\frac{1}{\alpha_0 \beta_0}\right)$$

$$\alpha_s(0) = \alpha_0$$



Λ_{QCD} is determined through $\alpha_s(M_Z^2) = 0.1183$

$$V_{G\Lambda}(p,k) = \frac{4\pi\alpha_s(Q^2)}{m_g^2 - q^2}\left(\frac{\Lambda^2 - m_g^2}{\Lambda^2 - q^2}\right)^{n_G} \qquad V_{G\Lambda}(p,k) = \frac{4\pi\alpha_s(Q^2)}{m_g^2 - q^2} e^{-q^4/\Lambda^4}$$

- First results do not show an overall improvement over fixed α_s
- Heavy quarkonia decay constants are smaller, but δ_{rms} of mass spectrum increases

Summary

- With the simplest, one-channel CST equation and a few global parameters, we get a very nice description of the heavy and heavy-light meson spectrum
- (S+PS) confining kernel with ~ 0%—30% admixture of Lorentz-vector coupling is compatible with the data
- A more careful scaling of form factor cutoffs with quark masses can significantly improve the description of the mass spectra
- Decay constants are very sensitive to details — place stronger constraints on the kernel. Further improvements are needed!
- We have generalized our OGE kernel for a running coupling $\alpha_s(Q^2)$.
  So far no positive impact on the results, but many more tests are still to be done.

Outlook

Next steps:

- Further exploration of cutoff scaling
- Comprehensive study of the effect of a running quark-gluon coupling
- Dynamical quark mass (mass function) from quark self-interaction
- Tensor mesons (spin ≥ 2)
- Extension to the light-quark sector (4-channel CSE)
- Parton distribution functions
- Relativistic quark-antiquark states with exotic J^{PC}