

Removing the Wigner Bound in non-perturbative effective field theory

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Effective field theory (EFT) - A brief reminder

- Usually in physics we have an underlying theory with a typical scale M_{high} but we want to study it in a lower scale $M_{low} \ll M_{high}$.
- EFTs provides a framework to construct the interactions **systematically**.
- In EFT, the high energy degrees of freedom are integrated out and the details of the interactions are encoded in the coupling constants.
- In **pionless EFT** ($\not\propto$ EFT) the degrees of freedom are baryons where the pions are integrated out. The interactions become contact.

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Structure-less bosonic fields

In the naïve power counting

- Non-relativistic fields count as $(M_{low}/M_{high})^{3/2}$.
- Derivatives count as (M_{low}/M_{high}) .

The Lagrangian is expanded in (M_{low}/M_{high})

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \dots$$

For instance, the **LO** Lagrangian is

$$\mathcal{L}_0 = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi - \frac{C_0}{4} (\psi^\dagger \psi)^2,$$

and the **NLO** Lagrangian is

$$\mathcal{L}_1 = -\frac{C_2}{4} [(\psi^\dagger \psi) (\psi^\dagger \nabla^2 \psi) + h.c.].$$

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Naïve power counting in 3,4-body boson systems

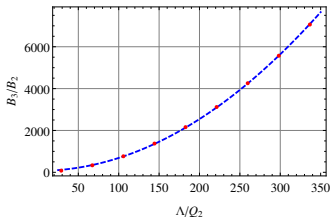
Thomas collapse:

- When trying to calculate the 3-body binding energy we get, already in LO:

$$B_3 \propto \frac{\hbar\Lambda^2}{m}$$

In order to fix it, the 3-body force $D_0 (\psi^\dagger \psi)^3$ is **promoted from higher orders to the LO**.

- Again, the 4-body binding energy diverges in NLO and the 4-body force is **promoted to NLO** to fix it¹.



Unfortunately, the naïve power counting is not sufficient.

¹B. Bazak et al. Phys. Rev. Lett. 122, 143001 (2019)

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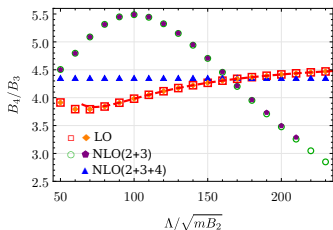
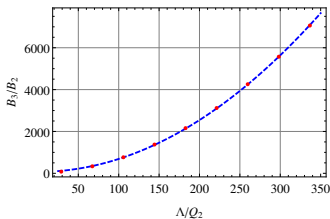
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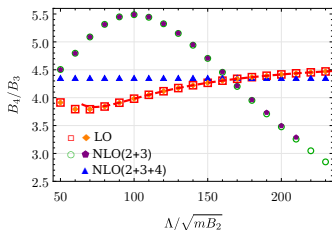
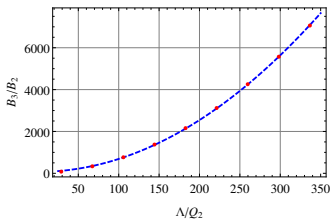
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Wigner bound

Phillips and Cohen showed that²:

$$r_{\text{eff}} \leq 2R \left(1 - \frac{R}{a_s} + \frac{R^2}{3a_s^2} \right)$$

In EFT, $R_{\text{EFT}} \sim \Lambda^{-1}$. Thus $r_{\text{eff}} \leq \frac{W}{\Lambda}$ as $\Lambda \rightarrow \infty$.

We can "reverse" this inequality to get

$$\Lambda \leq \Lambda_{\text{max}} \equiv \frac{W}{r_{\text{eff}}}$$

Does Λ_{max} increase as more EFT orders are taken into account? i.e. can we restore RG invariance order by order?

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Lippmann-Schwinger equation

In order to find r_{eff} we need to solve the Lippmann-Schwinger (LS) equation:

$$T = V + VGT,$$

where

$$V_{\text{N}^{\text{LO}}} = C_0 + C_2 (p^2 + p'^2) + C_4 (p^4 + p'^4) + C_{22} p^2 p'^2 + \dots$$

- The LS equation is analytically solvable for separable potential.
- We regularize the theory with a cutoff regulator $F(p^2/\Lambda^2)$.

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Solving Lippmann-Schwinger equation

The potential can be written as

$$V = F(p^2/\Lambda^2) \left(\sum_{i,j=0}^n p^{2i} \lambda_{i,j} p'^{2j} \right) F(p'^2/\Lambda^2)$$

where

$$\lambda_{\text{NLO}} = \begin{pmatrix} C_0 & C_2 \\ C_2 & 0 \end{pmatrix} \quad \lambda_{\text{N}^2\text{LO}} = \begin{pmatrix} C_0 & C_2 & C_4 \\ C_2 & C_{2,2} & 0 \\ C_4 & 0 & 0 \end{pmatrix} \quad \dots$$

The T-matrix assumes the form

$$T = F(p^2/\Lambda^2) \sum_{i,j=0}^n p^{2i} \tau_{ij}(E) p'^{2j} F(p'^2/\Lambda^2).$$

The LS equation is reduced to the matrix equation

$$\tau = \lambda + \lambda \mathcal{I} \tau, \quad \mathcal{I}_{ij} \equiv \int \frac{d^3q}{(2\pi)^3} \frac{F^2(q^2/\Lambda^2) q^{2(i+j)}}{E + i\epsilon - \frac{q^2}{m}}.$$

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Solving Lippmann-Schwinger equation

The solution is easily found to be

$$\tau = \frac{1}{1-\lambda\mathcal{I}}\lambda$$

The matrix elements of \mathcal{I} depend only on the sum of the indices $\mathcal{I}_{i,j} = \mathcal{I}_{2(i+j)}$ and admit the recursive relations

$$\mathcal{I}_{2k} = mE\mathcal{I}_{2(k-1)} + I_{2k+1}$$

where

$$I_{2k+1} = -m \int \frac{d^3q}{(2\pi)^3} F^2(q^2/\Lambda^2) q^{2k-2} \quad \mathcal{I}_0(E) = \int \frac{d^3q}{(2\pi)^3} \frac{F^2(q)}{E+i\epsilon-\frac{q^2}{2\mu}}.$$

For example, using Gaussian regulator in NLO:

$$\frac{1}{\tau} = e^{\frac{2mE}{\Lambda^2}} \left(\frac{(C_2 I_3 - 1)^2}{C_0 + C_2^2 I_5 + \frac{mE}{I_3} (1 - (C_2 I_3 - 1)^2)} - \mathcal{I}_0(E) \right).$$

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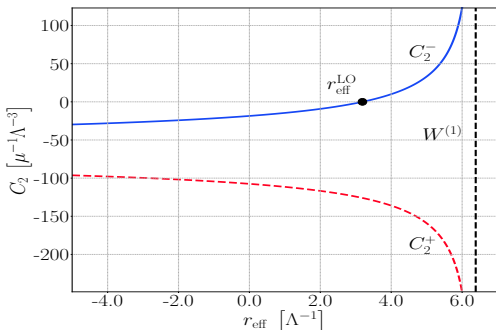
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$$\frac{1}{T} \approx -\frac{m}{4\pi} \left(-\frac{1}{a_s} + \frac{1}{2} r_{\text{eff}} p^2 + \dots - ip \right).$$

Match the LECs to the observables:

$$C_0, C_2 \longrightarrow a_s, r_{\text{eff}}$$

Two solutions for C_2 exist:



Black dot - r_{eff} obtained at LO for $a_s\Lambda = 10^3$.

$W^{(n)}$

The matching gives

$$r_{\text{eff}} = \frac{\tau'_{00}(0) + m(\tau_{01}(0) + \tau_{10}(0)) - \frac{8\pi a_s}{\Lambda^2}}{2\pi a_s^2}$$

To find $W^{(n)}$ we need to find its maximum using the parameter space $\{C_{pq}\}$ with the constraint $a_s = \frac{m}{4\pi}\tau_{00}(0)$.

$$\{C_0, C_2, C_4, C_{2,2}, \dots\} \longrightarrow \{a_s, C_2, C_4, C_{2,2}, \dots\}$$

Imposing the condition $C_{p,q}/l_{p+q+1} \propto \Lambda$

$W^{(n)}$ as a function of the EFT order n :

Order	1	2	3	4	5	6
Gaussian	$8\sqrt{\frac{2}{\pi}}$	$\frac{32\sqrt{\frac{2}{\pi}}}{3}$	$\frac{64\sqrt{\frac{2}{\pi}}}{5}$	$\frac{512\sqrt{\frac{2}{\pi}}}{35}$	$\frac{1024\sqrt{\frac{2}{\pi}}}{63}$	$\frac{4096\sqrt{\frac{2}{\pi}}}{231}$
Sharp	$\frac{16}{\pi}$	$\frac{256}{9\pi}$	$\frac{1024}{25\pi}$	$\frac{65536}{1225\pi}$	$\frac{262144}{3969\pi}$	$\frac{4194304}{53361\pi}$

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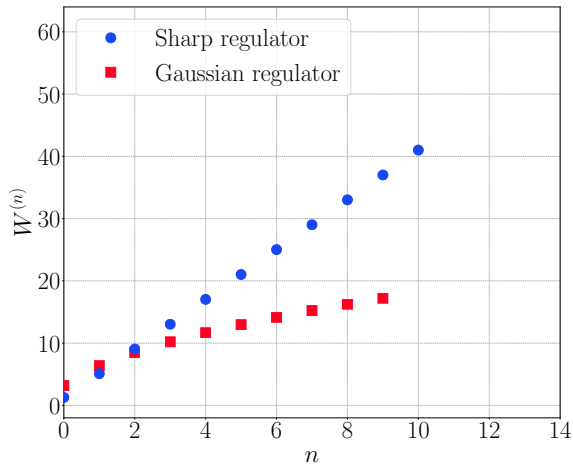
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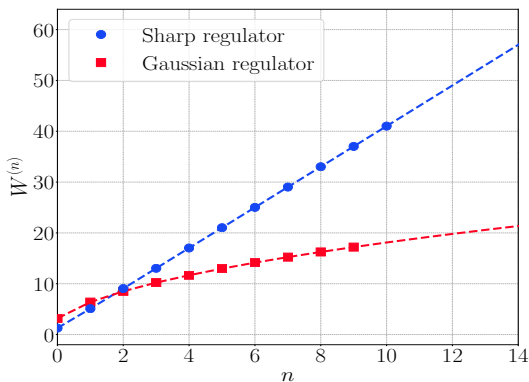
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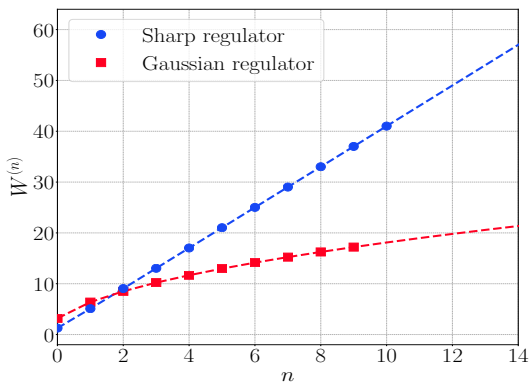
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$W^{(n)}$ 

Conjecture for $W^{(n)}$ 

$$W_{Gauss}^{(n)} = 4\sqrt{2} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \qquad W_{Sharp}^{(n)} = 4 \left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \right)^2$$

Conjecture for $W^{(n)}$ 

$$W_{Gauss}^{(n)} = 4\sqrt{2} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}$$

$$\lim_{n \rightarrow \infty} W_{Gauss}^{(n)} = 4\sqrt{2n}$$

$$W_{Sharp}^{(n)} = 4 \left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \right)^2$$

$$\lim_{n \rightarrow \infty} W_{Sharp}^{(n)} = 4n$$

Degeneracy of $C_{2,2}$

At NLO, two renormalization conditions were needed, i.e. a_s, r_{eff} .

Note, however, that the T-matrix provides **one** condition each order

$$\frac{1}{T} \approx -\frac{m}{4\pi} \left(-\frac{1}{a_s} + \frac{1}{2} r_{\text{eff}} p^2 + \sum_{n \geq 2} S_n p^2 - ip \right).$$

The potential, on the other hand, gives rise to more LECs. Already in **N²LO** we get **2 more** coefficients: $C_4, C_{2,2}$

$$V_{\text{N}^2\text{LO}} = C_0 + C_2 (p^2 + p'^2) + C_4 (p^4 + p'^4) + C_{22} p^2 p'^2 + \dots$$

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Volkov potential as an example

The Volkov potential has the form

$$V(r) = V_R e^{-\frac{r^2}{R_1^2}} + V_A e^{-\frac{r^2}{R_2^2}}$$

with the effective range parameters

$$a_s = 10.08 \text{ fm} \quad r_{\text{eff}} = 2.37 \text{ fm} \quad S_2 = 0.43 \text{ fm}^3.$$

The **maximum** r_{eff} at different orders and cutoffs is

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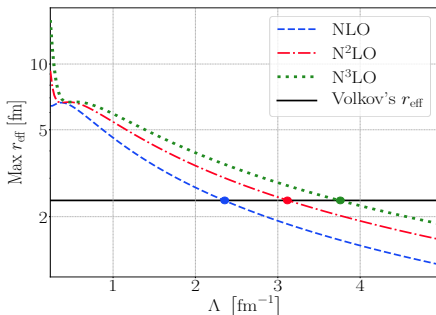
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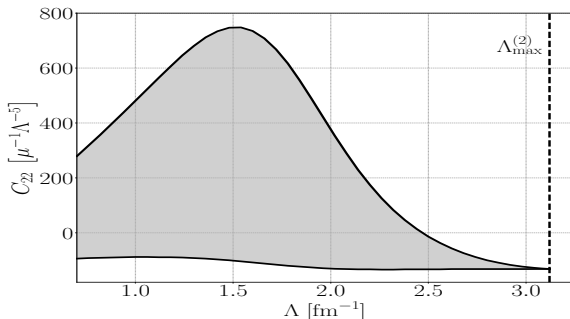


The freedom in C_{22}

Does fitting the LECs to the effective range expansion parameters constrain Λ_{\max} ?

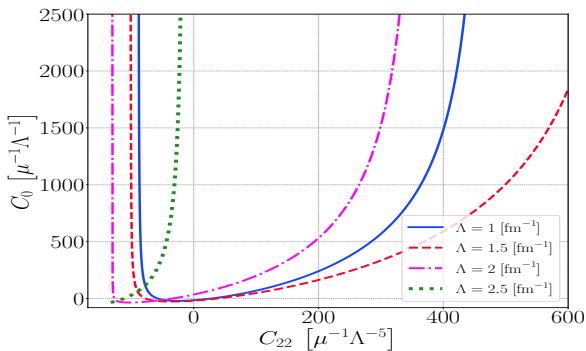
Match the LECs to the observables in N^2 LO:

C_0	C_2	C_4	C_{22}
↓	↓	↓	↓
a_s	r_{eff}	S_2	?



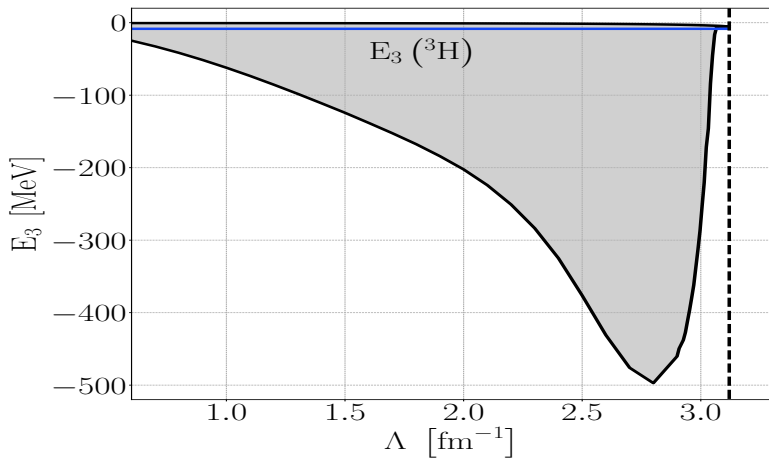
LECs dependence on the cutoff

The dependence of the LECs on the cutoff explodes near the extremal allowed values of C_{22} :

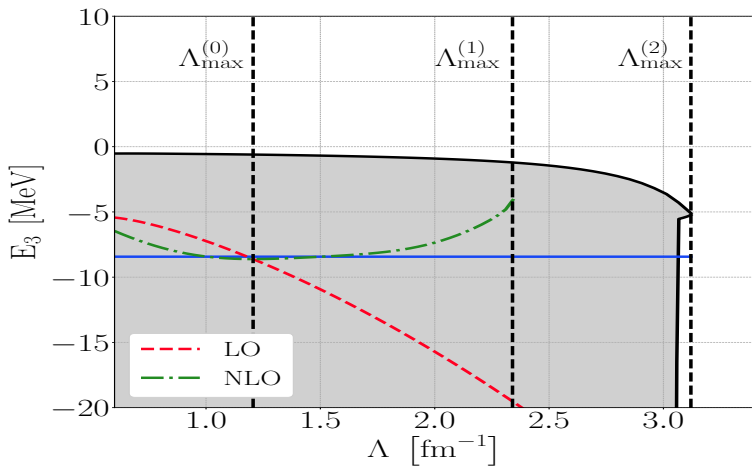


Unstable numerical results at high cutoff

3-body binding energy



3-body force



Summary

- The Wigner bound on the effective range **increases** with the non-perturbative EFT orders.
- The number of possible solutions **increases** with increasing orders. Only one of them is physical.
- Numerical calculations become unstable at Λ_{\max} .
- With the right choice of LECs, the promotion of the 3-body force may be suspended!

Saar Beck, Betzalel Bazak, and Nir Barnea, arXiv:1907.11886

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