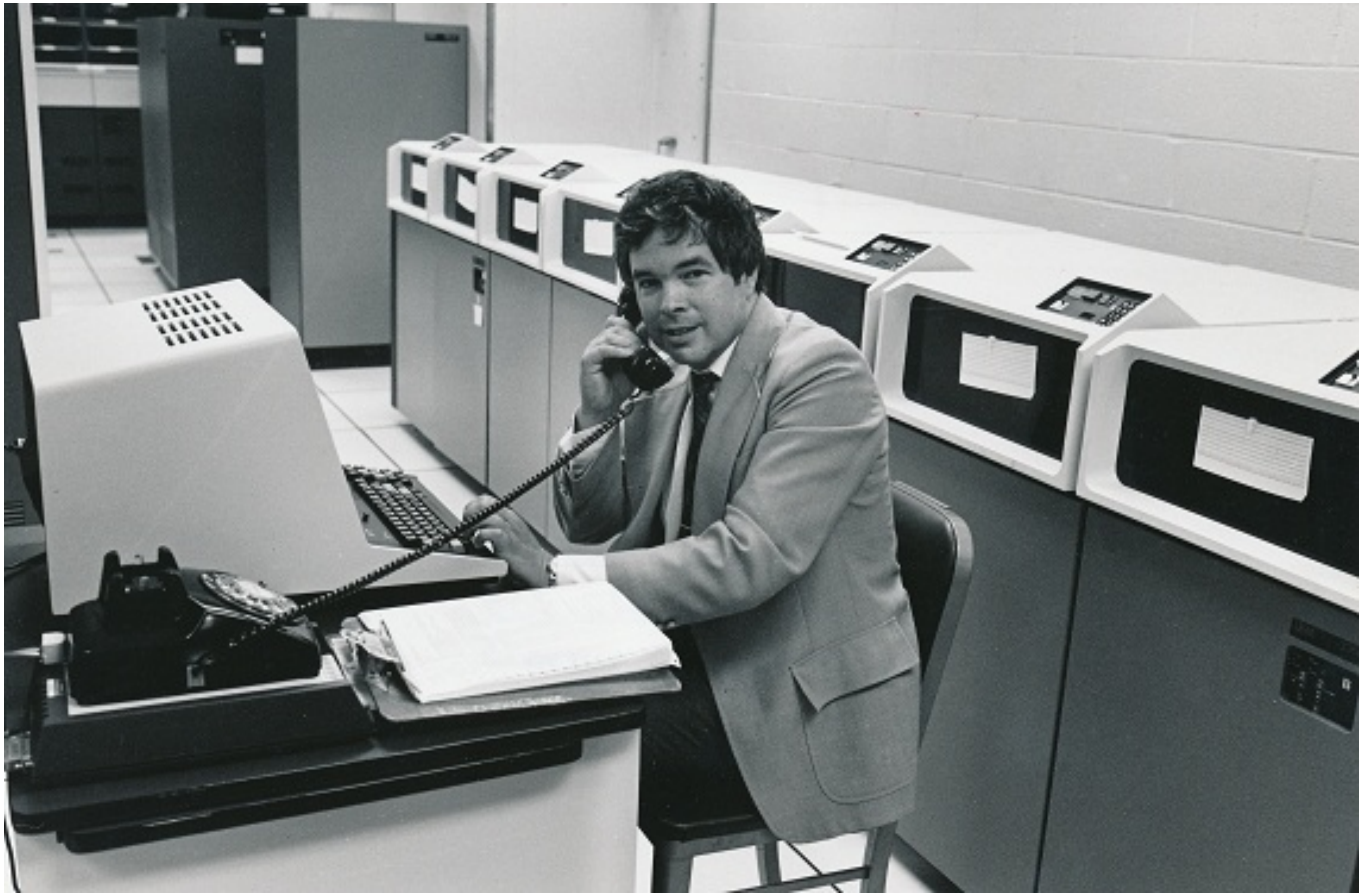
40 Years of Lattice QCD

G. Peter Lepage

Cornell University

February 2015

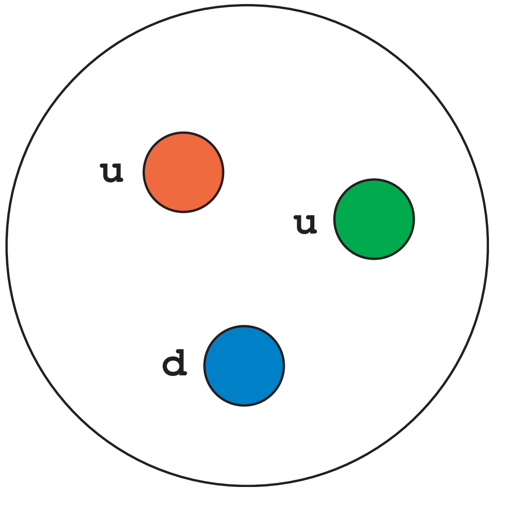


Ken Wilson: 1936–2013

A Short History of the Strong Interaction

Quark Model (1960s)

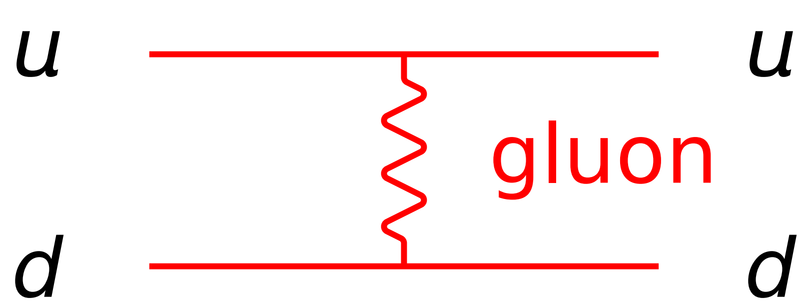
Proton =



No interactions; no dynamics!

Also no quarks visible.

Interactions — QCD (1970s)



Gauge theory (like QED) ⇒ complete theory!

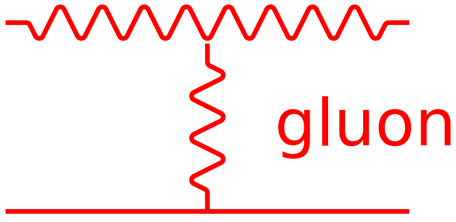
gluons $\to F_{\mu\nu}(x)$ (tensor of traceless 3×3 color matrices)

$\to \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$

quarks $\to \psi(x)$ (Dirac spinor of color 3-vectors)

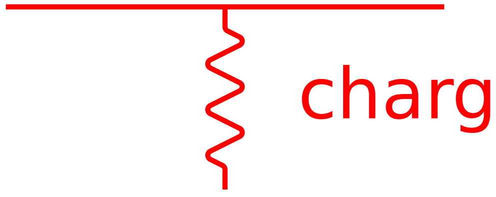
But unlike QED ...

Nonlinear:



gluon carries QCD charge

Strongly Interacting:



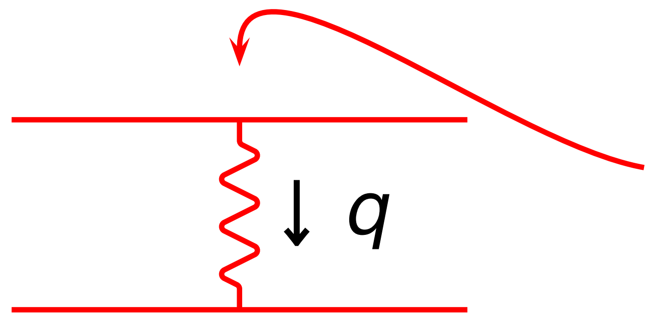
charge g big

$\Rightarrow$ Can’t solve QCD.

$\Rightarrow$ Adds nothing to understanding of proton structure?

$\Rightarrow$ Theory useless??

Asymptotic Freedom (1973)



$$g_{\mathrm{eff}} = g(q) \to 0 \quad \text{as} \quad q \to \infty.$$

$\Rightarrow$ Solve QCD for high-energy (short-distance) processes by expanding in powers of:

$$\alpha_s(q) \equiv \frac{g^2(q)}{4\pi}$$

$\Rightarrow$ Detailed experimental verification of QCD at high-energy accelerators (1980s–2000s).

But still no insight into proton, neutron, pion … structure since low-energy (<1 GeV) QCD is nonperturbative.

Lattice QCD

PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Confinement of quarks*

Kenneth G. Wilson

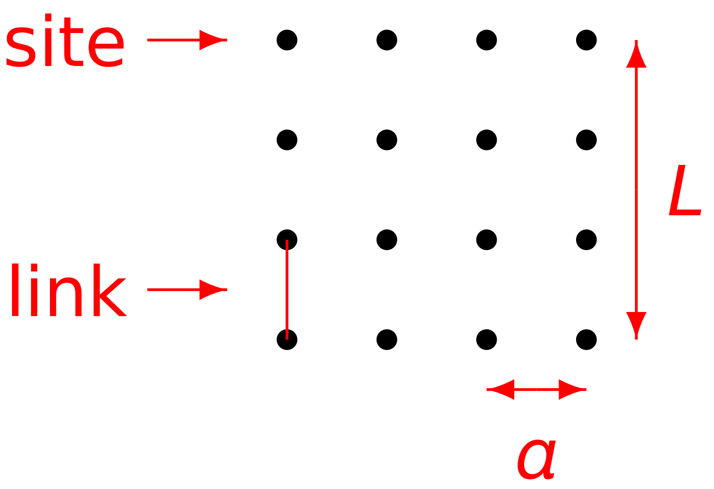
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850

(Received 12 June 1974)

A mechanism for total confinement of quarks, similar to that of Schwinger, is defined which requires the existence of Abelian or non-Abelian gauge fields. It is shown how to quantize a gauge field theory on a discrete lattice in Euclidean space-time, preserving exact gauge invariance and treating the gauge fields as angular variables (which makes a gauge-fixing term unnecessary). The lattice gauge theory has a computable strong-coupling limit; in this limit the binding mechanism applies and there are no free quarks. There is unfortunately no Lorentz (or Euclidean) invariance in the strong-coupling limit. The strong-coupling expansion involves sums over all quark paths and sums over all surfaces (on the lattice) joining quark paths. This structure is reminiscent of relativistic string models of hadrons.

Lattice Approximation

Continuous
Space & Time $\longrightarrow$



$\Rightarrow$ Fields $\psi(x)$, $A_\mu(x)$ specified only at grid sites (or links); interpolate for other points.

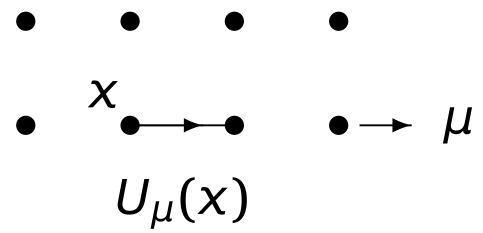
$\Rightarrow$ Solving QCD → multidimensional integration:

$$\boxed{\int \mathcal{D}A_\mu \ldots \mathrm{e}^{-\int L dt} \longrightarrow \int \prod_{x_j \epsilon \text{grid}} dA_\mu(x_j) \ldots \mathrm{e}^{-a \sum L_j}}$$

Wilson's Gluon Action

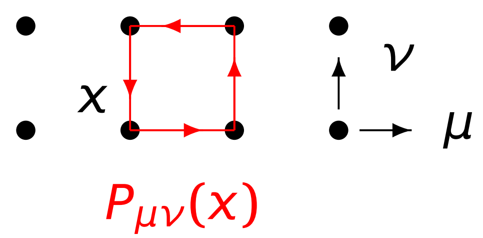
- Integration variables are link variables (SU_3 matrices):

$$U_\mu(x) = \mathrm{P}\exp\left(-i\int_x^{x+a\hat{\mu}} gA(y)\cdot dy\right)$$



- Action:

$$S_{\mathrm{gluon}} = \frac{6}{g^2(\pi/a)}\sum_{x,\mu>\nu}(1 - P_{\mu\nu}(x))$$



where

$$P_{\mu\nu} \equiv \frac{1}{3}\mathrm{ReTr}\left(U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\mu}+a\hat{\nu})U_\nu^\dagger(x)\right)$$

- Compare continuum S —

$$S^{\text{cont}}_{\text{gluon}} = \int d^4x \sum_{\mu>\nu} \mathrm{Tr} F^2_{\mu\nu}(x)$$

- Wilson's action is very complicated but gauge invariant without gauge fixing.

- QCD as a spin model.

- Expand in $1/g(\pi/a) \Rightarrow \boxed{\text{Confinement!!!}}$

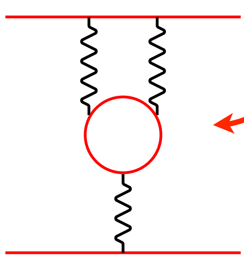
- But large $g(\pi/a)$ implies
  - large lattice spacing (asymptotic freedom);
  - large lattice artifacts (*e.g.*, breaks rotation & Lorentz invariance);
  - useless for phenomenology.
- Use numerical integration of path integral instead.

  N.B., 64^3x192 lattice has 1.6x10^9 integration variables (gluonic), so need Monte Carlo integration/simulation.

Theory “Stalls” for 20 Years

- Lattice spacing errors too large in simulations.
  - $O(a^1)$ errors in Wilson’s discretization ⇒ need very small a.
- Light-quark vacuum polarization too expensive for realistic (very small) u/d quark masses m_q:

$$\det\left((\partial - igA)\cdot\gamma + m_q\right)$$



- 10^8x10^8 matrix (today).
- Sparse; solve iteratively.
- Singular at $m_q = 0$.
- Extrapolate in m_q or omit.

- Computing cost $\propto$ (1/error)^8 for Wilson discretization.
  - 100x increase in computer power reduces error by only 44%.
- Wilson declares lattice QCD dead (BNL, 1986).

Quantum Field Theory on a Lattice

Approximate Derivatives

Numerical Analysis ⇒

$$\frac{\partial\psi(x_j)}{\partial x} = \Delta_x\psi(x_j) + \mathcal{O}(a^2)$$

$$\frac{\psi(x_j + a) - \psi(x_j - a)}{2a}$$

⇒ Use only ψ's at grid sites.

Large $a \Rightarrow$ need *improved discretizations.*

E.g.,

$$\frac{\partial\psi}{\partial x} = \Delta_x\psi - \frac{a^2}{6}\Delta_x^3\psi + \mathcal{O}(a^4)$$

10–15% for

a=0.4 fm

1–2% for

a=0.4 fm

$\Rightarrow$ a = 0.4 fm okay?

Except ...

quantum numerical analysis $\neq$ classical numerical analysis!

Ultraviolet Cutoff

$\lambda_{\text{min}} = 2a$ is smallest wavelength on lattice.

E.g.) $\psi =$ $+1$ -1 $+1$ -1 $+1$

$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$

$\Rightarrow$ all quark and gluon states with $p > \pi/a$ are excluded by the lattice since $p = 2\pi/\lambda$.

Lattice QCD $\equiv$ QCD + (nonperturbative) lattice UV regulator

$\equiv$ real QCD

But ∀p's important in quantum field theory!

(Consider ultraviolet divergences.)

Renormalization Theory ⇒ mimic effects of $p > \pi/a$ excluded states by adding a-dependent *local* terms to the field equations, Lagrangians, currents, operators, *etc*.

K. Wilson late 1960s, early 1970s.

$\Rightarrow$ $$\partial\psi \to \Delta\psi + c(a) a^2 \Delta^3 \psi + \cdots.$$

where

$$c(a) = -\frac{1}{6} + \text{Contribution from } p > \pi/a \text{ physics}$$

Numerical Analysis.

Theory & context specific $\Rightarrow$ not universal!

Bad News: Need a^2 corrections when a large, but *Numerical Recipes* can't tell you values of $c(\alpha)$ …

Good News: $p > \pi/a$ QCD is perturbative if a small enough (asymptotic freedom).

$\Longrightarrow$ compute $c(\alpha)$ … using perturbation theory.

$\Longrightarrow$ Perturbation theory fills gaps in lattice.

$\Longrightarrow$ Continuum results without $a \to 0$!

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E.g.,

Renormalization constant.

$$\mathcal{L}^{(\Lambda)} = Z(a)\,\overline{\psi}(\Delta\cdot\gamma - m(a))\psi + c(a)\,a^2\,\overline{\psi}\Delta^3\cdot\gamma\psi + \cdots$$

Finite-a correction.

where

$$c(a) = -\frac{1}{6} + c_1\,\alpha_s(\pi/a) + \cdots$$

Numerical Analysis

Mimics effects of $p > \pi/a$ states excluded by grid.

Lattice QCD Strategy

Asymptotic freedom in QCD $\Rightarrow$

- short-distance physics simple (perturbative);
- long-distance difficult (nonperturbative).

Lattice separates “short” from “long”:

- $p > \pi/a$ QCD $\rightarrow$ corrections δL computed in perturbation theory (a must be small enough).
- $p < \pi/a$ QCD $\rightarrow$ nonperturbative, numerical Monte Carlo integration.

Two QCD Breakthroughs

1990s: Larger a.

Before $\Rightarrow$ need $a \leq 0.05$ fm.

Now, better discretizations $\Rightarrow$ $a = 0.1$–0.4 fm works.

Simulations cost $\propto (1/a)^6$

$\Rightarrow$ new simulations cost 10^2–10^6 times less!

2000s: Smaller u/d quark masses.

Before $\Rightarrow$ $m_{u/d}$ 10–20x too big; vac. pol'n impossible.

Now, better discretizations $\Rightarrow$ correct masses.

Vac. pol'n enters at 15–30%

$\Rightarrow$ high-precision (few %) possible for first time.

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Now, better discretizations ⇒ faster ⇒ correct masses.

Vac. pol'n enters at 15–30%

⇒ high-precision (few %) possible for first time.

Does it work?

Example: Quarks and Relativity

Standard discretizations of the quark action have $O(a^2)$ errors.

$$\mathcal{L}_{\mathrm{lat}} \approx \overline{\psi}(D\cdot\gamma + m)\psi + \frac{a^2}{6}\sum_\mu \overline{\psi} D_\mu^3 \gamma^\mu \psi + \cdots$$

$O(a^2)$ error violates rotation/Lorentz invariance; removed by adding correction term.

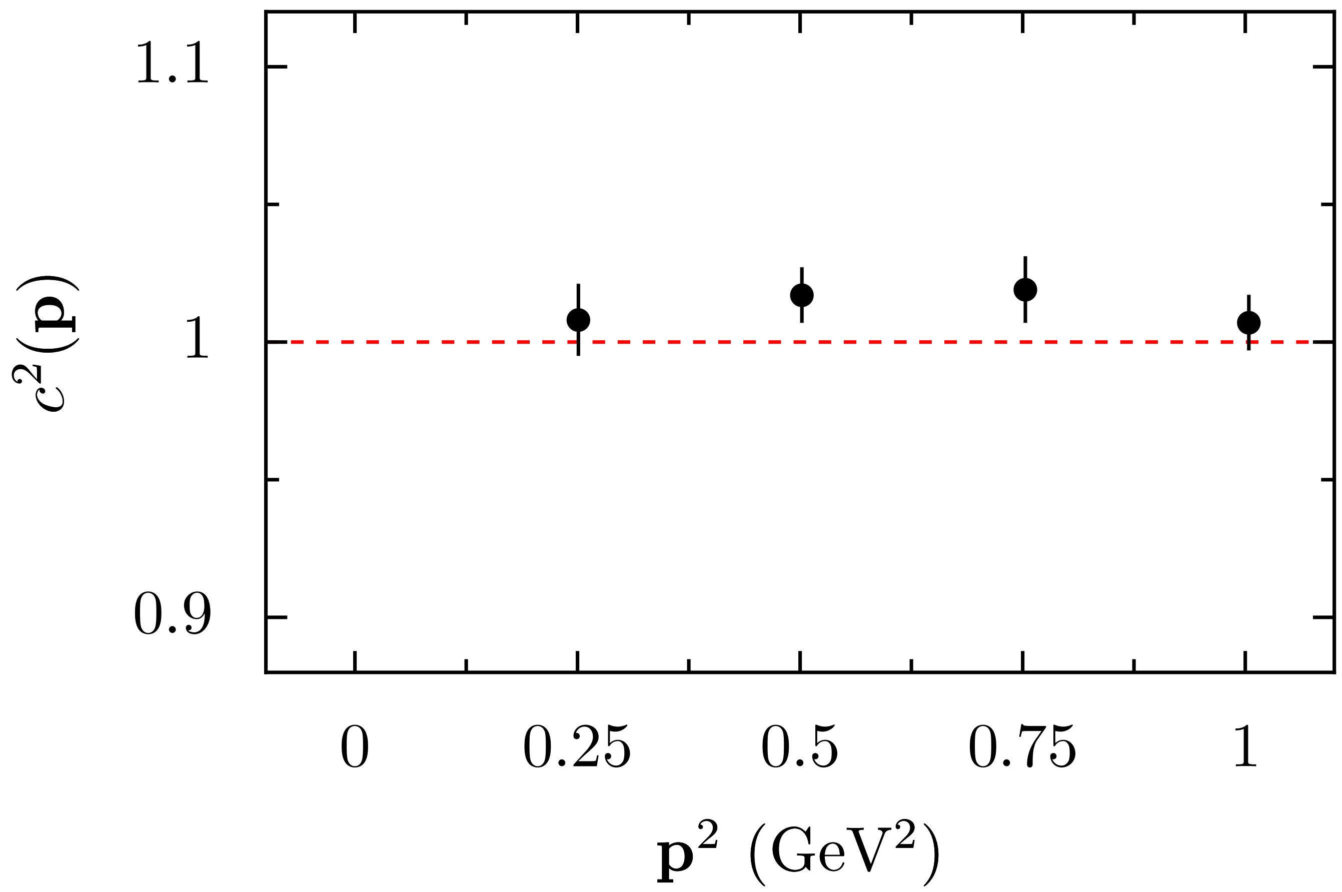
Test by computing

$$c^2(\mathbf{p}) \equiv \frac{E^2(\mathbf{p}) - m^2}{\mathbf{p}^2}$$

Lorentz invariance implies:

$$c^2(\mathbf{p}) = 1 \qquad \forall \mathbf{p}$$

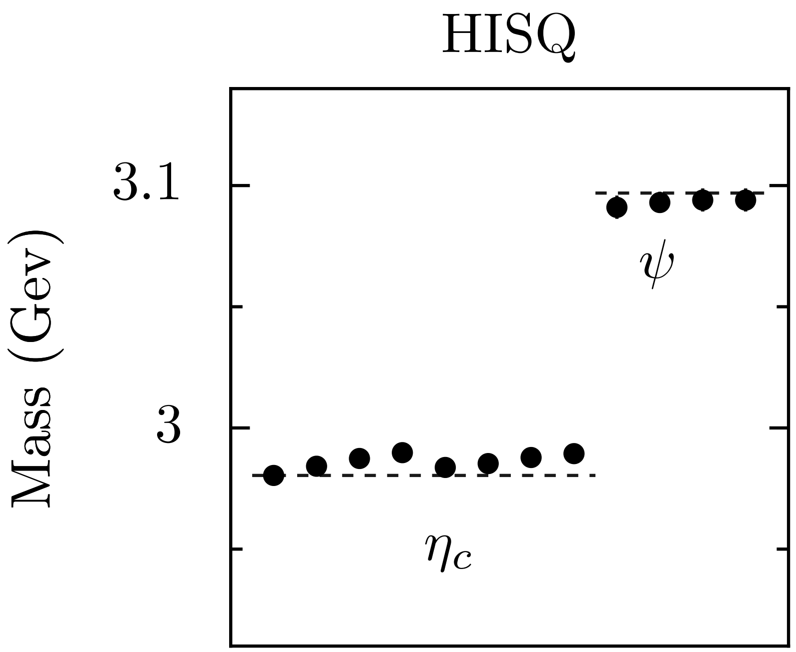
E.g., c^2 for η_c with $m_c = 0.67/a$ using highly improved HISQ discretization:

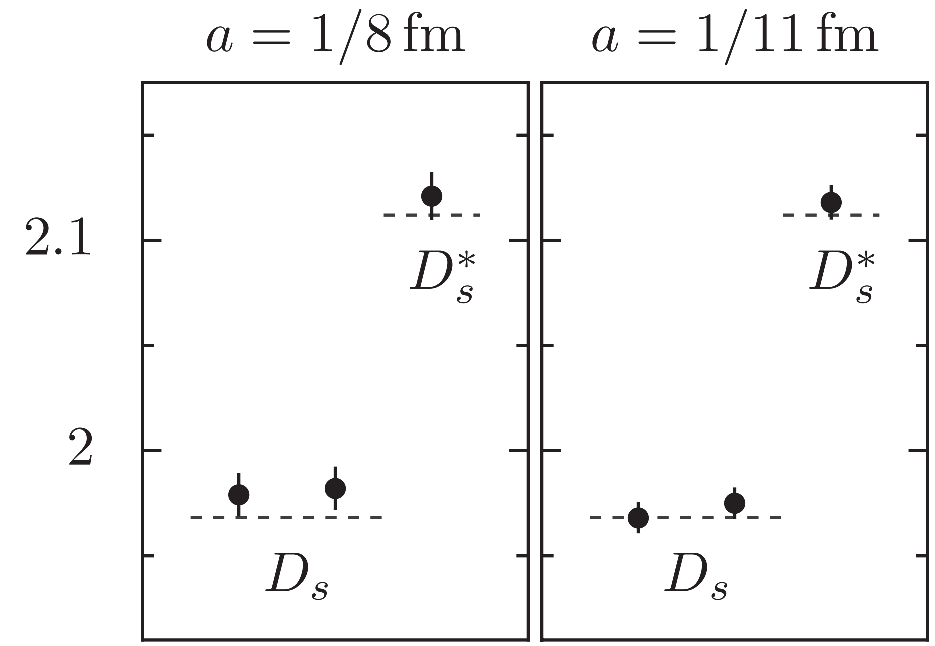


N.B. Gives 0.56 without $O(a^2)$ corrections.

Follana *et al* (2007) .

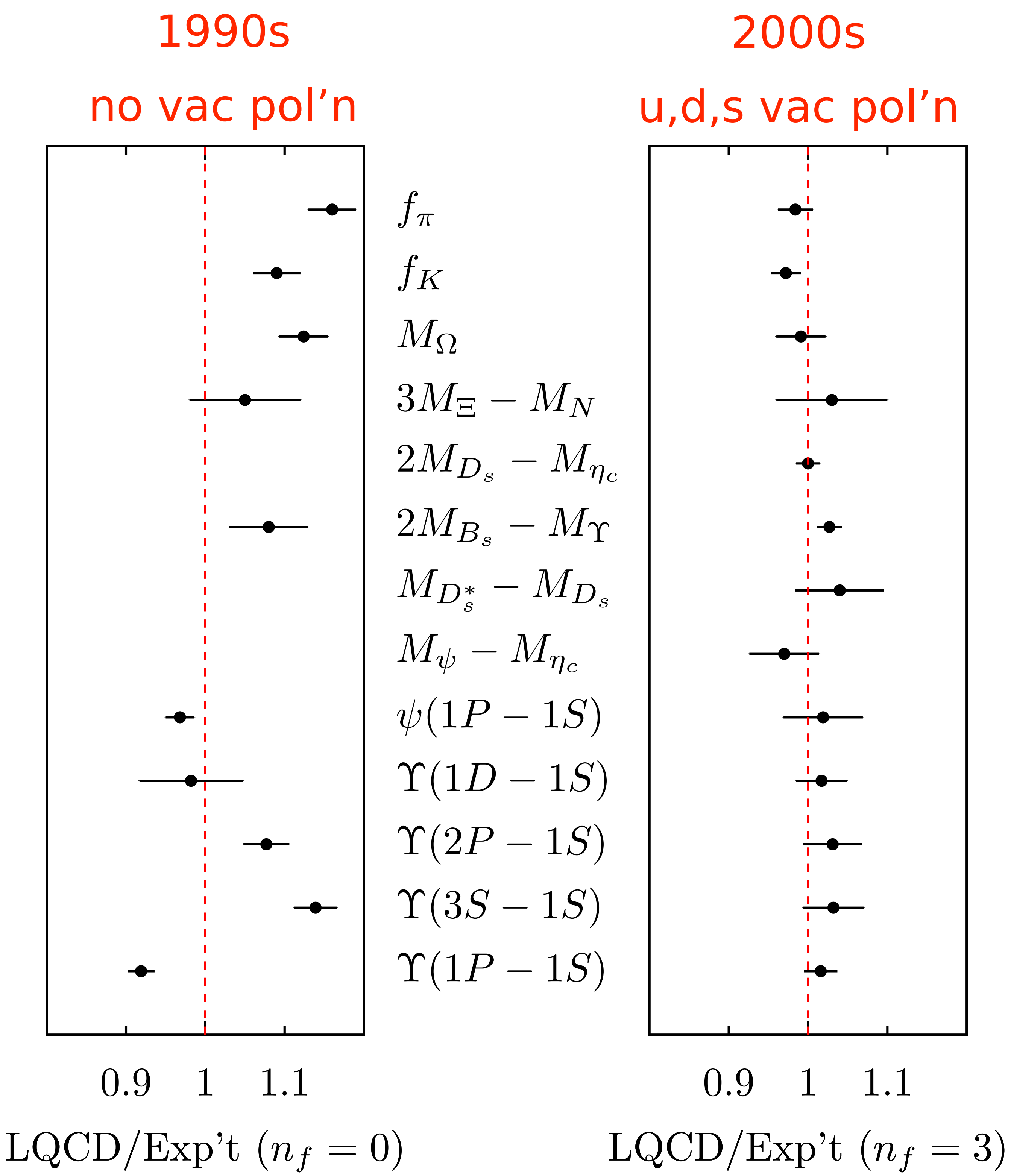
Test relativity in hyperfine spin-splittings of heavy-quark mesons; compare with experiment:





N.B. Few MeV precision with no free parameters!

Example: Add u,d,s Vacuum Polarization



Lattice QCD/Experiment

- Correct answer is 1.
- Focus on well measured quantities.
- Only 5 parameters: *e.g.*, tune quark masses from m_π, m_K, $m_{\eta c}$, $m_{\eta b}$; tune bare coupling from Υ(*2S-1S*) (or ...)

  ⇒ no free parameters!

Davies et al (2004).

Lattice QCD since 2004

Physics Focus

1) Heavy-quark physics.

- Major experimental program to measure weak decays of *c* and *b* quarks to few % (BaBar, Belle, CLEO-c, Fermilab, LHCb, ...).

- Push Standard Model to point of failure (SUSY, extra dim. ... ??)

- Lattice QCD essential:

  quark decay = weak-interactions x QCD

- CKM matrix unitary?

$$\begin{pmatrix} \begin{array}{c} V_{ud} \\ \pi \to l\nu \end{array} & \begin{array}{c} V_{us} \\ K \to l\nu \\ K \to \pi l\nu \end{array} & \begin{array}{c} V_{ub} \\ B \to \pi l\nu \end{array} \\ \begin{array}{c} V_{cd} \\ D \to l\nu \\ D \to \pi l\nu \end{array} & \begin{array}{c} V_{cs} \\ D_s \to l\nu \\ D \to K l\nu \end{array} & \begin{array}{c} V_{cb} \\ B \to D l\nu \end{array} \\ \begin{array}{c} V_{td} \\ \langle B_d|\overline{B}_d\rangle \end{array} & \begin{array}{c} V_{ts} \\ \langle B_s|\overline{B}_s\rangle \end{array} & V_{tb} \end{pmatrix}$$

2) Hadronic spectrum, structure, QCD parameters …

- Major experimental programs at DESY, JLab …
- Structure functions, form factors …
- Low-energy nuclear physics, small nuclei on the lattice.
- Exotic/hybrid mesons, glueballs …
- High-precision quark masses, $\alpha_s \Rightarrow$ precise Higgs decays.

3) QCD at finite temperature and density. (RHIC)

4) Strong coupling beyond QCD. (LHC?, ILC?)

- 2 of 3 known interactions strongly coupled. (QCD, gravity)
- Generic in non-abelian gauge theories …
- … unless gauge symmetry spontaneously broken ($\Rightarrow$ strong coupling)

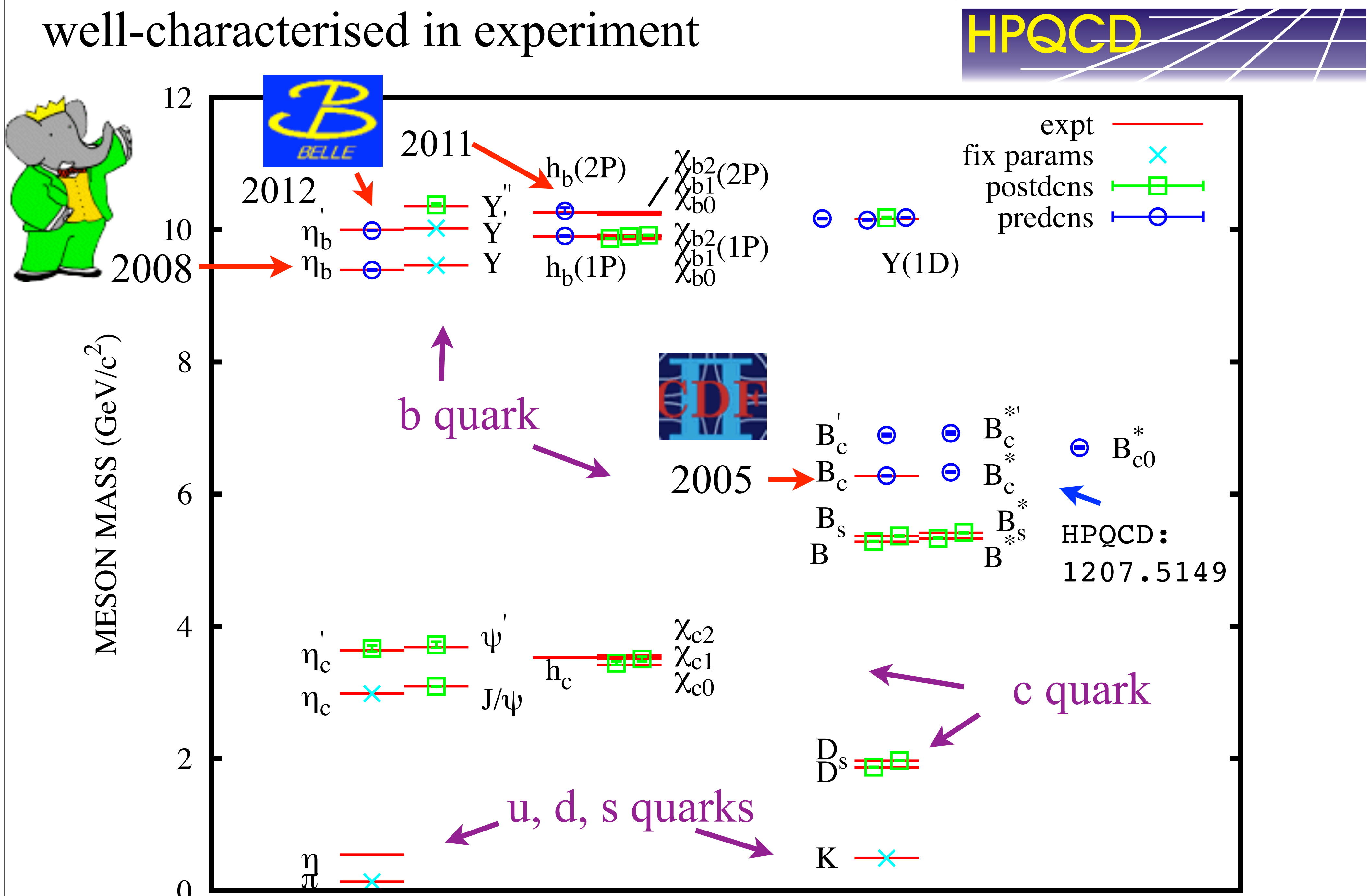
Sampler

Slides from C. Davies review

2013 Lepton-Photon Symposium

San Francisco

Results for the masses of mesons that are long-lived and so can be well-characterised in experiment



Agreement very good - errors typically a few MeV, need to worry about em, mu-md ..

Tuesday, 25 June 2013

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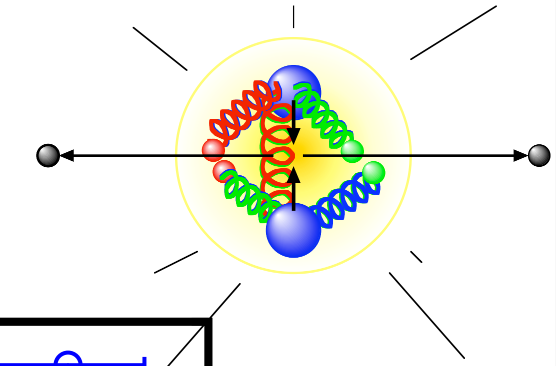
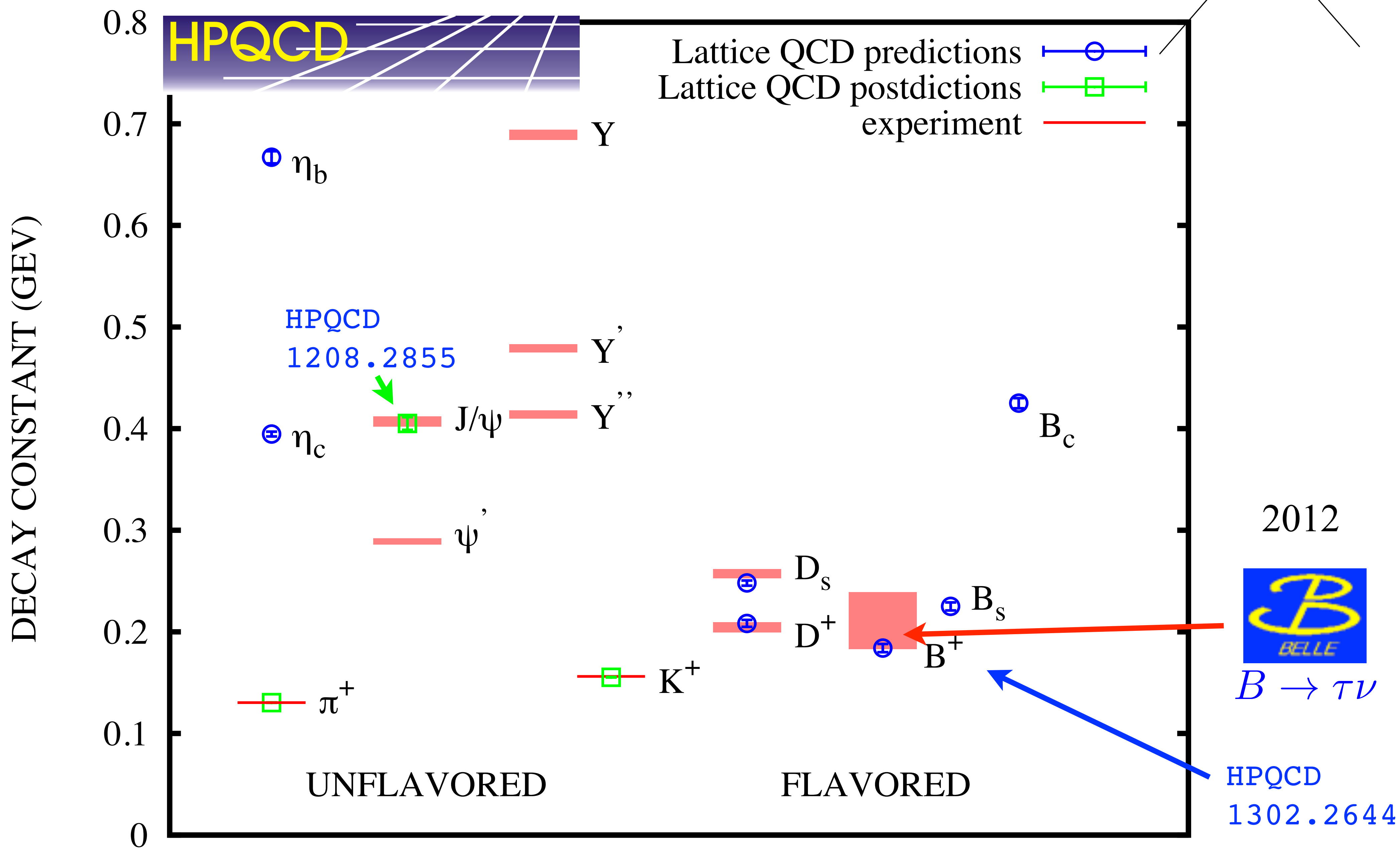
Meson decay constants

Parameterises hadronic information needed for annihilation rate to W or photon: $\Gamma \propto f^2$





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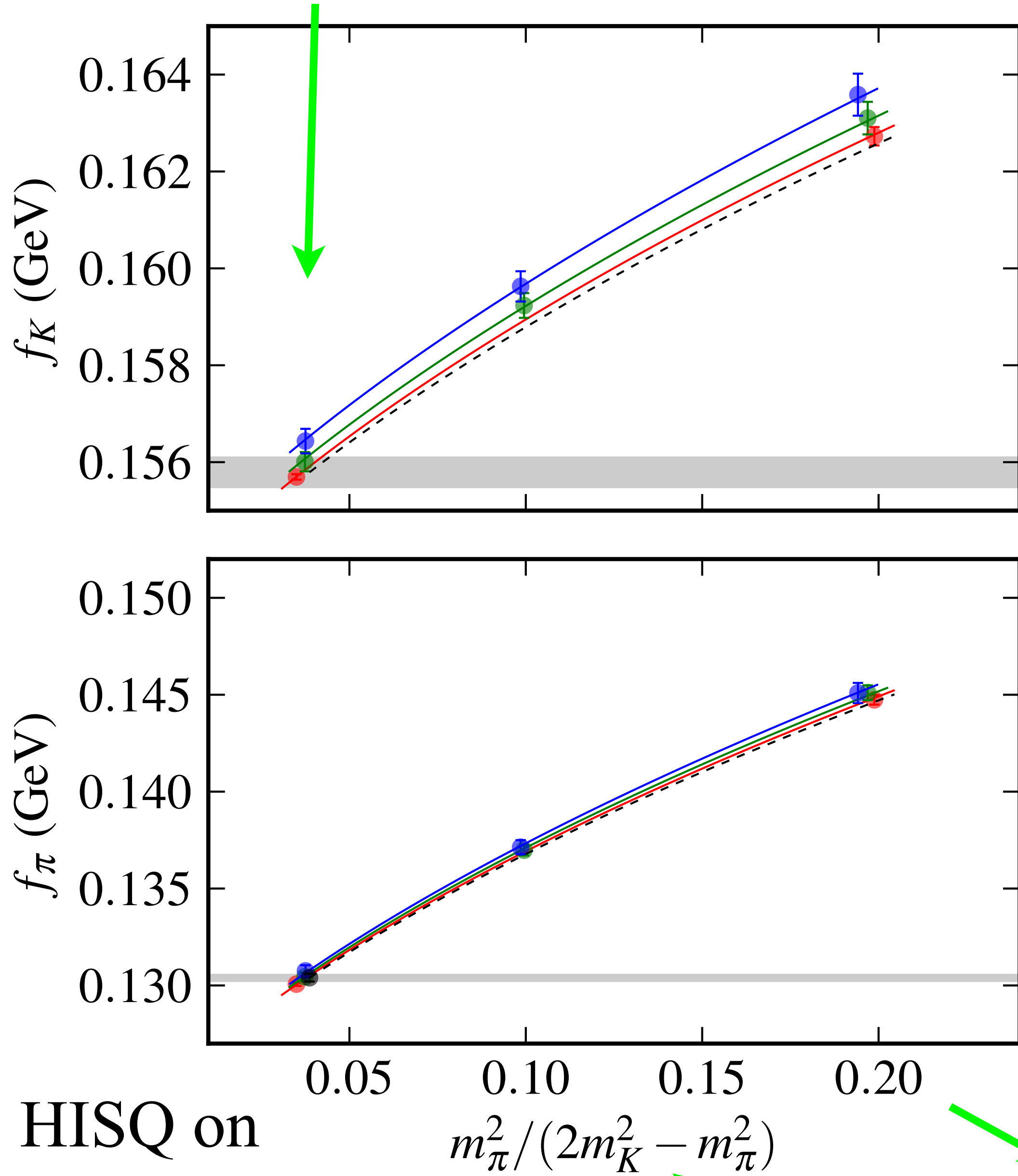
Constraining new physics with lattice QCD

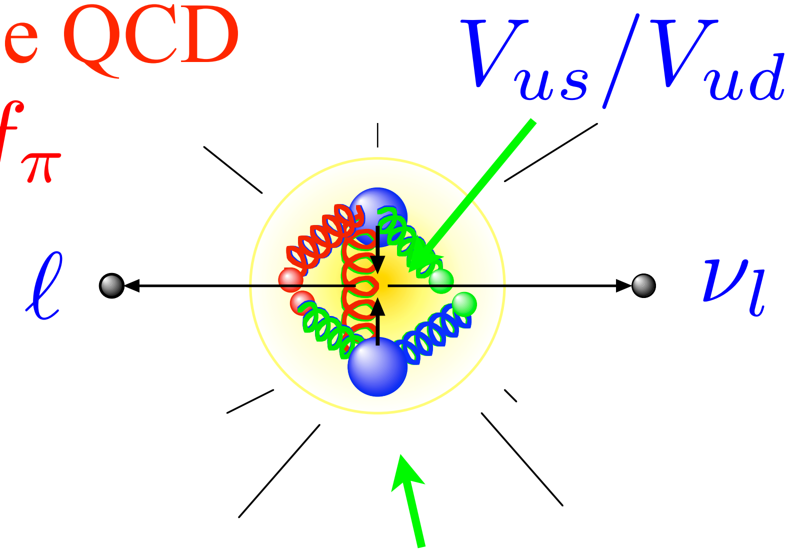
* results at physical u/d quark masses*

f_K/f_π



$= m_{u,d}/m_s$

HISQ on
MILC configs
HPQCD:1301.1670



Annihilation of K/π to W allows CKM element determination given decay constants from lattice QCD

expt for $\dfrac{\Gamma(K^+ \to \ell\nu)}{\Gamma(\pi^+ \to \ell\nu)}$

$$\frac{|V_{us}|f_{K^+}}{|V_{ud}|f_{\pi^+}} = 0.27598(35)_{\mathrm{Br}(K^+)}(25)_{EM}$$

$\dfrac{f_{K^+}}{f_{\pi^+}}$ from lattice gives CKM

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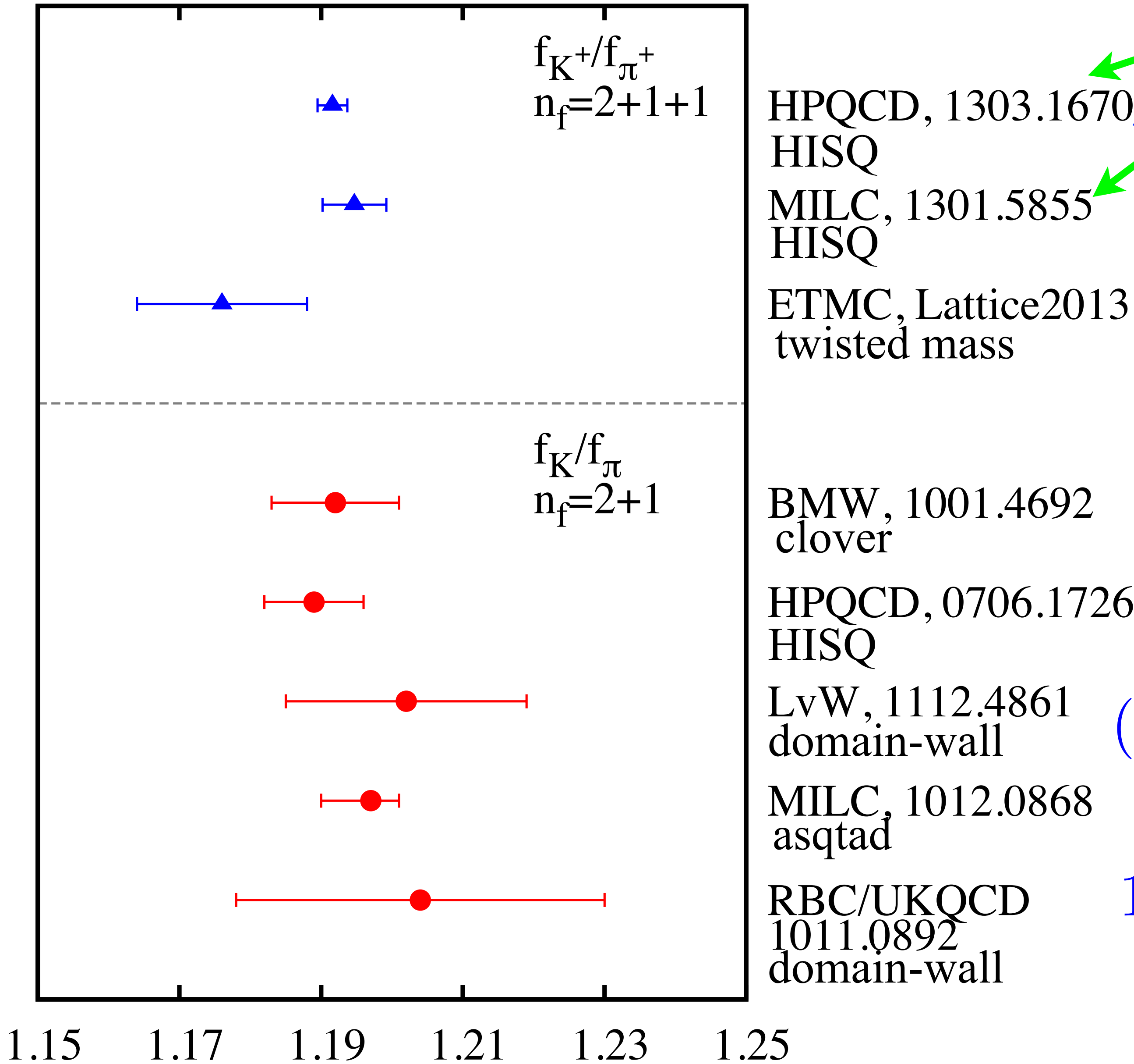
Comparison of results (note: $f_{K^+} < f_K$)

RM123:1303.4896
gives by 0.40(4)%

good agreement from different formalisms



* results at physical u/d quark masses*

$$\frac{f_{K^+}}{f_{\pi^+}} = 1.1916(21)$$

$$\frac{|V_{us}|}{|V_{ud}|} = 0.23160(29)_{expt}(21)_{EM}(41)_{latt}$$

$$|V_{us}| = 0.22564$$

$$(28)_{\mathrm{Br}}(20)_{EM}(40)_{latt}(5)_{V_{ud}}$$

$$1 - |V_{ud}|^2 - |V_{us}|^2 - |V_{ub}|^2$$

$$= -0.00009(51)$$

V_{ud} from nuclear β decay now needs improvement for unitarity test!

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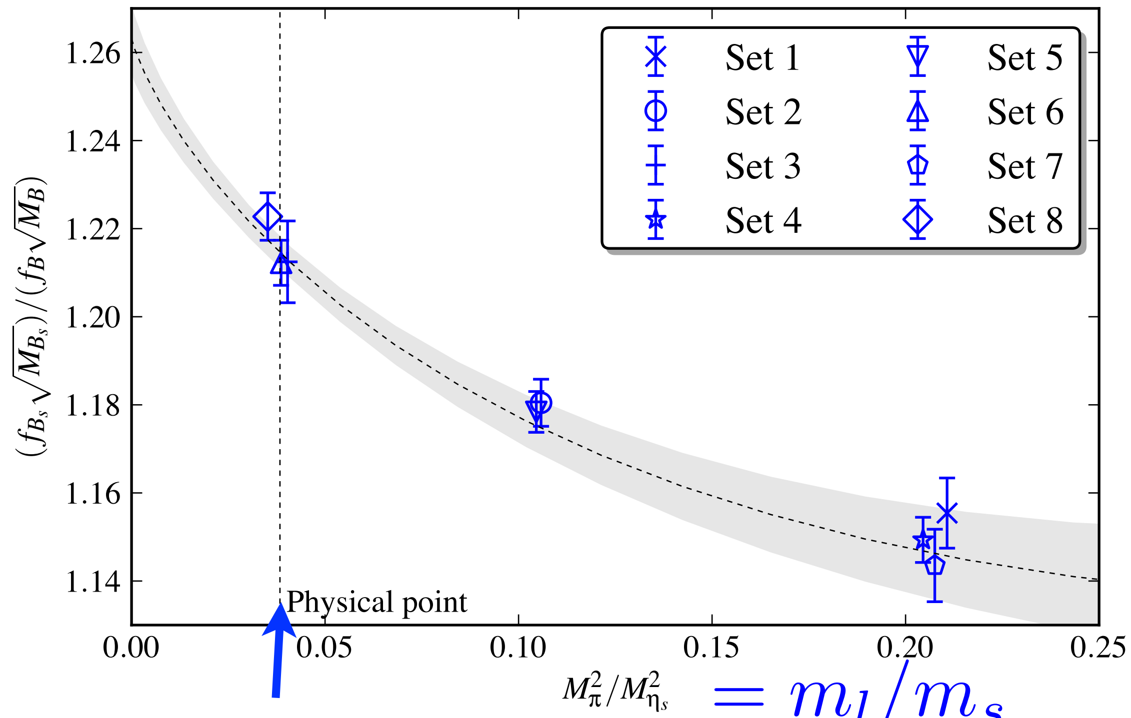
Constraining new physics with lattice QCD: f_{B_s}, f_B



$= m_l/m_s$

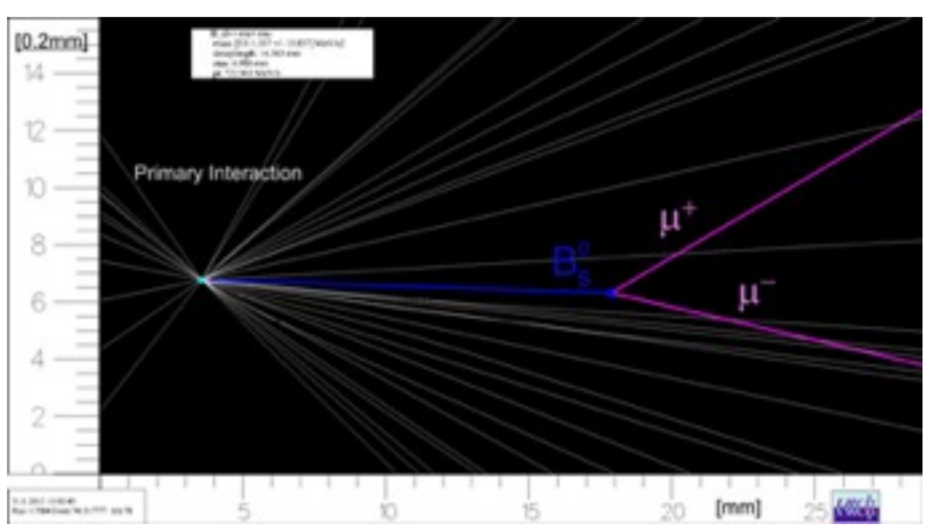
* results at physical u/d quark masses*

HPQCD: 1302.2644.

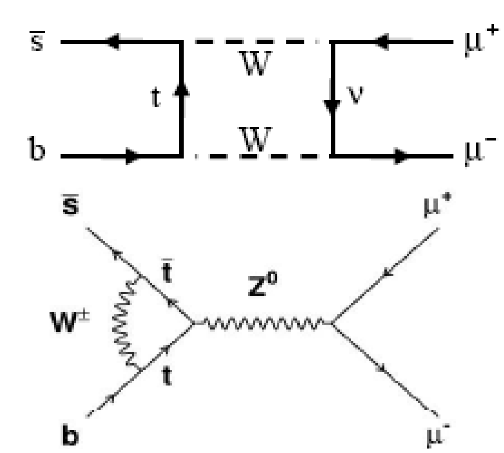
Uses improved NRQCD for b quark and HISQ u/d and s quarks on HISQ 2+1+1 gluon configs

$$f_{B_s} = 224(5)\mathrm{MeV}$$

$$\mathrm{Br}(B_s \to \mu^+\mu^-) = 3.47(19) \times 10^{-9}$$



Standard Model processes:



Nov. 2012

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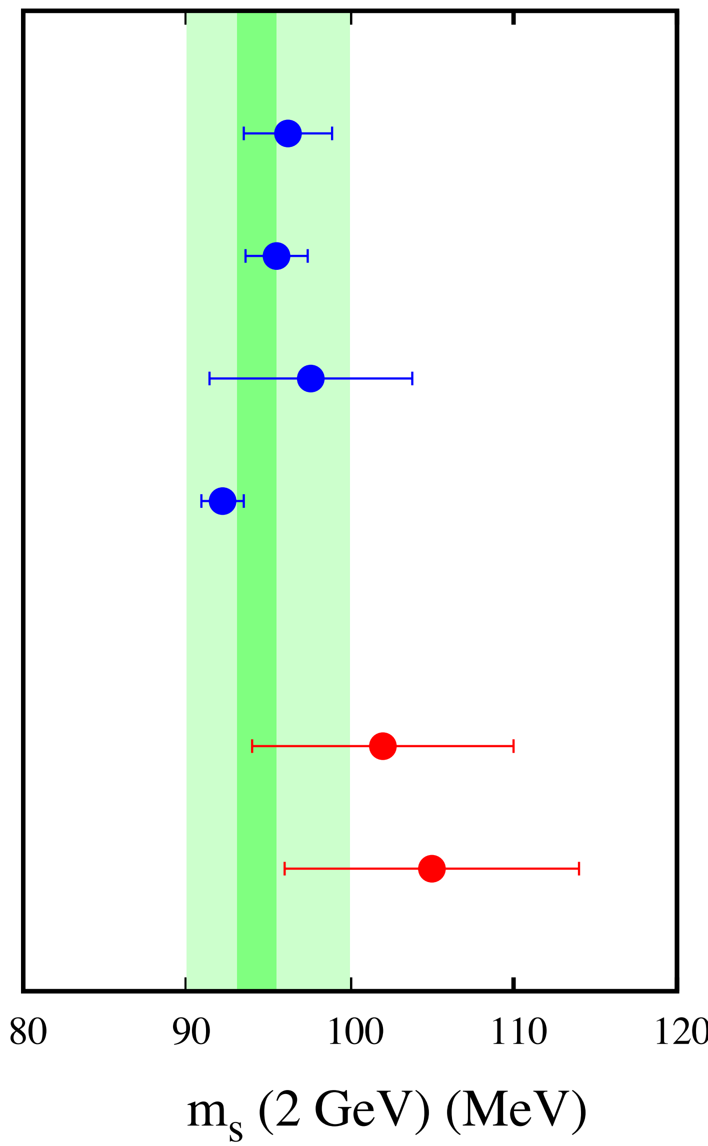
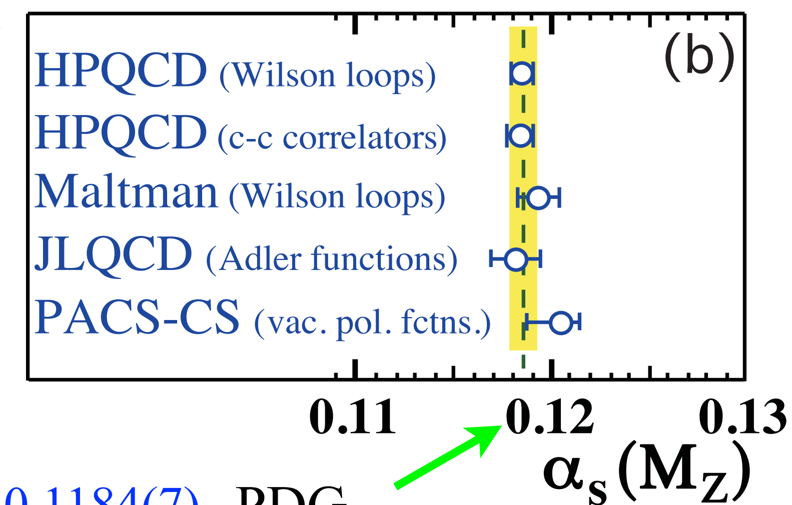
Lattice QCD sets world averages for quark masses and α_s

Direct access to parameters in QCD Lagrangian means systematic errors smaller

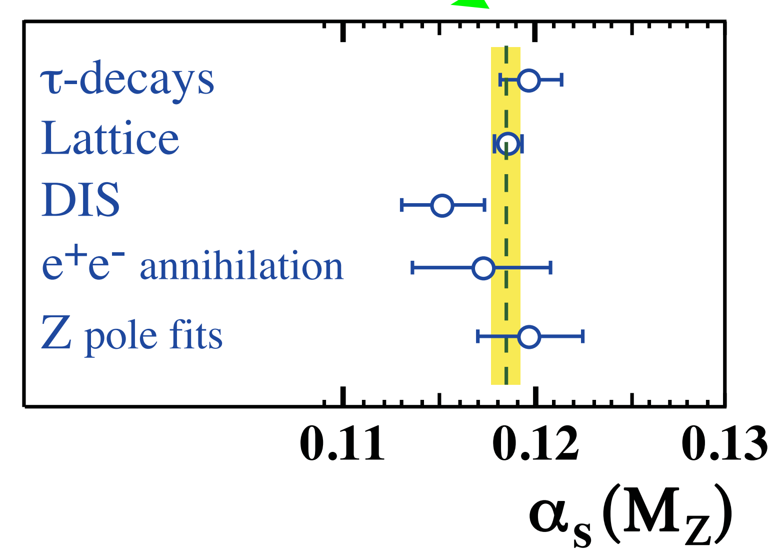
PDG av:94.3(1.2) MeV



a variety of lattice methods agree



av:0.1184(7) PDG



non-lattice methods have larger errors

Lattice calcs now adding QED for accurate m_u/m_d

`Izubuchi:Lattice2012; RM123: 1303.4896`

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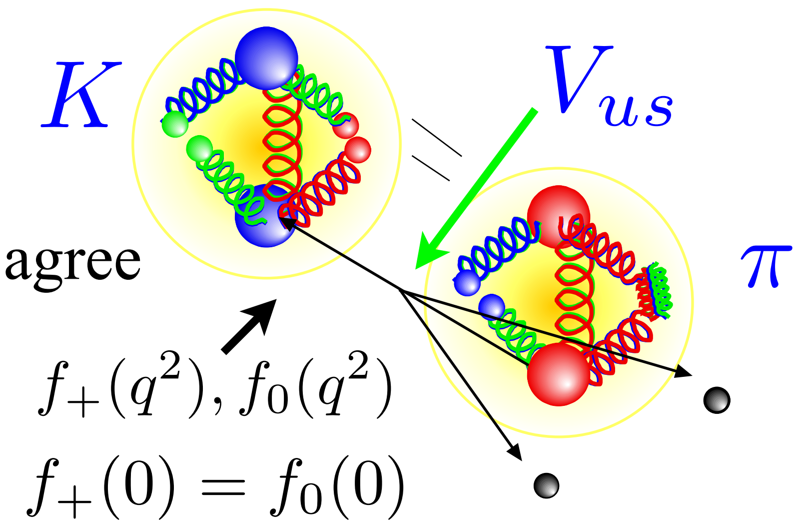
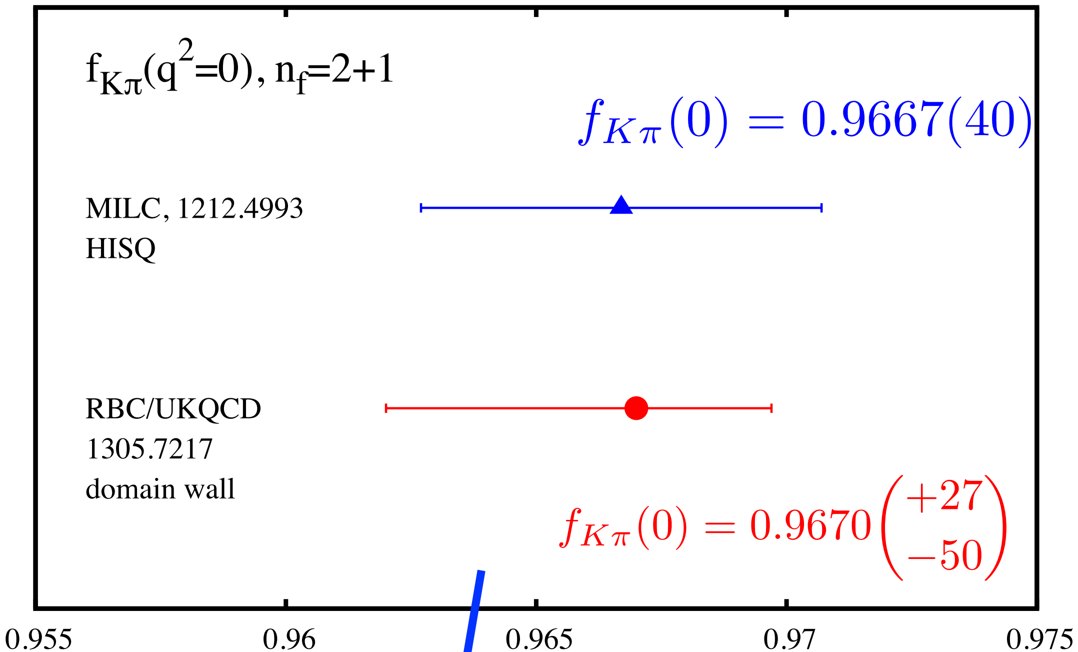
Constraining new physics with lattice QCD: form factors

$K \to \pi \ell \nu$



NEW - now results with full continuum
and chiral extrapolation. Different formalisms agree



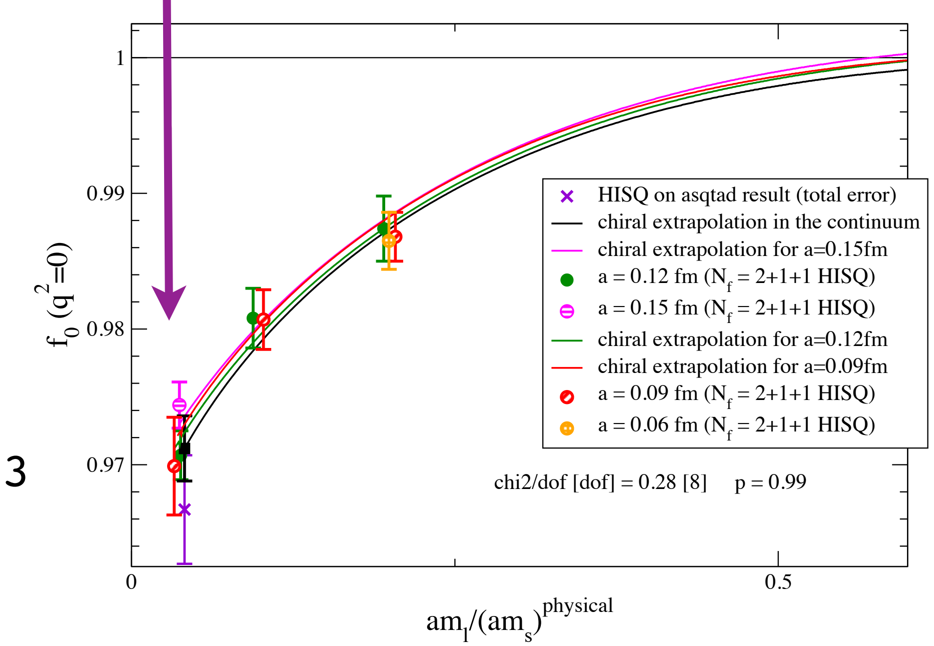
with experiment:

$$|V_{us}|f_{K\pi}(0) = 0.2163(5)$$

1005.2323

$$|V_{us}| = 0.2238(11)$$

MILC in progress: HISQ results with physical u/d quarks - expect ~0.3% error



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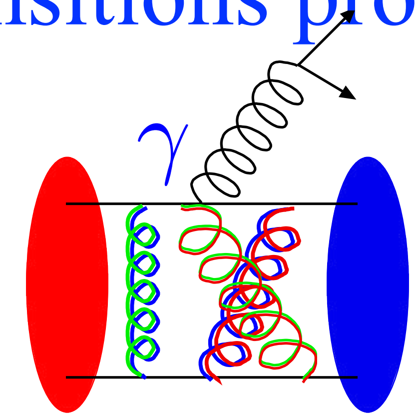
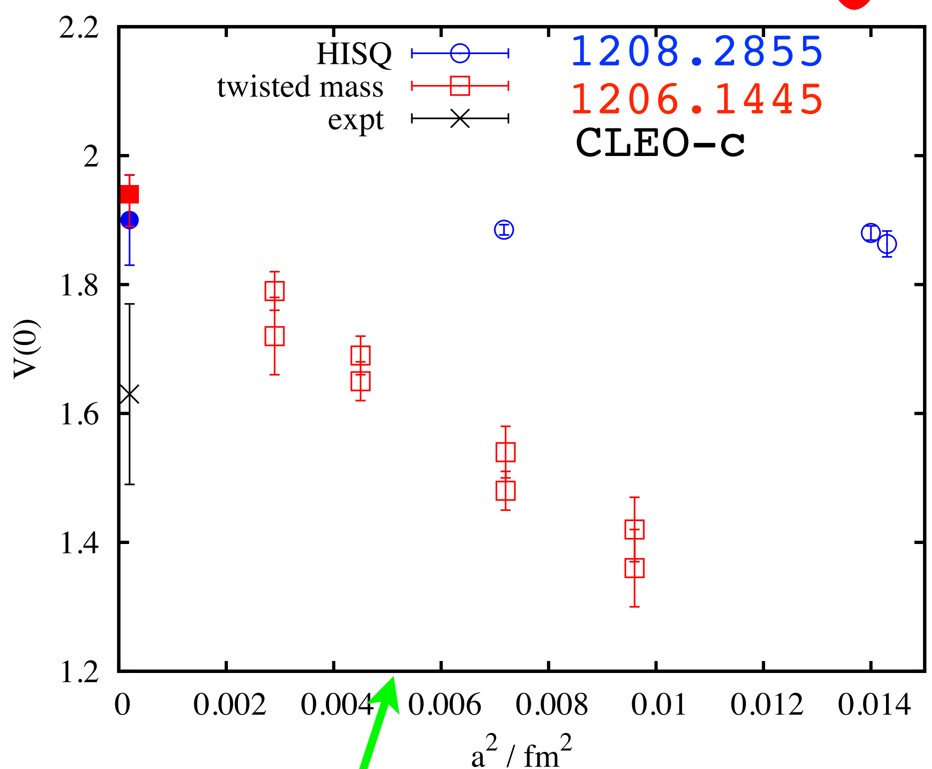
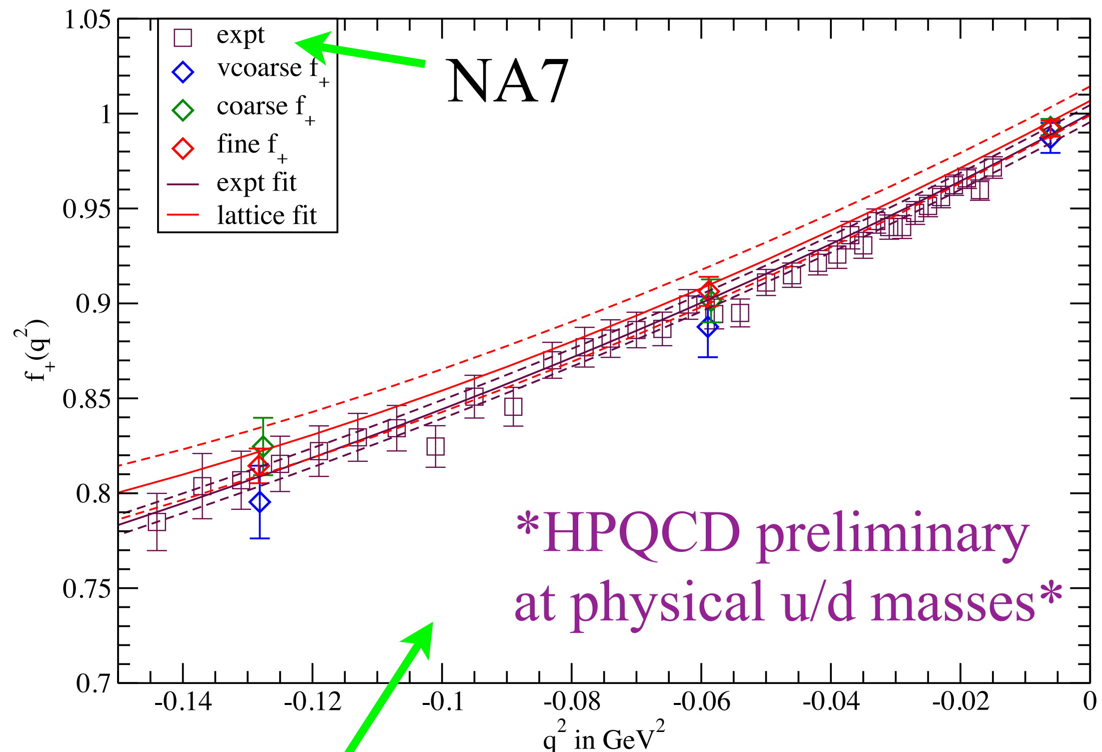
Electromagnetic transitions provide important tests

$J/\psi \to \gamma\eta_c$



pion electromagnetic form factor





Different quark formalisms agree. Lattice more accurate than experiment.

Test lattice QCD vs direct experiment at low spacelike q^2 ; $\langle r^2 \rangle$ sensitive to light quark mass

`ETMC: 0812.4042; JLQCD:0810.2590; Mainz:1306.2916`

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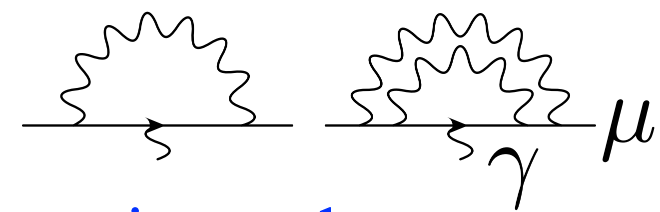
Muon anomalous magnetic moment

$$\vec{\mu} = g\frac{e}{2m}\vec{S}$$

anomaly $a_\mu = \dfrac{g-2}{2} = \mathcal{O}(10^{-3})$

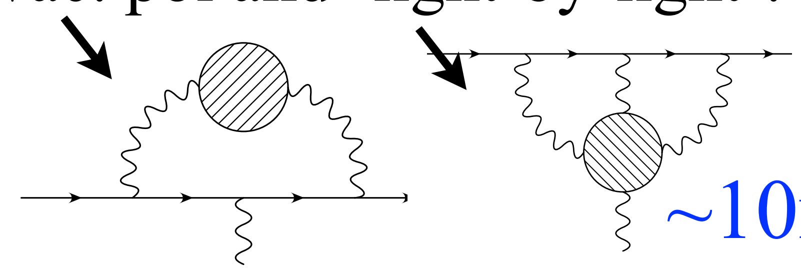
e.g. from QED:



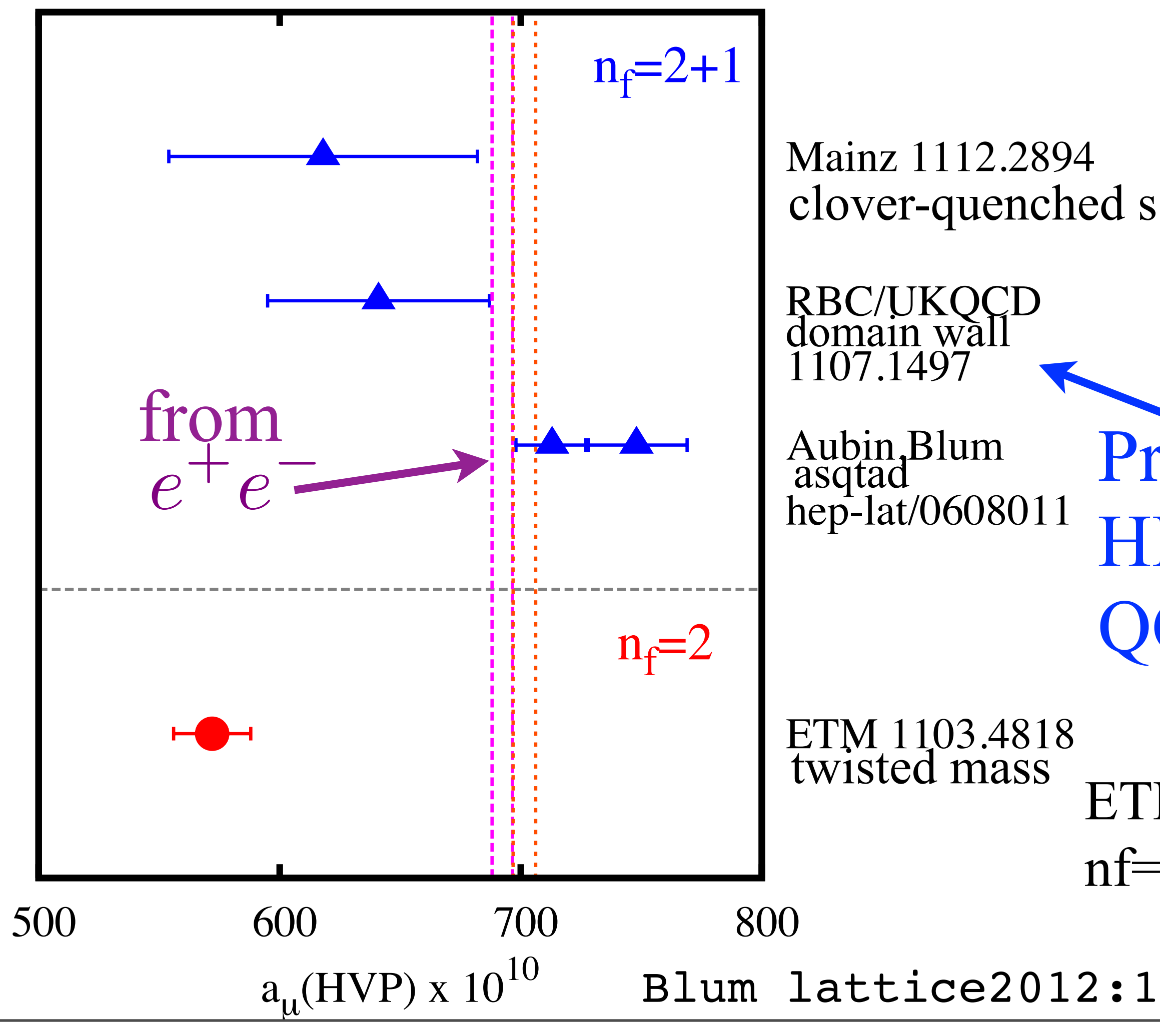
$a_\mu^{expt} - a_\mu^{SM} \approx 29(8) \times 10^{-10} = 3\sigma$ mismatch

BNL, to be improved at FNAL and J-PARC

QCD contribn from hadronic vac. pol and ‘light-by-light’.



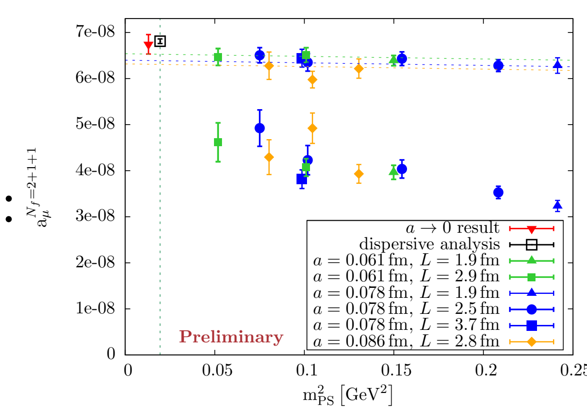
~10x 10^{-10}



Progress on HVP

HLBL needs QCD+QED

ETM underway : nf=2+1+1



Blum lattice2012:1301.2607

Tuesday, 25 June 2013

Slide from C. Davies review talk, 2013 Lepton-Photon Symposium, San Francisco

Conclusions

- Lattice QCD now a standard tool for strong interaction physics, both theoretical and experimental.
  - Most accurate strong-interaction calculations in history.
  - Landmark in history of quantum field theory: high-precision quantitative verification of nonperturbative technology (for a real theory).
  - Essential for weak interaction phenomenology, Beyond the Standard Model physics, … — QCD backgrounds.
- Problems that remain: hadronization of jets, quark matter, axial gauge theories, SUSY …
  - Need methods that don’t rely upon Monte Carlo integration.
- Ready for strong coupling beyond QCD?

- Ken Wilson’s “*Homage to Lattice Gauge Theory Today*”:

  “The current knowledge base in lattice gauge theory dwarfs the state of knowledge in 1974 and even ... in 1985. The accuracy and reliability of lattice gauge computations is vastly improved thanks in part to improved algorithms, in part to increased computer power, and in part to the increased scale of the research effort underway today. The breadth of topics that have been researched is also greater today ...”

  (from “The Origins of Lattice Gauge Theory”, 2004)

- Ken Wilson’s contribution to nonperturbative QCD:
  - Renormalization group makes lattice theories (& QCD) possible.
  - Discretization that preserves exact gauge invariance.
  - Strong coupling expansion and proof of quark confinement.
  - Monte Carlo simulation/integration of path integrals.
  - Supercomputers to do the Monte Carlo simulations.