

Renormalization Group Flows and Boundary States in 2d Systems

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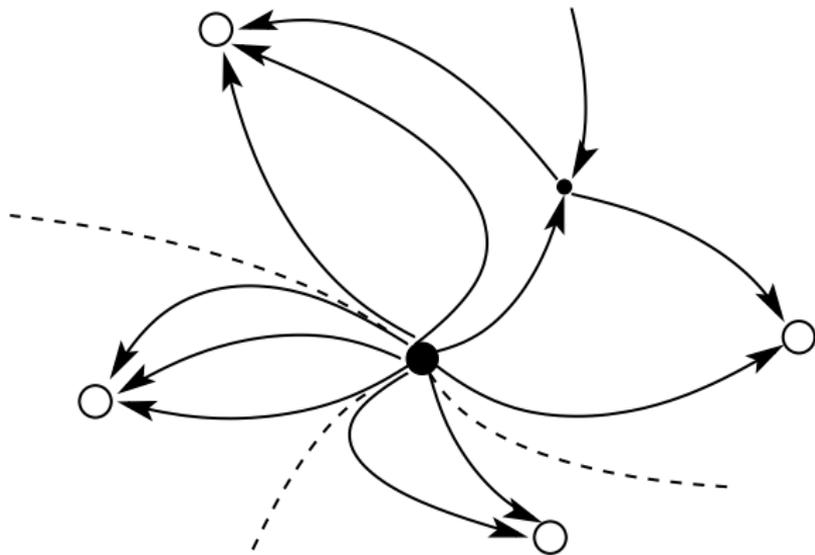
University of Oxford

Boundary and Defect CFT 2017

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See also talk by A. Konechny

RG fixed points and sinks



The unstable fixed points correspond to CFTs

Each stable sink corresponds to a phase

The trajectories leading to them correspond to massive QFTs

The general problem

Given a CFT and a set of relevant operators $\{\Phi_j\}$

$$\hat{H} = \hat{H}_{CFT} + \sum_j \lambda_j \int \hat{\Phi}_j(x) d^D x$$

where do the RG flows end up for different choices of the $\{\lambda_j\}$?

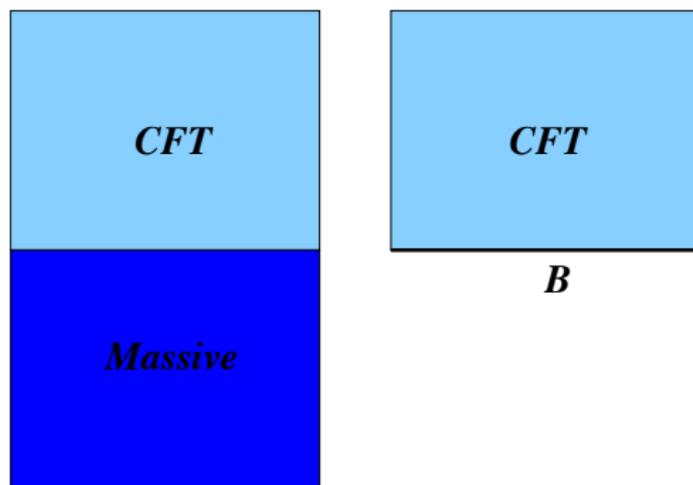
What are the properties of the massive QFTs along the trajectories?

How do we relate UV and IR physics?

In general, understanding this requires *non-perturbative* methods

Boundary states

Another way of understanding the physics is through the different possible *boundary conditions* which may be imposed on the CFT.



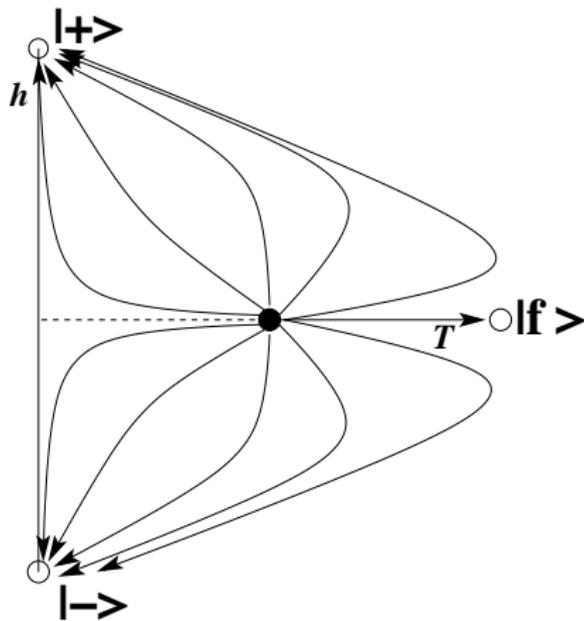
A special set of boundary conditions are *conformal*, corresponding to fixed points of the *boundary* RG flows.

In the language of QFTs in $D + 1$ dimensions, quantized on a constant time slice, these correspond to *boundary states* $|\mathcal{B}\rangle$ satisfying

$$\hat{T}_{0k}(x) |\mathcal{B}\rangle = 0, \quad (k = 1, \dots, D)$$

We conjecture that the conformal boundary states label the possible sinks of bulk RG flows.

e.g. for the Ising field theory there are 3 such states,
 $|\text{free}\rangle, |+\rangle, |-\rangle$:



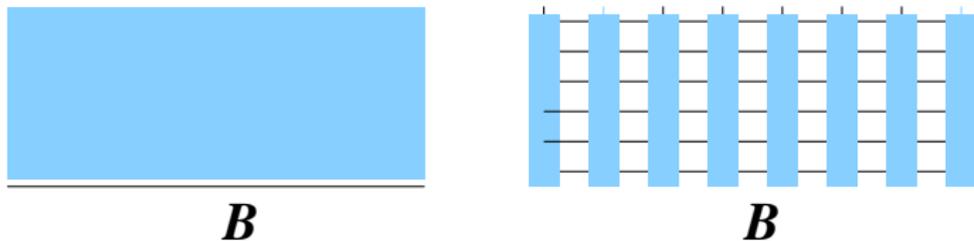
However boundary states by themselves have no scale. In QFT they have infinite energy and are not even normalizable. They have zero overlap with the true ground state in the $L \rightarrow \infty$ limit.

One way around this is to consider *smeared* boundary states

$$e^{-\tau \hat{H}_{\text{CFT}}} |\mathcal{B}\rangle$$

These have finite correlation length $\propto \tau$ and finite energy $\propto 1/\tau$, and the same short-distance behavior as the CFT

They can be viewed as a continuum version of matrix product states



We therefore propose a variational approach (just as for matrix product states): take a general smeared boundary state

$$|\Psi\rangle = \sum_a \alpha_a e^{-\tau_a \hat{H}_{CFT}} |\mathcal{B}_a\rangle$$

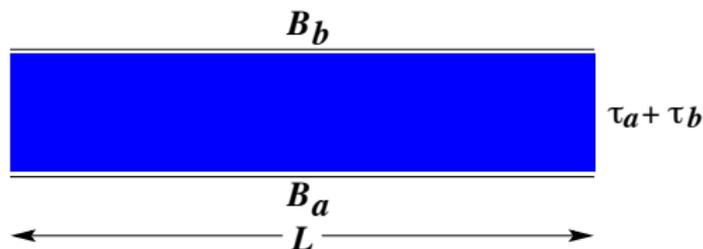
and minimize

$$E_{var} = \frac{\langle \Psi | \hat{H}_{CFT} + \sum_j \lambda_j \int \hat{\Phi}_j(x) d^D x | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

with $\{\alpha_a\}, \{\tau_a\}$ as variational parameters.

[Cf. Konechny (2017) who compares ratios of overlaps between different boundary states and numerical approximations to exact ground state]

Normalization $\langle \Psi | \Psi \rangle$



$$\langle \mathcal{B}_a | e^{-\tau_a H_{\text{CFT}}} e^{-\tau_b H_{\text{CFT}}} | \mathcal{B}_b \rangle$$

is the partition function Z_{ab} in a slab.

If $a = b$ this is dominated by the Casimir energy
 $Z_{aa} \sim \exp(\sigma_a (L/2\tau_a)^D)$

For $a \neq b$, Z_{ab} is exponentially smaller than $(Z_{aa} Z_{bb})^{1/2}$ due to the interfacial energy.

So the off-diagonal terms are suppressed – similarly in the numerator.

So the hamiltonian is diagonal in this subspace with elements

$$E_a/L^D = \frac{\sigma_a}{(2\tau_a)^{D+1}} + \sum_j \lambda_j \langle \Phi_j \rangle_a$$

where $\langle \Phi_j \rangle_a = \frac{A_j^a}{(2\tau_a)^{\Delta_j}}$

is the one-point function of Φ_j in the center of a strip of width $2\tau_a$ with boundary condition a on each edge.

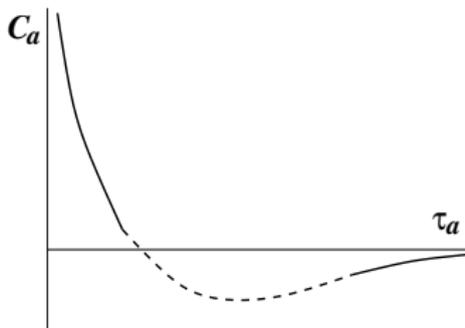
A_j^a is a universal amplitude.

We should minimize each E_a wrt τ_a and choose the a which gives the absolute minimum.

$$E_a \propto \frac{\sigma_a}{(2\tau_a)^{D+1}} + \sum_j \lambda_j \frac{A_j^a}{(2\tau_a)^{\Delta_j}}$$

Since $\Delta_j < D + 1$, $E_a \rightarrow +\infty$ as $\tau_a \rightarrow 0$

As $\tau_a \rightarrow \infty$ $E_a \rightarrow 0$ and is dominated by the most relevant operator with $\lambda_j \neq 0$. [At least in 2d] we can show that there always exists an a such that the approach is from below, so that there is always a minimum at finite τ_a



RG flows

$$E_a \propto \frac{\sigma_a}{(2\tau_a)^{D+1}} + \sum_j \lambda_j \frac{A_j^a}{(2\tau_a)^{\Delta_j}}$$

E_a scales multiplicatively under

$$\lambda_j \rightarrow e^{(D+1-\Delta_j)\ell} \lambda_j, \quad \tau_a \rightarrow e^{-\ell} \tau_a$$

so once we have found the absolute minimum a for a particular set of couplings $\{\lambda_j\}$, it is the same along the RG trajectory ☺

2d minimal CFTs

Unitary 2d CFTs with $c < 1$ are well understood, and give the scaling limits of simple 2d universality classes.

Bulk operators Φ_j are labelled by entries $j = (r, s)$ in the Kac table with $1 \leq s \leq r \leq m - 1$, with m an integer ≥ 3 and $c = 1 - 6/m(m + 1)$.

In the diagonal A_m models each value of (r, s) occurs just once.

The physical boundary states B_a are also labelled by entries in the Kac table, one for each value of (r, s) .

1-point amplitudes are also known [Lewellen + JC 1991]:

$$A_a^j = \frac{S_a^j}{S_a^0} \left(\frac{S_0^0}{S_j^0} \right)^{1/2}$$

where S_a^j is the modular S -matrix – symmetric, orthogonal, with $S_j^0 > 0$

For any j we can always choose a so that $\lambda_j A_a^j < 0$, so there is always a minimum for some a .

We can also show that for a particular state b there is a choice of the $\{\lambda_j\}$ so that

$$\sum_j \lambda_j A_a^j < 0 \quad (a = b); \quad \sum_j \lambda_j A_a^j > 0 \quad (a \neq b)$$

So every boundary state b represents an achievable RG sink.

Aside for *afficionados*: Ishibashi states and fusion rules

In 2d the physical states are linear combinations of Ishibashi states

$$|a\rangle = \sum_i S_a^i |i\rangle\rangle \quad \text{where} \quad |i\rangle\rangle \propto \sum_N |i, N\rangle \otimes \overline{|i, N\rangle}$$

Between physical states

$$\langle a | e^{-\tau H} \hat{\phi}_j e^{-\tau H} | b \rangle \propto \frac{1}{(2\tau)^{\Delta_j}} \frac{S_a^j}{S_a^0} \delta_{ab}$$

$$\text{So} \quad \langle\langle i | e^{-\tau H} \hat{\phi}_j e^{-\tau H} | k \rangle\rangle \propto \sum_a \frac{S_a^i S_a^j S_a^k}{S_a^0} = N_{ijk}$$

Verlinde formula – N_{ijk} are fusion rule coefficients: non-negative integers

Example 1: the Ising model

$$\hat{H} = \hat{H}_{CFT} + t \int \varepsilon dx + h \int \sigma dx$$

$\{\Phi_j\} = (\varepsilon, \sigma)$, boundary states $(+, -, f)$.

$$C_+ = \frac{1}{48\tau^2} + \frac{t}{\tau} - 2^{1/4} \frac{h}{\tau^{1/8}}$$

$$C_- = \frac{1}{48\tau^2} + \frac{t}{\tau} + 2^{1/4} \frac{h}{\tau^{1/8}}$$

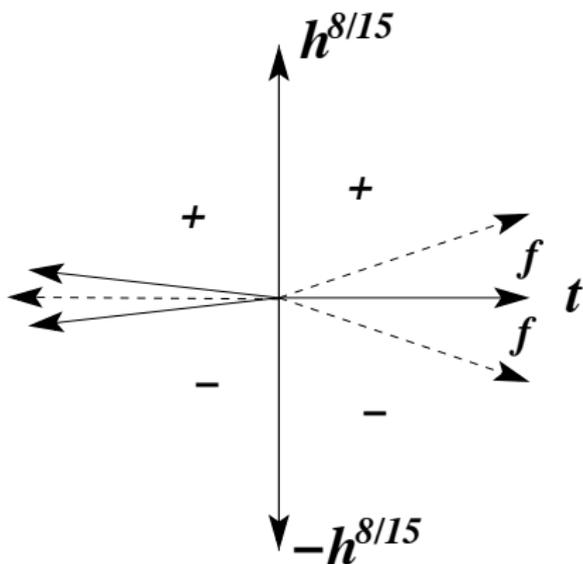
$$C_f = \frac{1}{48\tau^2} - \frac{t}{\tau}$$

[In units where $2\pi = 1$.]

For $t > 0$, $h = 0$, f wins

For $t < 0$, $h > 0$, $-$ wins

For $t < 0$, $h < 0$, $+$ wins.

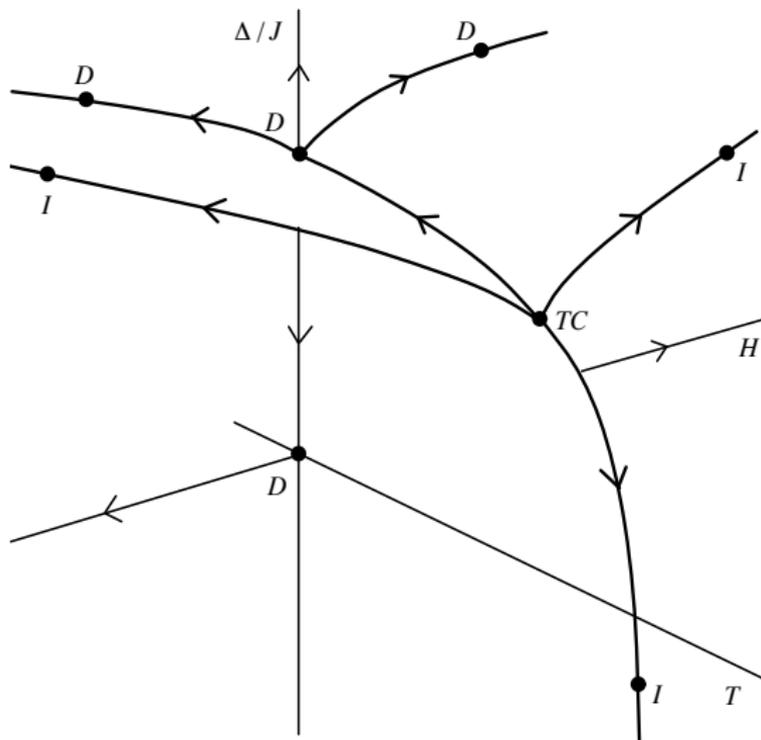


The \pm sinks do not extend all the way to $h = 0$ for $t > 0$
 There is an unphysical phase boundary along $h^{8/15}/t \approx 0.1$.

A general feature of this simple variational approximation:
 1st-order transitions between different sinks. ☹️☹️

Example 2: Tricritical Ising (A_4) model

This can be viewed as scaling limit of Ising model with vacancies, or spin 1 Ising model (Blume-Capel), or as ϕ^6 scalar field theory.



$(+)$ ●	(0) ●	$(-)$ ●
$(0+)$ ●	(m) ●	ϕ^2 ●
$(0-)$ ●	ϕ ●	ϕ^4 ●
1 ●	ϕ^3 ●	ϕ^6 ●

Figure: Kac table of bulk fields and identification of boundary states.

Results of variational method:

ϕ : (+) or (-) according to sign

ϕ^2 : (0) or [(+), (-)] according to sign

ϕ^3 : [(0+), (+)]* or [(0-), (-)]* according to sign

ϕ^4 : [(0), (+), (-)]* or [(0+), (0-), (m)] according to sign

* should be massless flows to $c = \frac{1}{2}$ CFT, but ansatz incapable of describing this: instead it suggests phase coexistence.

Similar checks can be made for general A_m models and their lattice analogues.

Summary

- boundary states give a simple way of understanding the end points of relevant RG flows for CFTs
- they give a rigorous upper bound on the free energy (ground state energy) of the massive theory
- for 2d minimal models every boundary state corresponds to the end point of an RG flow, but these have finite width with possibly unphysical first-order transitions between them
- the method cannot account for massless flows, suggesting first-order rather than continuous transitions
- however the general physics seems to be correct for the A_m models
- the variational states could be improved, and some of these features possibly removed, at a considerable cost in computational effort.