Defects in the tricritical Ising Model

Gerard Watts

King's College London

Chicheley Hall, September 7, 2017

Based on arXiv:1703.09148 with Isao Makabe

The Problem

- Can we find non-topological non-factorising conformal defects in the tricritical Ising model?

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Well-understood classes are
Topological:

$$L_m D = D L_m \ , \ \ \bar{L}_{-m} D = D \bar{L}_{-m}$$

Factorised:

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Is there evidence of any others?

Perturbations of defects

- One can perturb defects by fields that live on the defect.

  In Virasoro minimal models, the simplest defect is $D_{1,2}$ which can be perturbed by two fields, $\phi_{h,0}$ and $\phi_{0,h}$

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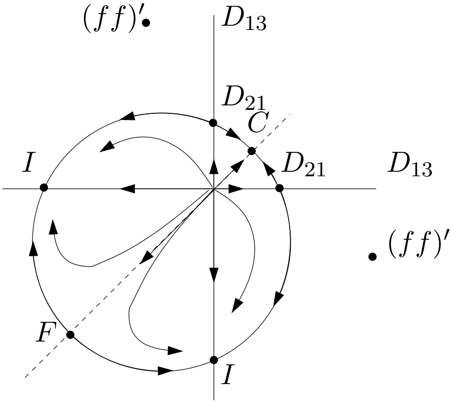
- The same result has been seen in numerical studies using the truncated conformal space approach
- Can we find any exact results?

Space of flows from TCSA

We found evidence of ten non-trivial fixed points:



The folding trick

- A conformal defect in a CFT is equivalent to a conformal boundary condition on CFT$^{\otimes 2}$.

  If the boundary conditions on CFT$^{\otimes 2}$ are known then the problem is solved.

  The central charge of the TCIM is $7/10 < 1$ so the boundary conditions are classified by conformal symmetry alone.

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- Idea: the superconformal minimal models sVir_3 and sVir_{10} have central charges $7/10$ and $7/5$. The superconformal boundary conditions for sVir_{10} should be classifiable. Does this help?

Previous work

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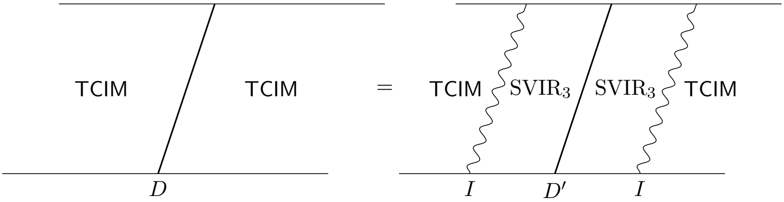
- Two problems:
- Predicted defects which did not satisfy the Cardy constraint
- We believe that the defects are not correctly GSO projected and so are not defects in TCIM
- New proposal: based on idea of Gaiotto for construction of RG defects.

NS sector and interfaces

- Use topological interfaces to construct defects in TCIM.



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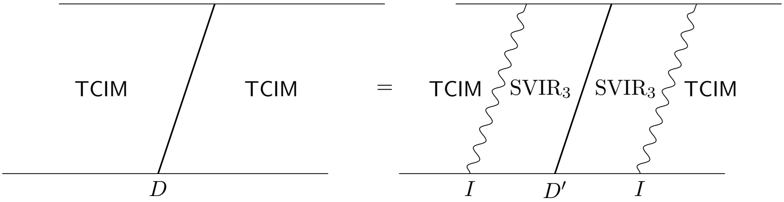

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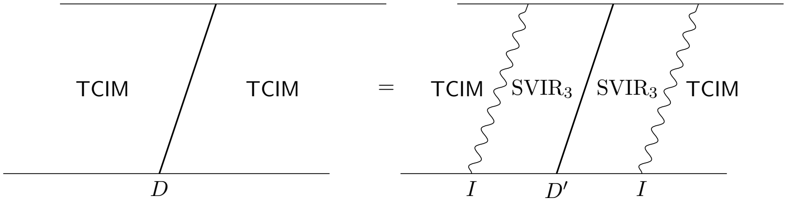

- Use only the Neveu-Schwarz subsector SVIR_3 of the superconformal theory.
- Advantage: do not need Ramond sector
- Disadvantage: cannot expect to produce elementary defects

Warm-up exercise: the free fermion

The ingredients:
defects in the free-fermion model (NS sector only);
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We ask a defect $D_{\epsilon,\epsilon'}$ to satisfy

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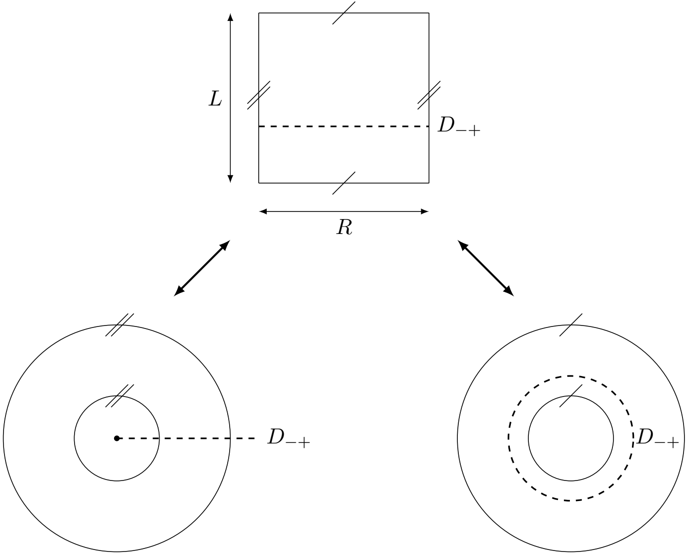
Requiring cylinder partition functions with a defect inserted to be expressed as a trace, we get

$$D_{++} = \mathbf{1} \ , \ D_{+-} = \sqrt{2}\,(-1)^{\bar{F}} \ , \ D_{-+} = \sqrt{2}\,(-1)^{F} \ , \ D_{--} = (-1)^{F+\bar{F}}$$



$$\mathrm{Tr}_{\mathcal{H}_{-+}}(q^{L_0+\bar{L}_0-c/12}) = 2\chi_R(q)\chi_{NS}(\bar{q})$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathsf{FF}}}(\tilde{q}^{L_0+\bar{L}_0-c/12}D_{-+}) = \sqrt{2}\chi_{\widetilde{NS}}(\tilde{q})\chi_{NS}(\bar{\tilde{q}})$$

Interfaces between Ising and FF

Topological interfaces must intertwine equivalent Virasoro representations.

The Ising and FF Hilbert spaces are

$$\begin{aligned} \mathcal{H}_{\mathsf{Ising}} &= (\mathcal{H}_0 \otimes \bar{\mathcal{H}}_0) \oplus (\mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_{1/2}) \oplus (\mathcal{H}_{1/16} \otimes \bar{\mathcal{H}}_{1/16}) , \\ \mathcal{H}_{\mathsf{FF}} &= (\mathcal{H}_0 \oplus \mathcal{H}_{1/2}) \otimes (\bar{\mathcal{H}}_0 \oplus \bar{\mathcal{H}}_{1/2}) , \end{aligned}$$

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$$\begin{array}{rlcrl} & \text{Ising} & & & \text{FF} \\ \hline & (\mathcal{H}_0 \otimes \bar{\mathcal{H}}_0) & \longrightarrow & & (\mathcal{H}_0 \otimes \bar{\mathcal{H}}_0) \\ & & & \oplus & (\mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_0) \\ & & & \oplus & (\mathcal{H}_0 \otimes \bar{\mathcal{H}}_{1/2}) \\ \oplus & (\mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_{1/2}) & \longrightarrow & \oplus & (\mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_{1/2}) \\ \oplus & (\mathcal{H}_{1/16} \otimes \bar{\mathcal{H}}_{1/16}) & & & \end{array}$$

$$I_{a,b} = aP_0\bar{P}_0 + bP_{1/2}\bar{P}_{1/2}$$

For consistency, we need $I \cdot I^\dagger$ to be a topological defect in the Ising model and $I^\dagger \cdot I$ to be a topological defect in the FF model.

The defects in the Ising model are

$$\text{Ising}: \quad \begin{aligned} D_0 &= P_0\bar{P}_0 + P_{1/2}\bar{P}_{1/2} + P_{1/16}\bar{P}_{1/16}\,, \\ D_{1/2} &= P_0\bar{P}_0 + P_{1/2}\bar{P}_{1/2} - P_{1/16}\bar{P}_{1/16}\,, \\ D_{1/16} &= \sqrt{2}(P_0\bar{P}_0 - P_{1/2}\bar{P}_{1/2}) \end{aligned}$$

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Since the $h = 1/16$ sector is absent from the FF model, the simplest option is that the defect satisfies

$$I \cdot I^\dagger = D_0 + D_{1/2}$$

We can choose $I = I_{\sqrt{2},\sqrt{2}}$

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The result is the closed algebra

$$
\begin{array}{ll}
I\, I^\dagger = D_0 + D_{1/2}\ , & I\, I'^\dagger = 2 D_{1/16}\ , \\
I^\dagger\, I = D_{++} + D_{--}\ , & I'^\dagger\, I = D_{+-} + D_{-+}\ , \\
I\, D_{++} = I\, D_{--} = I\ , & I\, D_{+-} = I\, D_{-+} = I'\ , \\
D_0\, I = D_{1/2}\, I = I\ , & D_{1/16}\, I = I'\ .
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Algebra of defects and interfaces

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$$\begin{array}{ll} I\,I^\dagger = D_0 + D_{1/2}\ , & I\,I'^\dagger = 2D_{1/16}\ , \\ I^\dagger\,I = D_{++} + D_{--}\ , & I'^\dagger\,I = D_{+-} + D_{-+}\ , \\ I\,D_{++} = I\,D_{--} = I\ , & I\,D_{+-} = I\,D_{-+} = I'\ , \\ D_0\,I = D_{1/2}\,I = I\ , & D_{1/16}\,I = I'\ . \end{array}$$

Note that

$$I' I'^\dagger = 2D_0 + 2D_{1/2}$$

The "2" in front of D_0 means this is not a conventional elementary defect; it is possible that this counts one fermionic and one bosonic weight zero field on the defect.

The tricritical Ising model

We now repeat these for the tricritical Ising model:

- Construct a suitable fermionic theory
- Find defects in the fermionic theory
- Find interfaces between the TCIM and the fermionic theory
- Compose the interfaces and defects to get new defects in TCIM

The tri-critical Ising model (TCIM) is the unitary minimal model of the Virasoro algebra with $c = 7/10$. All the fields have integer spin.

The Virasoro algebra has six unitary highest weight representations with weights

$$0, \frac{3}{2}, \frac{1}{10}, \frac{3}{5}, \frac{7}{16}, \frac{3}{80}$$

TCIM has

- six primary fields
- six elementary topological defects
- six elementary conformal boundary conditions
- thirty-six elementary factorised defects

We want to relate it to a fermionic theory we call SVIR_3.

SVIR_3

The superVirasoro algebra has four unitary highest-weight representations. Two are Neveu-Schwarz and two are Ramond.

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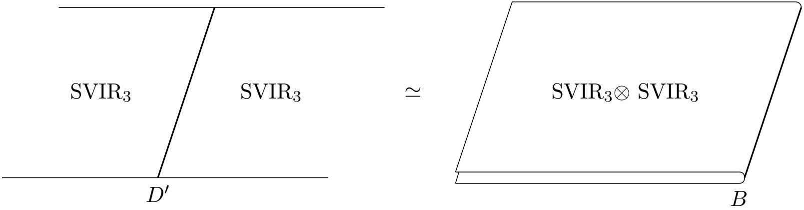
Its partition function can be written in Virasoro and superVirasoro characters as

$$\begin{aligned}&|\chi_0^{(4)}(q) + \chi_{3/2}^{(4)}(q)|^2 + |\chi_{1/10}^{(4)}(q) + \chi_{3/5}^{(4)}(q)|^2 \\ =& |\mathrm{ch}_0^3(q)|^2 + |\mathrm{ch}_{1/10}^3(q)|^2 \ .\end{aligned}$$

It is not fully modular invariant, but it is invariant under $\tau \to -1/\tau$ and $\tau \to \tau + 2$.

Defects as boundary conditions

By the folding trick, defects in SVIR_3 are boundary conditions for $\mathsf{SVIR}_3{}^{\otimes 2}$:



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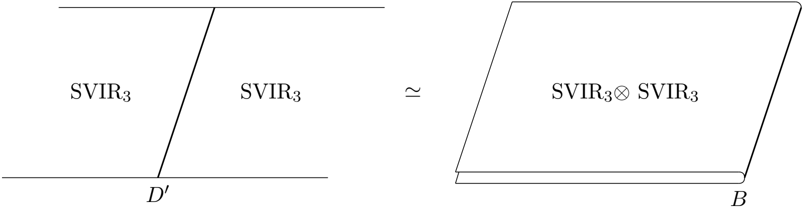

This hides quite a few subtleties

To obtain a defect from a boundary condition we need

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We choose

$$\begin{aligned}
\rho(G^1_m \| B\rangle\rangle) &= G_m \, \hat{D} \, , \\
\rho(\bar{G}^1_m \| B\rangle\rangle) &= \bar{G}_m \, \hat{D} \, , \\
\rho(G^2_m \| B\rangle\rangle) &= -i(-1)^{F+\bar{F}} \hat{D} (-1)^{F+\bar{F}} \bar{G}_{-m} \, , \\
\rho(\bar{G}^2_m \| B\rangle\rangle) &= i(-1)^{F+\bar{F}} \hat{D} (-1)^{F+\bar{F}} G_{-m} \, .
\end{aligned}$$

and two different maps:

$$\rho(|h_1\rangle \otimes |h_2\rangle) = |h_1\rangle\langle h_2|$$

$$\rho'(|h_1\rangle \otimes |h_2\rangle) = |h_1\rangle\langle h_2|(-1)^F$$

$\mathrm{SVIR}_3{}^{\otimes 2}$ is a theory with $c = 7/5$ and superconformal symmetry. There are again choices of four signs $\alpha\beta\gamma\delta$ to fix this embedding

$$\iota_{\alpha\beta\gamma\delta}(G_m) = \alpha G_m^1 + \beta G_m^2 \; , \;\; \iota_{\alpha\beta\gamma\delta}(\bar{G}_m) = \gamma \bar{G}_m^1 + \delta \bar{G}_m^2 \; .$$

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Given an embedding, the space of $\text{SVIR}_3{}^{\otimes 2}$ can be decomposed into representations of sVir_{10} which are labelled (r, s)

For example the square of the vacuum character

$$(\text{ch}_0^3)^2 \equiv (\text{ch}_{1,1}^3)^2 = \text{ch}_{1,1}^{10} + \text{ch}_{1,5}^{10} + \text{ch}_{1,7}^{10} + \text{ch}_{1,11}^{10}$$

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The weights are $h^{(10)}_{1,5} = 3/2$, $h^{(10)}_{1,7} = 7/2$, $h^{(10)}_{1,11} = 10$ and so the chiral algebra contains three super-primary fields $\mathcal{W}^{(3/2)}$, $\mathcal{W}^{(7/2)}$, and $\mathcal{W}^{(10)}$

more subtleties

- Each boundary condition for sVir_{10} has a gluing condition

$$(G_m + i\epsilon\bar{G}_{-m})|h,\epsilon\rangle\rangle = 0\ .$$

so that with the embedding $\iota_{\alpha\beta\gamma\delta}$, this state satisfies

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- Each topological defect has two sign choices

$$\left.\begin{array}{l} G_m D = \eta D G_m \\ \bar{G}_m D = \eta' D \bar{G}_m \end{array}\right\} \Rightarrow \begin{cases} (G^1_m + i\eta\bar{G}^2_{-m})\|B\rangle\rangle = 0 \\ (G^2_m + i\eta'\bar{G}^1_{-m})\|B\rangle\rangle = 0 \end{cases}$$

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- Each factorising defect has two sign choices

$$\left.\begin{array}{l} (G_m + i\eta\, \bar{G}_{-m})\|A,\eta\rangle\rangle\langle\langle B,\eta'\| = 0 \\ \|A,\eta\rangle\rangle\langle\langle B,\eta'\|(G_m - i\eta'\, \bar{G}_{-m}) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} (G^1_m + i\eta \bar{G}^1_{-m})\|B\rangle\rangle = 0 \\ (G^2_m - i\eta' \bar{G}^2_{-m})\|B\rangle\rangle = 0 \end{array}\right.$$

Two classes of embeddings

The result is that we cannot obtain all defects with one embedding.

- If $\alpha\beta\gamma\delta = 1$ then $\eta\eta' = 1$ for transmitting defects and $\eta\eta' = -1$ for factorising defects
- If $\alpha\beta\gamma\delta = -1$ then $\eta\eta' = -1$ for transmitting defects and $\eta\eta' = 1$ for factorising defects

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  These two sets will not be expressible in terms of sVir_{10} at the same time.
- This is what we find

The expressions of the highest weight states with respect to sVir_{10} also depends on the embedding.

For example, the state $|1,5\rangle$ of weight $(3/2,3/2)$ is given by

$$\iota_{\alpha\beta\gamma\delta}(|1,5\rangle) = \frac{i\eta_{1,5}}{4c/3}(\alpha G^1_{-3/2} - \beta G^2_{-3/2})(\gamma \bar{G}^1_{-3/2} - \delta \bar{G}^2_{-3/2})|0\rangle \ ,$$

where $c = 7/10$ and $\eta_{1,5}$ is a free sign,

The boundary states of Gang and Yamaguchi

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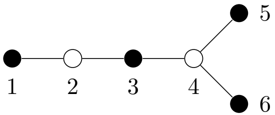
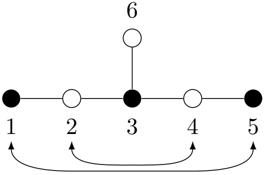
They are labelled by a pair of nodes from the Dynkin diagrams of D_6 and E_6. The diagrams are bi-coloured as here:



D_6



E_6
(Showing map $a \mapsto r(a)$)

The boundary states are

- $\|(a,b)_{NS}\rangle\rangle = \sqrt{2} \sum_{\substack{r\in\{1,3,5,5',7,9\} \\ s\in\{1,7\}}} \Psi^{(a,b)}_{(r,s)} \, |(r,s),+\rangle\rangle$
- $\|(a,b)_{\widetilde{NS}}\rangle\rangle = (-1)^F \|(a,b)_{NS}\rangle\rangle$

where $\psi^r_a(G)$ are eigenvectors of the adjacency matrices of the Dynkin diagram of G and $S^{(k)}_{rs}$ are the S-matrices for $su(2)$ characters at level k.

These states are invariant under $b \mapsto r(b)$, the symmetry of the E_6 diagram. This would be broken by the addition of Ramond sectors, as originally proposed by Gang and Yamaguchi.

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The two distinct sets of states are:

For (a,b) the same colour, we take $\alpha\beta\gamma\delta = -1$ and the map ρ'

For (a,b) opposite colour, we take $\alpha\beta\gamma\delta = 1$ and the map ρ

Properties of the defects

We can calculate easily two properties of the defects:

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- The entropy or $g - value$.

  This decreases along RG flows. The entropy of a superposition of defects is the sum of their energies.

Properties of the defects

We can calculate easily two properties of the defects:

- The entropy or $g - value$.

  This decreases along RG flows. The entropy of a superposition of defects is the sum of their energies.

- The transmission coefficient.

  This takes values $0 \leq T \leq 1$ in a unitary theory. It is 1 for a topological defect and 0 for a factorising defect. A non-topological non-factorising defect will have $0 < T < 1$.

We find that defects labelled $(a, 1)$ and $(a, 6)$ are either topological or factorising; those labelled $(a, 2)$ and $(a, 3)$ are neither. The defect of lowest entropy is $(1, 2)$ with $g = 1.015..$ and $T = 0.633..$

We can identify all the topological and factorising defects in terms of superconformal topological and factorising defects for SVIR_3.

We found a complete set of eight elementary defects,

$$D_I,\ D_\varphi,\ \sqrt{2}(-1)^F D_I,\ \sqrt{2}(-1)^F D_\varphi,$$

$$\sqrt{2}(-1)^{\bar{F}} D_I,\ \sqrt{2}(-1)^{\bar{F}} D_\varphi,\ (-1)^{F+\bar{F}} D_I,\ (-1)^{F+\bar{F}} D_\varphi$$

and a complete set of four superconformal boundary states

$$\|I_{NS}\rangle\!\rangle,\ \|\varphi_{NS}\rangle\!\rangle = D_\varphi \|I_{NS}\rangle\!\rangle,$$

$$\|I_R\rangle\!\rangle = \sqrt{2}(-1)^F \|I_{NS}\rangle\!\rangle,\ \|\varphi_R\rangle\!\rangle = \sqrt{2}(-1)^F \|\varphi_{NS}\rangle\!\rangle$$

Explicitly

$$D_I = (P_0 + P_{3/2})(\bar{P}_0 + \bar{P}_{3/2}) + (P_{1/10} + P_{3/5})(\bar{P}_{1/10} + \bar{P}_{3/5}) \ ,$$

$$D_\varphi = \tfrac{1+\sqrt{5}}{2}(P_0+P_{3/2})(\bar{P}_0+\bar{P}_{3/2})+\tfrac{1-\sqrt{5}}{2}(P_{1/10}+P_{3/5})(\bar{P}_{1/10}+\bar{P}_{3/5})$$

and

$$\|I_{NS}\rangle\!\rangle = \left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{4}} |0,+\rangle\!\rangle + \left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{4}} |\tfrac{1}{10},+\rangle\!\rangle$$

e.g. we could identify

- The identity defect

$$D_I = \rho'_{++++}(\|(2,1)_{NS}\rangle\rangle)$$

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- The identity defect

$$D_I = \rho'_{++++}(\|(2,1)_{NS}\rangle\rangle)$$

- The factorising defect

$$\|I_{NS}\rangle\rangle\langle\langle I_{NS}\| = \rho_{+++-}(\|(1,1)_{NS}\rangle\rangle)$$

These two defects correspond to different embeddings and so the overlap of the two states $\|(2,1)_{NS}\rangle\rangle$ and $\|(1,1)_{NS}\rangle\rangle$ has no meaning and does not represent a trace.

Interfaces

To complete the construction we need interfaces between the tricritical Ising model and SVIR_3.

We find that the elementary interface has the simple form

$$I = \sqrt{2}(P_0\bar{P}_0 + P_{3/2}\bar{P}_{3/2} + P_{1/10}\bar{P}_{1/10} + P_{3/5}\bar{P}_{3/5})$$

It has the following algebra

$$I\,I^\dagger = D_0 + D_{3/2}\ , \quad I^\dagger\,I = D_I + (-1)^{F+\bar{F}} D_I\ ,$$

$$D_{3/2}\,I = I \cdot (-1)^{F+\bar{F}} = I\ .$$

Defects in TCIM

We can now consider defects in TCIM using the construction

$$D_{TCIM} = I \cdot D_{\mathrm{SVIR}_3} \cdot I^\dagger$$

The T and g values of D_{TCIM} are easy to find:

$$T(\hat{D}_{TCIM}) = T(\hat{D}_{\mathrm{SVIR}_3}) \ , \ \ g(\hat{D}_{TCIM}) = 2g(\hat{D}_{\mathrm{SVIR}_3}) \ .$$

We have been able to identify the images of all the factorising and topological defects. eg

$$\begin{aligned}
&I \cdot D_1 \cdot I^\dagger = D_0 + D_{3/2} \ , \\
&I \cdot D_\varphi \cdot I^\dagger = D_{1/10} + D_{3/5} \ , \\
&I \cdot \sqrt{2}(-1)^F \cdot I^\dagger = 2D_{7/16} \ , \\
&I \cdot \sqrt{2}(-1)^F D_\varphi \cdot I^\dagger = 2D_{3/80} \ , \\
&I \cdot |I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}| \cdot I^\dagger = \big(|B_0\rangle\!\rangle + |B_{3/2}\rangle\!\rangle\big)\left(\langle\!\langle B_0| + \langle\!\langle B_{3/2}|\right)
\end{aligned}$$

New defects

We have also found a new non-topological non-factorising defect.

The boundary state $\|\rangle(1,2)_{NS}$ leads to a defect in TCIM with

$$T = \frac{3-\sqrt{3}}{2} = 0.633.. \,, \quad g = (1+\sqrt{3})\sqrt{\frac{\sqrt{5}-1}{\sqrt{5}}} = 2.031..$$

This last g value is not the sum of the g-values of any combination of known topological or factorising defects.

Overlaps of the new defects

We cannot easily find the overlaps of the new defects. Even the overlaps of the Ishibashi states are hard to find.

As an example, consider the Ishibasi state $|(1,1)+\rangle\rangle$ for sVir_{10}.

This overlap with itself the character

$$\mathrm{ch}^{10}_{(1,1)}(q) = q^{-7/120}\,[1+q^{3/2}+q^2+q^{5/2}+q^3+2q^{7/2}+3q^4+3q^{9/2}+3q^5+...]$$

However, when the interfaces are included, up to a factor of two the overlap is

$$q^{-7/120}\,[1+\tfrac{1}{2}q^{3/2}+q^2+\tfrac{1}{2}q^{5/2}+q^3+q^{7/2}+\tfrac{49123}{17689}q^4+$$

$$+\tfrac{3}{2}q^{9/2}+\tfrac{49123}{17689}q^5+\tfrac{5}{2}q^{11/2}+\tfrac{102941115}{16999129}q^6+...]$$

Relation to W-algebras

There are three chiral superprimary fields in ${\text{SVIR}_3}^{\otimes 2}$ of weights $(3/2)$, $7/2$ and 10.

The boundary states we used preserve the super W-algebra $SW(3/2, 10)$, the extension of the superVirasoro algebra by the field of weight 10.

There are eight representations of this W-algebra in ${\text{SVIR}_3}^{\otimes 2}$, four of which admit four choices of gluing conditions and four which allow two choices, giving 24 Ishibashi states in all.

Hence it seems that the boundary states we have used are exactly those that preserve the $SW(3/2, 10)$ algebra.

Conclusions

We have given convincing evidence for non-topological non-factorising conformal defects in TCIM

We have not been able to find the partition function owing the breaking of superconformal symmetry by the Interface operators.

We are currently calculating the perturbative corrections to the transmission coefficients in minimal models to compare with the values found here.

We would like to look further into the use of extended algebras to construct new conformal defects.

It would be interesting to derive the boundary conditions for the free fermion and SVIR_3 from first principles or using the methods of Novak and Runkel.

Boundary conditions for the free fermion

Looking just at the NS sector it is impossible to find a set of boundary states which strictly solve Cardy's constraint.

The best one can do is find two states $\|A\rangle\!\rangle$ and $\|B\rangle\!\rangle$ whose mutual overlaps can be expressed as sums of characters but for which only $\|A\rangle\!\rangle$ is elementary; $\|B\rangle\!\rangle$ has two identity operators.

To be precise, we can take

$$\|A\rangle\!\rangle = |NS,+\rangle\!\rangle \ , \quad \|B\rangle\!\rangle = \sqrt{2}|NS,-\rangle\!\rangle$$

Then

$$\begin{aligned} Z_{AA} &= \langle\!\langle A\|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}\|A\rangle\!\rangle = \chi_{NS}(\tilde{q}) \ , \\ Z_{AB} &= \langle\!\langle A\|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}\|B\rangle\!\rangle = 2\chi_{R}(\tilde{q}) \ , \\ Z_{BB} &= \langle\!\langle B\|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}\|B\rangle\!\rangle = 2\chi_{NS}(\tilde{q}) \ . \end{aligned}$$

This agrees with the analysis of Ghoshal and Zamolodchikov in which the free fermion is stated to have two boundary conditions, "free" and "fixed" distinguished by having two and one ground states respectively, with the "fixed" boundary condition having a two-dimensional space of weight zero fields, one bosonic and one fermionic.

[Note that this is not the same as the usual naming conventions for the Ising model, for which "free" and "fixed" would have one and two ground states respectively, but since these are interchanged by duality this should not be of concern]

This fermionic weight zero field also arises in the Ising model, where the "ϵ" defect can end at the junction of a $(+)$ and and a $(-)$ boundary condition in a field of weight zero. Such a space with one bosonic and one fermionic degree of freedom, $\mathbb{C}^{1|1}$, also arises when constructing spin fields in the Ising model by Novak and Runkel, and we think it is probable that a treatment of boundary conditions in their manner will give our result.