CFTs on real projective spaces

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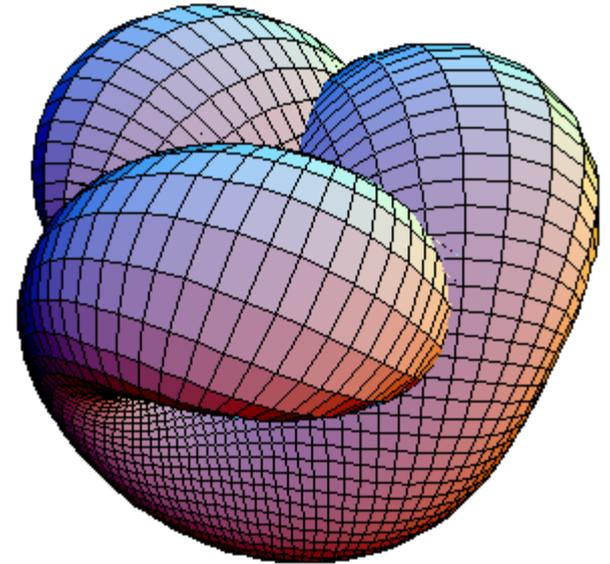
In part, collaboration with H. Ooguri
arXiv:1605.00334 1507.04130

and C. Hasegawa **1611.06373**

Never, never, never, never give up

- I'm not talking about boundaries nor defects

- But closely related, I hope.

Boy's surface: immersion of RP2 in three dimensions.
Obtained from the assignment by Hilbert that
he should prove RP2 cannot be immersed in three dimensions.

How much can almighty do?

Suppose we know all the CFT data on flat space-time, how much can we solve CFTs on non-trivial space-time (w or w/o boundaries)?

- Action in curved space-time is defined up to some finite(?) choices (curvature coupling, counter-terms etc...)

- I even don't know how much correlation functions are determined on T^d from bulk CFT data
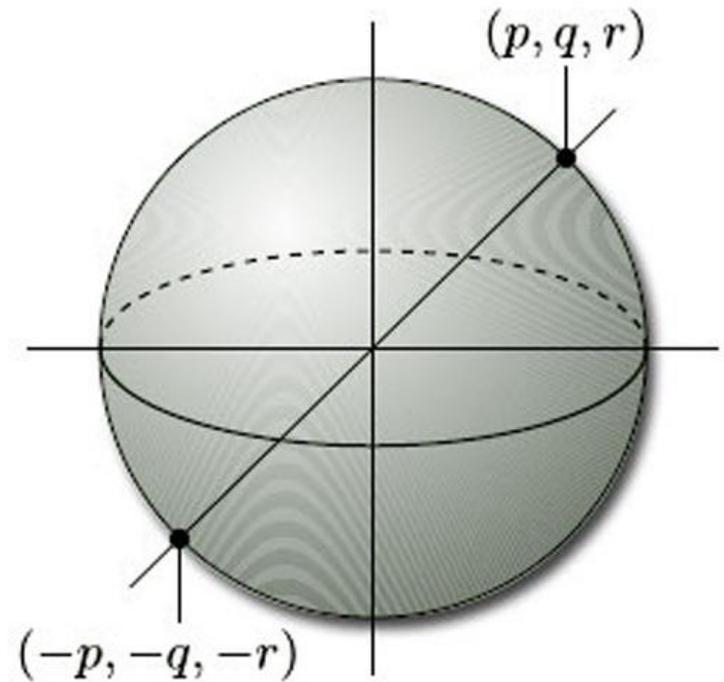- I'll rather try RP^d

Real Projective Space 1

- Take d-dim sphere.
- Identify the antipodal points

- No fixed points

  (no-local defects)
- Locally conformal flat → SO(d,1) symm
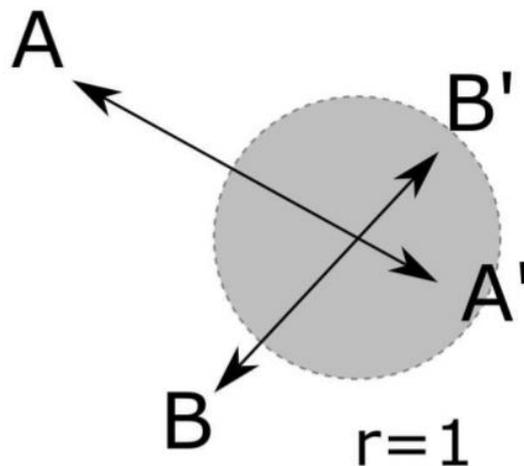

- When d=even, non-orientable (un-oriented strings, time-reversal anomaly…)

Real Projective Space 2

- Conformally map to a plane

- Involution: $\vec{x} \rightarrow -\dfrac{\vec{x}}{\vec{x}^2}$

A

B'

B

A'

r=1

- Operator identification

$$O_i(\vec{x}) \sim O_i\left(-\dfrac{\vec{x}}{\vec{x}^2}\right)$$

- Up to choice of involution and conformal factor $(\vec{x}^2)^{-\Delta}$

Real Projective Space 3

- Conformally map to a cylinder $R_t \times S^{d-1}_\Omega$

- Involution:
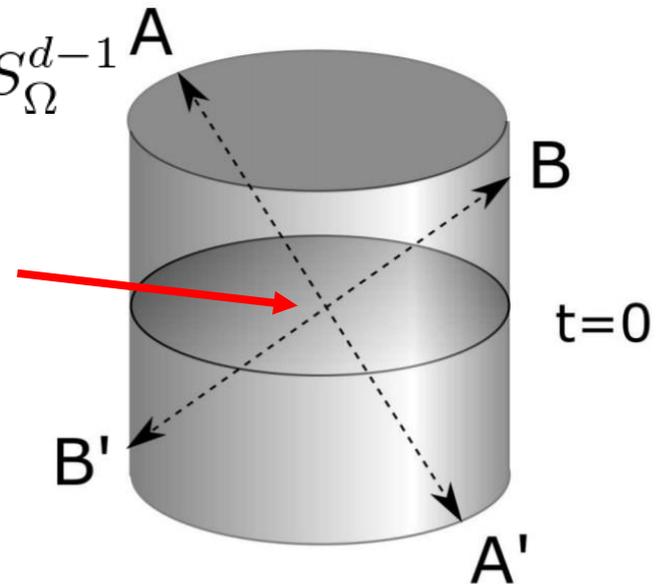$$(t, \vec{\Omega}) \rightarrow (-t, -\vec{\Omega})$$

- PT transformation

- Will define crosscap state $|C\rangle$

$$M_{ab}|C\rangle = 0$$

$$(P_a + K_a)|C\rangle = 0$$

- If we fill in the cylinder (c.f. AdS space), then we have a fixed point! $\quad (\rho, t, \vec{\Omega}) \rightarrow (\rho, -t, -\vec{\Omega})$
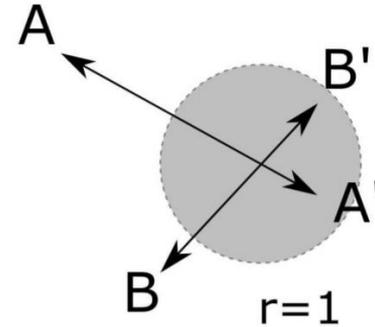
- → Relation between crosscap state and a bulk point in AdS (YN-Ooguri, Takayanagi et al, Verlinde et al)

CFTs on real projective space

- New CFT data

- Scalar one-point function

$$\langle O_i(\vec{x}) \rangle = \frac{A_i}{(1 + \vec{x}^2)^{\Delta_i}}$$

- With bulk CFT data, all the correlation functions are fixed

- Crosscap states (Cardy from Ishibashi)

$$|C\rangle\rangle = \sum_i A_i \Gamma(\Delta_i - d/2 + 1) \left(\frac{\sqrt{p^2}}{2}\right)^{\frac{d}{2} - \Delta_i} J_{\Delta_i - \frac{d}{2}}(\sqrt{p^2})|O_i\rangle$$

- Global Ishibashi condition is explicitly solved (=AdS bulk-boundary propagator)

Bootstrap on real projective space

> Suppose we know all the CFT data on flat space-time, how much can we solve CFTs on real projective space?
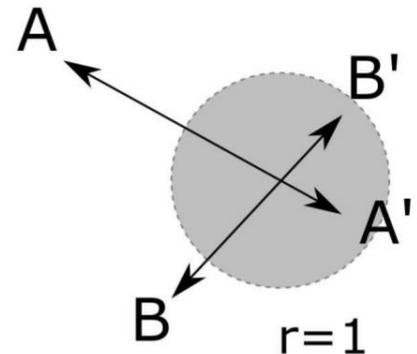
- Consider two-point functions:

$$\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\rangle = \frac{(1+\vec{x}_1^2)^{\frac{-\Delta_1+\Delta_2}{2}}(1+\vec{x}_2^2)^{\frac{-\Delta_2+\Delta_1}{2}}}{(\vec{x}_1-\vec{x}_2)^{2(\frac{\Delta_1+\Delta_2}{2})}}G_{12}(\eta) \ ,$$

$$\eta = \frac{(\vec{x}_1-\vec{x}_2)^2}{(1+\vec{x}_1^2)(1+\vec{x}_2^2)}$$

- Crossing symmetry $\quad O_i(\vec{x}) \sim O_i(-\frac{\vec{x}}{\vec{x}^2})$

$$\left(\frac{1-\eta}{\eta^2}\right)^{\frac{\Delta_1+\Delta_2}{6}} G_{12}(\eta) = \left(\frac{\eta}{(1-\eta)^2}\right)^{\frac{\Delta_1+\Delta_2}{6}} G_{12}(1-\eta) \ .$$

Exactly solvable Virasoro case (d=2)

- One may happily realize the bootstrap philosophy in d=2 minimal models

- We do NOT know Virasoro crosscap state in general, but we do know them in minima models

- Example: Virasoro OPE in Ising model

$$[\sigma] \times [\sigma] = [1] + [\epsilon]$$

- Virasoro block decomposition

$$G_{\sigma\sigma}(\eta) = (1-\eta)^{3/8} {}_2F_1(\frac{3}{4}, \frac{1}{4}; \frac{1}{2}; \eta) + c_I \eta^{1/2}(1-\eta)^{3/8} {}_2F_1(\frac{3}{4}, \frac{5}{4}; \frac{3}{2}; \eta),$$

- Demanding crossing equation → $c_I = \dfrac{\sqrt{2}-1}{2} = 0.20711$

Strategy in the other dimensions

- Global conformal blocks are known

$$G_{12}(\eta) = \sum_i C_{12i} A_i \eta^{\Delta_i/2} \, _2F_1\left(\frac{\Delta_1 - \Delta_2 + \Delta_i}{2}, \frac{\Delta_2 - \Delta_1 + \Delta_i}{2}; \Delta_i + 1 - \frac{d}{2}; \eta\right) \, .$$

- C.f. AdS bulk-boundary-boundary 3pt function (geodesic Witten block)

$$\left(\frac{1-\eta}{\eta^2}\right)^{\frac{\Delta_1 + \Delta_2}{6}} G_{12}(\eta) = \left(\frac{\eta}{(1-\eta)^2}\right)^{\frac{\Delta_1 + \Delta_2}{6}} G_{12}(1-\eta) \, .$$

- We need infinite sum (except $\Delta = d/2 - 1$) to implement crossing at $\eta = 1$

  Idea of proof: use hypergeometric identity $\eta \to 1 - \eta$

  All the terms begin with $(1-\eta)^{d/2-1}$

- But convergence is exponentially fast at $\eta = 1/2$

Truncated bootstrap

- Global conformal blocks are known

$$G_{12}(\eta) = \sum_i C_{12i} A_i \eta^{\Delta_i/2} {}_2F_1\left(\frac{\Delta_1 - \Delta_2 + \Delta_i}{2}, \frac{\Delta_2 - \Delta_1 + \Delta_i}{2}; \Delta_i + 1 - \frac{d}{2}; \eta\right) .$$

- Truncate the sum over low-lying spectrum (with hopefully known scaling dimensions)

$$\sigma \times \sigma = 1 + \epsilon + \epsilon' + \epsilon'' + \cdots$$

- Impose crossing around $\eta = 1/2$

- Determine $C_{12i} A_i$

- If works, then our philosophy is good

Experiments in d=2

- Global conformal blocks are known

$$G_{12}(\eta) = \sum_i C_{12i} A_i \eta^{\Delta_i/2}\,_2F_1 \left(\frac{\Delta_1 - \Delta_2 + \Delta_i}{2}, \frac{\Delta_2 - \Delta_1 + \Delta_i}{2}; \Delta_i + 1 - \frac{d}{2}; \eta \right) \ .$$

- Truncate the sum over low-lying spectrum up to level n (spectrum is exactly known)

$$\sigma \times \sigma = 1 + \epsilon + \epsilon' + \epsilon'' + \cdots$$

- Impose crossing around $\eta = 1/2$

- Determine $C_{12i} A_i$

	Exact	2	3	4	5
$C_{\sigma\sigma\epsilon} A_\epsilon$	0.20711	0.20407	0.20757	0.20693	0.20710
$C_{\sigma\sigma\epsilon'} A_{\epsilon'}$	0.01563	0.01702	0.01539	0.01572	0.01563

Predictions in d=3

- Let's do it in d=3

$$G_{12}(\eta) = \sum_i C_{12i} A_i \eta^{\Delta_i/2} {}_2F_1\left(\frac{\Delta_1 - \Delta_2 + \Delta_i}{2}, \frac{\Delta_2 - \Delta_1 + \Delta_i}{2}; \Delta_i + 1 - \frac{d}{2}; \eta\right) \ .$$

- Truncate the sum over low-lying spectrum up to level n (use bulk bootstrap to get spectrum)

Set B: $\Delta_\sigma = 0.518151$, $\Delta_\epsilon = 1.4126605$, $\Delta_{\epsilon'} = 3.83034$, $\Delta_{\epsilon''} = 6.9994$, $\Delta_{\epsilon^3} = 10.88$, $\Delta_{\epsilon^4} = 15.56$, $\Delta_{\epsilon^5} = 22.37$

Set S: $\Delta_\sigma = 0.518149$, $\Delta_\epsilon = 1.412625$, $\Delta_{\epsilon'} = 3.82968$, $\Delta_{\epsilon''} = 6.8956$, $\Delta_{\epsilon'''} = 7.2535$,

- Impose crossing around $\eta = 1/2$

- Determine $C_{12i} A_i$

	(2,0)	(4,0)$_\mathrm{A}$	(4,0)$_\mathrm{B}$	(6,0)$_\mathrm{B}$	(4,0)$_\mathrm{S}$
$C_{\sigma\sigma\epsilon} A_\epsilon$	0.690	0.7015	0.7022	0.70197	0.6908
$C_{\sigma\sigma\epsilon'} A_{\epsilon'}$	0.054	0.0475	0.0470	0.04714	0.0549

Did it work a priori?

- Accuracy in d=2 is amazing

- Accuracy in d=3 should be reasonably good(?)

- Caution: there is a case in which this method (alone) should not work

- Free O(N) theory: Singlet spectrum is same for all N but CA depends on N, so one should not be able to get the unique answer in this way (alone)

- Degeneracy in spectrum makes the computation difficult in general

How did it work in ϵ expansions

- We can analytically solve crosscap bootstrap equation in $d = 4 - \epsilon$ with finite number of OPE summation

$$G(\eta) = 1 + (1 - a\epsilon)\eta^{1 - \frac{\epsilon}{2} + a\epsilon} {}_2F_1\left(1 - \frac{\epsilon}{2} + a\epsilon, 1 - \frac{\epsilon}{2} + a\epsilon, 2 - \epsilon + 2a\epsilon + 1 - \frac{4 - \epsilon}{2}; \eta\right)$$

$$+ \frac{a\epsilon}{2}\eta^{2 - \epsilon} {}_2F_1\left(2 - \epsilon, 2 - \epsilon, 4 - 2\epsilon + 1 - \frac{4 - \epsilon}{2}; \eta\right) + O(\epsilon^2)$$

- Satisfies crosscap bootstrap equation

$$\left(\frac{1 - \eta}{\eta^2}\right)^{\frac{\Delta_1 + \Delta_2}{6}} G_{12}(\eta) = \left(\frac{\eta}{(1 - \eta)^2}\right)^{\frac{\Delta_1 + \Delta_2}{6}} G_{12}(1 - \eta) \ .$$

- Can be checked by using

$${}_2F_1(a, b; c; \eta) = (1 - \eta)^{c - a - b} {}_2F_1(c - a, c - b; c; \eta)$$

$$= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} {}_2F_1(a, b; a + b + 1 - c; 1 - \eta) + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}(1 - \eta)^{c - a - b} {}_2F_1(c - a, c - b; 1 + c - a - b; 1 - \eta)$$
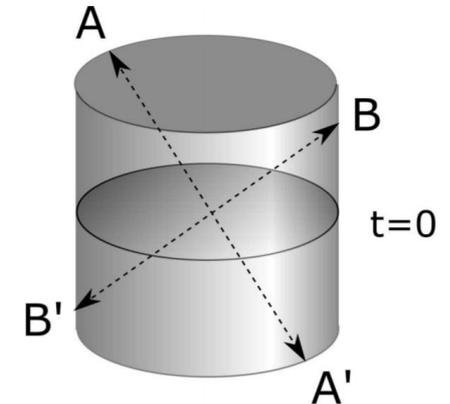
- We must provide $\gamma_{\phi^2} = 2a\epsilon$ (e.g. for O(N) theory $a = \frac{1}{2}\frac{N + 2}{N + 8}$)
- This worked because $\gamma_\phi = O(\epsilon^2)$
- Otherwise we needed infinite tower of OPE to satisfy crossing exactly (e.g. in $d = 6 - \epsilon$ case)

So, finally let's talk about defects (with future directions to pursue)

Bulk interpretation of conformal blocks

- (Global) Ishibashi state is bulk local operator in AdS

$$M_{ab}|C\rangle = 0$$
$$(P_a + K_a)|C\rangle = 0$$

- It has the same symmetry property
  as a point in AdS

- It is solved by

$$|C\rangle\rangle = \sum_i A_i \Gamma(\Delta_i - d/2 + 1) \left(\frac{\sqrt{p^2}}{2}\right)^{\frac{d}{2} - \Delta_i} J_{\Delta_i - \frac{d}{2}}(\sqrt{p^2})|O_i\rangle$$

- This can be written as a integral expression

$$|C\rangle\rangle_\phi = \hat{O}(z, x)|0\rangle = \int d^d x\, K(z, x|x')\phi(x')|0\rangle$$

- Known as KL(L) bulk reconstruction
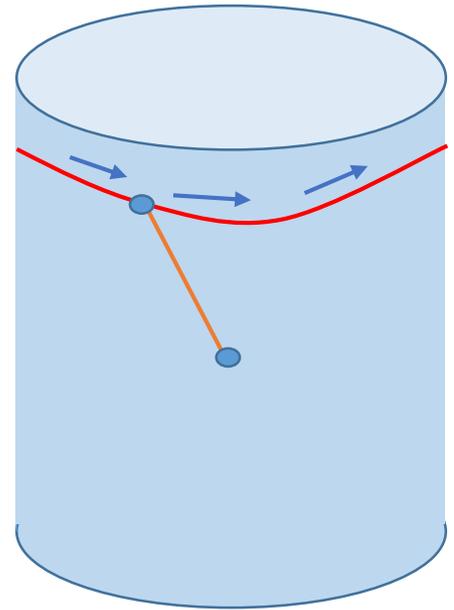
Bulk interpretation of conformal blocks

- Crosscap conformal block = Geodesic Witten diagram

$$\langle \phi(x)\phi(y)|C\rangle\rangle_\phi \sim G^\phi_{\phi\phi}(\eta)$$

$$|C\rangle\rangle = \sum_i A_i \Gamma(\Delta_i - d/2 + 1) \left(\frac{\sqrt{p^2}}{2}\right)^{\frac{d}{2}-\Delta_i} J_{\Delta_i - \frac{d}{2}}(\sqrt{p^2})|O_i\rangle$$
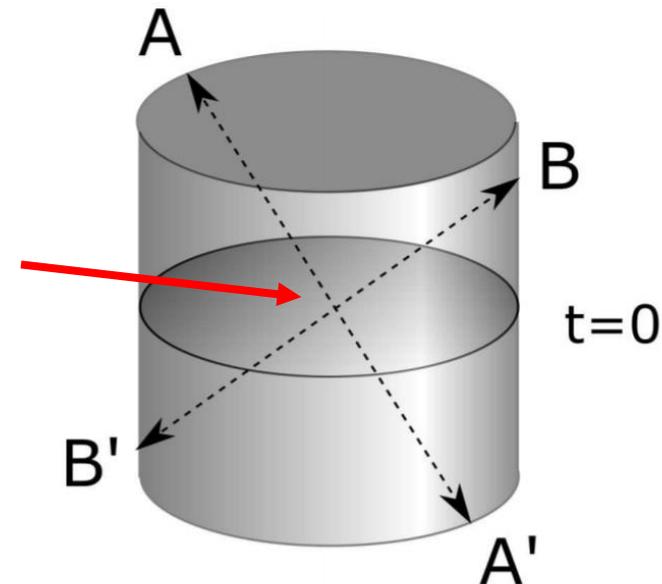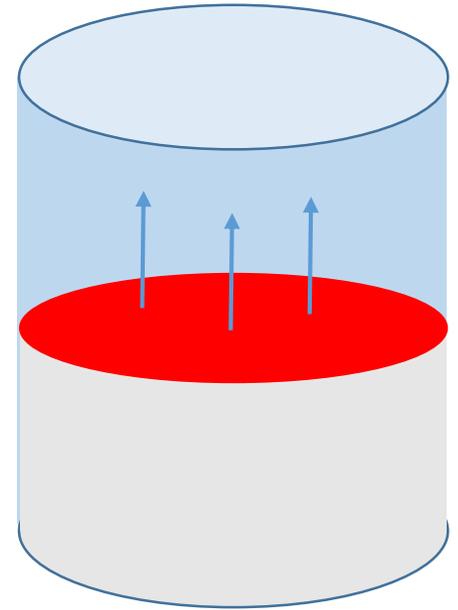
- Tree level bulk three-point function

  integrated only over geodesics

$$G(\eta) = \int_{\text{geodesics}} dl\, G(l|\rho = 0, \vec{0})$$

What happens at the center?

- (Cardy) Boundary states

= Beginning of AdS Universe

= Quench in CFT (yet another Cardy conj)

- (Cardy) Crosscap states

 = Bulk space-time with singularity or bounce

 = CFT interpretation?

- Any local degrees of freedom?
- Quantum gravity well-defined?
- We know Liouville crosscap state
- Any AdS3 interpretation?

Summary

- CFT on real projective space may be solved once we know CFT data on flat space-time

- How about N=4 SYM?

- Bulk interpretation?

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