RG interfaces and boundaries in Ising field theory
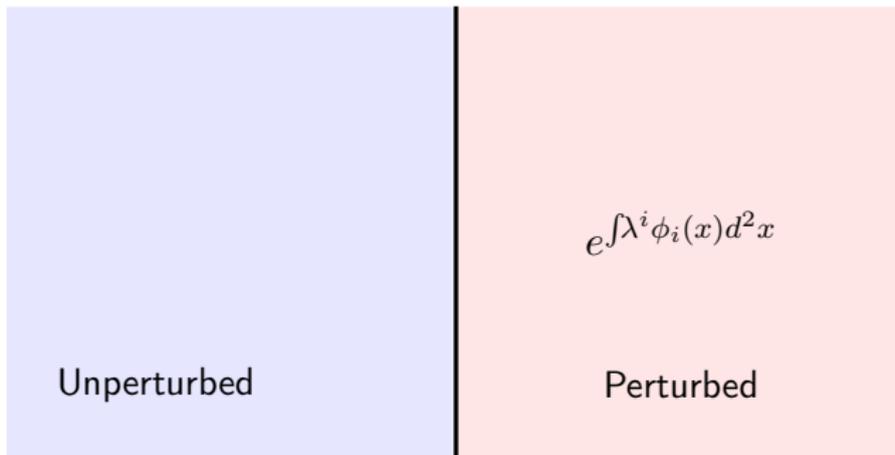
Anatoly Konechny

Heriot-Watt University

September 7, 2017

Boundary and Defect CFT: Open Problems and Applications,
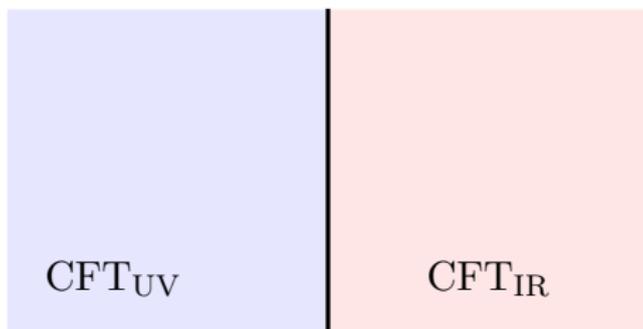Chicheley Hall

- Motivation and background.
- The results at a glance
- Details

We are interested in RG flows of 2D Euclidean QFTs. We would like to have a tool for establishing a global picture of flows that start from a given UV fixed point. An interesting object called RG interface was proposed by I. Brunner and D. Roggenkamp (2007). If the flow is triggered by a perturbation $\Delta S = \int d^2x\, \lambda^i \phi_i(x)$ of the UV fixed point one can consider putting this perturbation on a half plane and letting it flow with the renormalization group flow.

$$e^{\int \lambda^i \phi_i(x) d^2x}$$

Unperturbed

Perturbed

In the far infrared, if the RG flow end up in a non-trivial fixed point, we obtain a conformal interface that is a 1-dimensional object separating two conformal field theories and respecting the conformal symmetry.

$$\mathrm{CFT_{UV}} \qquad \qquad \mathrm{CFT_{IR}}$$

This object is conformal that imposes lots of restrictions on it and at the same time it must carry some information about RG flow between $\mathrm{CFT_{UV}}$ and $\mathrm{CFT_{IR}}$.
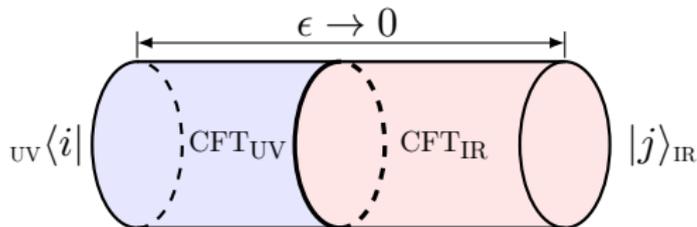
If we could somehow describe what special properties distinguish RG interfaces amongst all conformal interfaces we could potentially obtain selection rules of the type — there are no RG flows between a given pair of CFTs (as there are no suitable conformal interfaces between them). Various conformal interfaces between particular CFTs have been constructed in the literature. However even between the simplest CFTs, such as Virasoro minimal models, we do not have a description of *all* conformal interfaces (except for a handful of very special cases one of which I will discuss in detail later).

For flows to a trivial fixed point we obtain a conformal boundary condition which I call an RG boundary. The space of all massive flows then breaks up into regions according to their RG boundaries. We can think of the RG boundaries as labels of infrared phases of massive theories arising from the same UV fixed point.

For Virasoro minimal models we know all conformal boundary conditions. There are as many fundamental boundary conditions as there are Virasoro primary states in the spectrum. Any conformal boundary condition can be represented as a superposition of the fundamental ones. I will investigate how the space of massive flows that start at the critical Ising model breaks up according to their RG boundaries.

What quantities are associated with conformal interfaces and how can we determine them for a given RG flow?
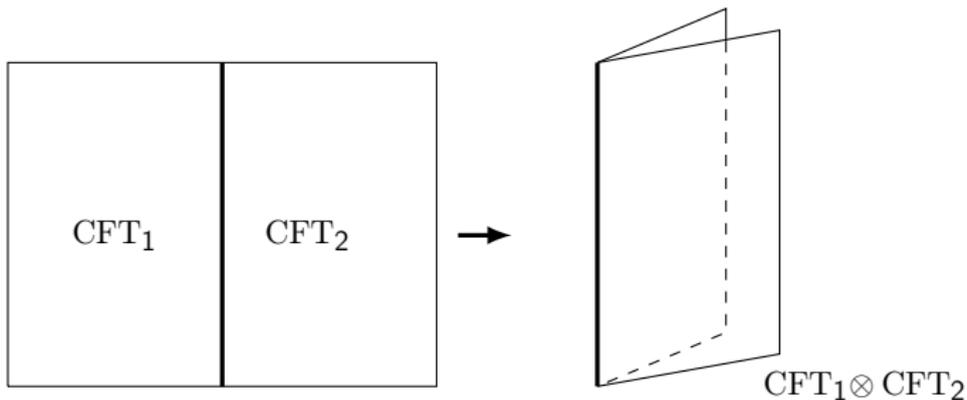
Conformal interfaces, being put on a circle, give pairings between states in the radial quantization in the two theories.

$$_{\text{UV}}\langle i | j \rangle_{\text{IR}}$$

Folding along the interface line we obtain a conformal boundary condition in $\mathrm{CFT}_1 \otimes \mathrm{CFT}_2$.

$$\mathrm{UV}\langle i|j\rangle_{\mathrm{IR}} = \langle \phi_i^{\mathrm{UV}} \phi_j^{\mathrm{IR}} \rangle$$

– a 1-point function on a half space.

For massive flows only the vacuum state survives and gives rise to components

$$_{\mathrm{UV}}\langle i|0\rangle_{\mathrm{IR}}$$

that are its overlaps with states in the UV theory. This gives a representation of the perturbed vacuum in the far infrared as a conformal boundary state in $\mathrm{CFT}_{\mathrm{UV}}$. This observation was made for particular perturbations solvable by a Bogolyubov transformation in P. Calabrese, F. Essler, M. Fagotti (2012) and observed in numerical solutions by G. Takács (2012).

TCSA

Truncated Conformal Space Approach (TCSA) was invented by V. Yurov and Al. Zamolodchikov in 1991. Consider perturbed Hamiltonian on a cylinder of radius R

$$H = \frac{2\pi}{R} \left(L_0 + \bar{L}_0 - \frac{c_{\mathrm{UV}}}{12} \right) + \lambda^i \int dx \, \phi_i(x, 0) \,,$$

restricted to a truncated finite-dimensional state space
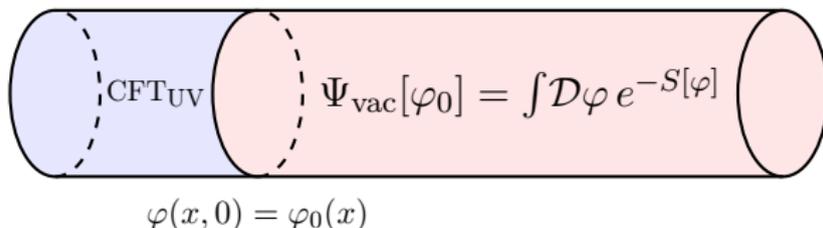
$$\Delta + \bar{\Delta} \leq 2n_c$$

The eigenvectors of H are found in the $\mathrm{CFT}_{\mathrm{UV}}$ state space. If we approach an IR fixed point the eigenvalues of H interpolate between Δ_{UV} and Δ_{IR}. The components of the corresponding eigenvectors give pairings

$$_{\mathrm{UV}}\langle i|j \rangle_{\mathrm{IR}}$$

In TCSA there is no canonical way to normalise the truncated eigenvectors so we consider component ratios

$$\Gamma^j_{i,k} = \frac{{}_{\mathrm{UV}}\langle i|j\rangle_{\mathrm{IR}}}{{}_{\mathrm{UV}}\langle k|j\rangle_{\mathrm{IR}}}$$

This pairing is the same as the RG interface pairing introduced before. This follows from the path integral representation of perturbed wave functionals in interaction representation.

$$\varphi(x,0) = \varphi_0(x)$$

Ising Field Theory

2D Ising field theory can be described as the critical theory that is the free massless fermion theory perturbed by temperature (fermion mass) and magnetic field.

$$S_{\mathrm{IFT}} = \frac{1}{2\pi} \int (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + im\bar{\psi}\psi) \, d^2 x + h \int \sigma \, d^2 x$$

For real couplings all RG flows are massive. For imaginary magnetic field the parameters can be fine-tuned to arrive at a non-trivial fixed point – Yang-Lee model that is a non-unitary Virasoro minimal model. The RG trajectories can be labelled by a dimensionless parameter

$$y = \frac{m}{|h|^{8/15}}$$

The flow to Yang-Lee model was found to occur at $y = y_{\mathrm{cr}} \approx -2.429$ P. Fonseca and A. Zamolodchikov (2001)

As shown by J. Cardy there are 3 fundamental conformal boundary conditions in the critical Ising model. They correspond to having boundary spin free or holding it fixed. In the letter case there are two way to do it: up or down. The corresponding conformal boundary states are

$$|\pm\rangle\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle\rangle + |\epsilon\rangle\rangle \pm 2^{1/4}|\sigma\rangle\rangle\right], \qquad |F\rangle\rangle = |0\rangle\rangle - |\epsilon\rangle\rangle$$
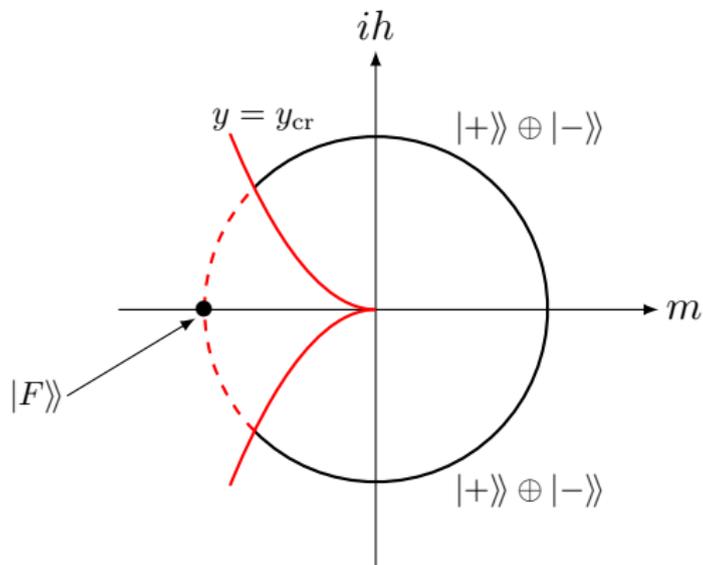
All other conformal boundary states are obtained as superpositions of these three boundary states, e.g.

$$|+\rangle\rangle \oplus |-\rangle\rangle$$

RG boundaries for IFT with imaginary magnetic field.

Everywhere on the dashed red line there is no limit except for the
$h = 0$ point. This includes the $y = y_{\mathrm{cr}}$ trajectory flowing to
Yang-Lee theory. Everywhere on the solid black line the RG
boundary is $|+\rangle\!\rangle \oplus |-\rangle\!\rangle$.

Now we pass to discussing some details of these results and some details of how these results were obtained.

The zero magnetic field case can be solved analytically via Bogolyubov's transformation between massless and massive fermonic creation and annihilation operators. The vacuum for massive fermions is represented as a state $|0\rangle_m$ in the Fock space of massless fermions. When $m < 0$ and $\nu = |m|R \to \infty$ we have (up to normalisation)

$$|0\rangle_m \to |F\rangle\rangle \quad \text{that means that} \quad |\text{RG}\rangle\rangle = |F\rangle\rangle .$$

For $m > 0$ the vacua in the NS and Ramond sectors asymptotically approach the same energies so that we get a degenerate vacuum subspace that is spanned by the two fixed spin boundary states

$$|\text{RG}\rangle\rangle = |+\rangle\rangle \oplus |-\rangle\rangle .$$

In both cases we obtain an analytic expression for the ratio of
components of the vacuum vector $|v_0\rangle$

$$\Gamma_\epsilon^0 = \frac{\langle\epsilon|v_0\rangle}{\langle 0|v_0\rangle} = \mathrm{sign}(m)\Big[\sqrt{1 + \Big(\frac{\pi}{\nu}\Big)^2} - \frac{\pi}{\nu}\Big]$$

Figure: The component ratio Γ_ϵ^0 for zero magnetic field, $m > 0$

$g(\nu)$ approaches its asymptotic value $g = 1$ very slowly. The function becomes larger than 0.9 past $\nu = 30$ and larger than 0.99 past $\nu = 313$. This presents a challenge for numerical calculation of the asymptotic value. Nevertheless TCSA numerics give reasonable results for generic perturbations. We present plots of the ratios

$$\Gamma_\epsilon^0 = \frac{\langle\epsilon|v_0\rangle}{\langle 0|v_0\rangle}\,, \qquad \Gamma_\sigma^0 = \frac{\langle\sigma|v_0\rangle}{\langle 0|v_0\rangle}$$

against the dimensionless scale

$$r = R|h|^{8/15}$$

$y = 3$ low temperature with magnetic field

The asymptotic values corresponding to the fixed boundary condition are

$$\Gamma_\epsilon^0(F) = 1\,, \qquad \Gamma_\sigma^0(F) = 2^{1/4} \approx 1.189$$

$y = -3$ high temperature with magnetic field

The Ising field theory taken at imaginary values of h is not unitary. However the corresponding Hamiltonian H enjoys the following symmetry

$$SHS = H^\dagger$$

where S is the operator that multiplies any Ramond sector vector by -1 and leaves any NS sector vector intact. As a consequence of this symmetry the energy eigenvalues are either real or they form complex conjugated pairs.

The results of Fonseca and Zamolodchikov indicate that for $y \leq y_{\mathrm{cr}}$ the vacuum eigenvalue is real while for $y > y_{\mathrm{cr}}$ it is complex and thus forms a conjugate pair.

These two cases have a very different behaviour in terms of the component ratios.

As another consequence of the symmetry operation S if the vacuum energy is real then Γ_σ^0 is purely imaginary. Since none of the conformal boundary states has a complex value of Γ_σ^0 *as long as the vacuum energy remains real the vacuum vector cannot approach a conformal boundary state as we move along the RG trajectory.*
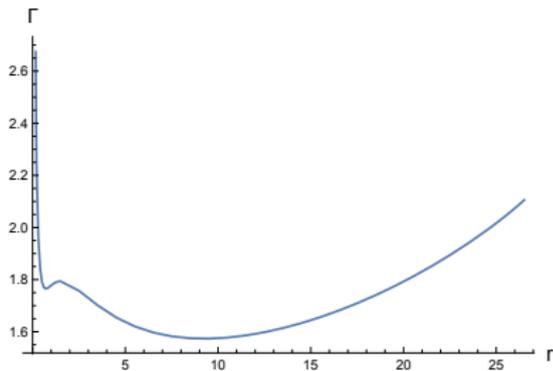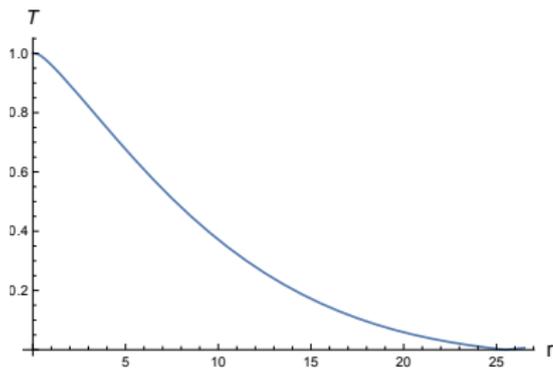
So, what happens?

Truncated Free Fermion Space Approach (TFFSA) was developed by Fonseca and Zamolodchikov. It treats the mass coupling exactly by working in the fermion massive space and using the finite size form factor expression for the matrix elements of σ. Combining it with the massive to massless Bogolyubov transformation we can calculate numerically the component ratios we are interested in.

Here is a plot of the component ratios for $y = -3.5$

For the quantities

$$T = \frac{|\langle 0|v_0\rangle|}{\|v_0\|}\,, \qquad \Gamma = \left|\frac{\Gamma_\epsilon^0}{\Gamma_\sigma^0}\right|\,.$$

For large negative values of y we can approximate the flow of the vacuum vector by a boundary RG flow: free boundary condition perturbed by imaginary boundary magnetic field.

$$\frac{1}{2\pi} \iint (\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi})\, d^2x + \int \left(\frac{i}{4\pi}\psi\bar{\psi} + \frac{1}{2}a\dot{a} + h_{\mathrm{b}}(\omega\psi + \bar{\omega}\bar{\psi})a \right) dy$$

where $\omega = e^{i\pi/4}$ and the boundary magnetic field coupling is taken here to be ih_{b} with h_{b} - real. This model is Gaussian and is exactly solvable.

Boundary magnetic field model

$$|h_{\rm b}\rangle\!\rangle = \sqrt{\pi}e^{-\alpha\ln(R\mu)}\Big[\frac{1}{\Gamma(1/2-\alpha)}\exp\Big(-i\sum_{n=0}^{\infty}\frac{n+1/2+\alpha}{n+1/2-\alpha}a_{n+1/2}^{\dagger}\bar{a}_{n+1/2}^{\dagger}\Big)|0\rangle$$
$$\pm i\frac{2^{1/4}\sqrt{\alpha}}{\Gamma(1-\alpha)}\exp\Big(-i\sum_{n=1}^{\infty}\frac{n+\alpha}{n-\alpha}a_n^{\dagger}\bar{a}_n^{\dagger}\Big)|\sigma\rangle\Big]$$

where $\alpha = 2h_{\rm b}^2 R$. This exact solution gives us the following component ratios

$$\Gamma_{\epsilon}^0 = -\frac{1/2+\alpha}{1/2-\alpha}\,, \qquad \Gamma_{\sigma}^0 = \pm i2^{1/4}\frac{\sqrt{\alpha}\Gamma(1/2-\alpha)}{\Gamma(1-\alpha)}\,,$$

$$\Gamma = \frac{\Gamma(1-\alpha)(1/2+\alpha)}{2^{1/4}\sqrt{\alpha}\Gamma(3/2-\alpha)}\,.$$

Figure: $\Gamma(\alpha)$

Boundary magnetic field model

Asymptoticaly (up to an overall factor) we have

$$|h_{\mathrm{b}}\rangle\!\rangle \sim \cos(\pi\alpha)[|0\rangle\!\rangle + |\epsilon\rangle\!\rangle] \pm i\sin(\pi\alpha)2^{1/4}|\sigma\rangle\!\rangle + \ldots$$

where the ellipsis stands for terms that contain components of level larger than α. Thus as α goes to infinity we will see a never ending rotation of the two combinations of Ishibashi states.

The onset of the oscillations matches well even quantitatively with the TCSA numerics.

The lowest 3 energy eigenvalues correspond to the Yang-Lee operators: $\phi, \mathbf{1}, \partial\bar{\partial}\phi$ where ϕ is the only primary, apart from the identity, that has dimension -0.4.

$$\tilde{E}(r) \equiv \frac{E(r)}{|h|^{8/15}} = \frac{2\pi e(r)}{r}$$

In addition to $\Gamma^0_{\epsilon,\sigma}$ we can also define $\Gamma^1_{\epsilon,\sigma}$. Since the energy levels are real the ratios Γ^0_σ and Γ^1_σ must be imaginary.

All conformal interfaces between Ising and Yang-Lee theories are known due to the fact that the tensor product $(\mathrm{Ising}) \otimes (\mathrm{Yang-Lee})$ is itself a non-unitary minimal model $\mathcal{M}_{5,12}$ with E_6 modular invariant as shown by T. Quella, I. Runkel, G. Watts in 2006. It has 12 conformal boundary states (and thus there are 12 conformal interfaces) and none of them has imaginary ratio Γ^0_σ or Γ^1_σ.

Summary for imaginary magnetic field

- it is interesting to study all flows originating from the tricritical Ising model (4 relevant operators, 6 elementary conformal boundary conditions, a flow to the critical Ising model with RG interface candidate proposed by D. Gaiotto)

- understand the interface between the unperturbed and perturbed theory better (thickness, scattering, how the boundary degrees of freedom emerge)

- get a better grip on truncation corrections; why does the method work at all at such large values of the couplings?

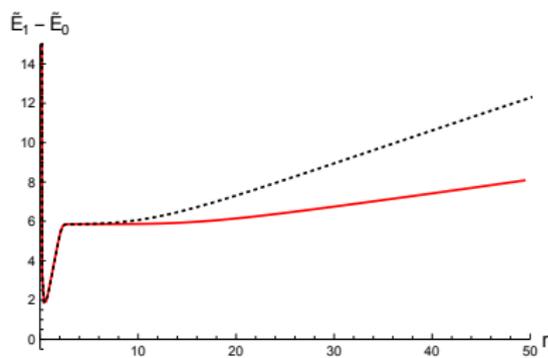The Γ_ϵ^0 numerics at $y = 1$ for different values of n_c

This is a plot of the components of the vacuum state in the NS sector for $y = 3$, $r = 50$, $n_c = 12$. The red numbers mark the conformal weights of the components. Only diagonal components are distinguishable above the i-axis.

For all (massive) flows we expect the vacuum energy to approach a straight line and a constant gap between the first excited and the vacuum levels. The vacuum energy and the energy gap at $y = 1$, $n_c = 12$.
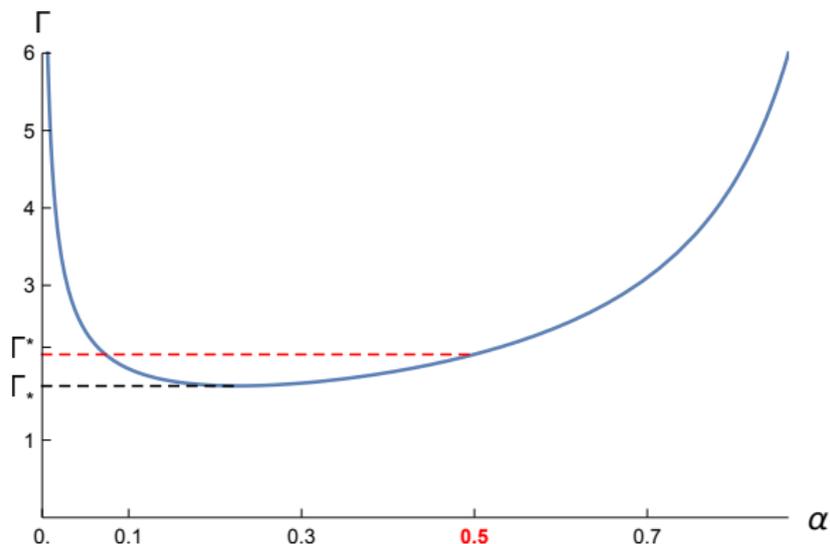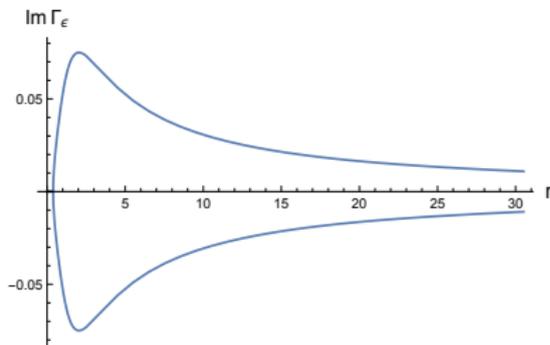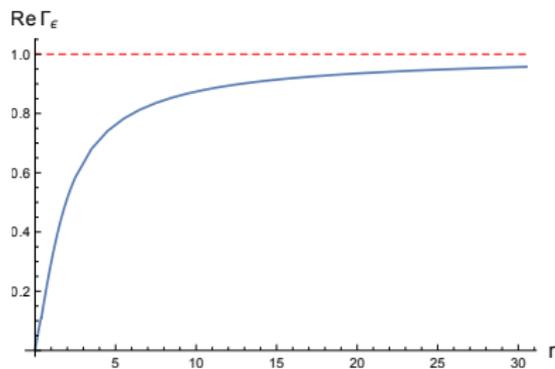
At the onset of these oscillations we have

that qualitatively matches with the TFFSA numerics. Moreover the quantities Γ_* and Γ^* are quite close numerically.
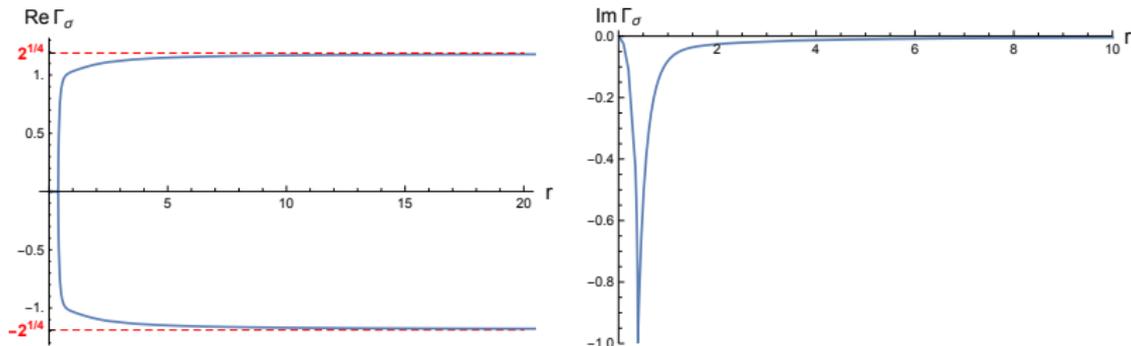
The numerics shows that in this case the RG boundary approaches asymptotically the superposition $|+\rangle\!\rangle \oplus |-\rangle\!\rangle$.

Real and Imaginary parts of Γ_ϵ^0 for the two vacuum vectors at $y = 2$, $n_c = 11$.

Real and Imaginary parts of Γ_σ^0 for the two vacuum vectors at $y = 2$, $n_c = 11$.