

# Energy-momentum multiplets for supersymmetric defects and the displacement operator

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Based on: [arXiv:1701.04323](#) - N.D., Dario Martelli and Itamar Shamir  
[arXiv:1709.?????](#) - N.D., Itamar Shamir and Cristian Vergu

Boundary and Defect Conformal Field Theory: Open Problems and  
Applications

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## Example of displacement operator

- Consider a 4d scalar  $\phi$  and a 3d scalar  $a$ , confined to a planar submanifold  $\Sigma$ . The 4d and 3d actions are

$$\int \mathcal{L}^{(4)} = \int \left( -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V_4(\phi) \right),$$

$$\int_\Sigma \mathcal{L}^{(3)} = \int_\Sigma \left( -\frac{1}{2} \partial^i a \partial_i a - V_3(a) \right).$$

- There are 4d and 3d terms in the energy-moment tensor

$$T_{\mu\nu}^{(4)} = \partial_\mu \phi \partial_\nu \phi + \eta_{\mu\nu} \mathcal{L}^{(4)}$$

$$T_{ij}^{(3)} = \partial_i a \partial_j a + \eta_{ij} \mathcal{L}^{(3)}.$$

- The full energy-momentum tensor includes both parts, which requires the embedding  $\mathcal{P}_\mu^i$  on the directions tangent to  $\Sigma$

$$T_{\mu\nu} = T_{\mu\nu}^{(4)} + \delta(x^n) \mathcal{P}_\mu^i \mathcal{P}_\nu^j T_{ji}^{(3)},$$

where  $x^n$  is the coordinate normal to  $\Sigma$ .

- Using the classical equations of motion one finds

$$\partial^\mu T_{\mu\nu} = 0.$$

- To make the system interesting, we need to couple the 3d and 4d fields. The simplest way to do that is

$$\int_{\Sigma} \mathcal{L}^{(I)} = - \int_{\Sigma} V_I(\phi, a),$$

with an arbitrary coupling potential  $V_I$ .

- Now the 3d term in the energy-moment tensor is

$$T_{ij}^{(3)} = \partial_i a \partial_j a + \eta_{ij} (\mathcal{L}^{(3)} + \mathcal{L}^{(I)}).$$

- Repeating the calculation, we now find a violation of conservation

$$\partial^\mu T_{\mu\nu} = n_\nu \delta(x^n) \partial_\phi V_I(\phi, a) \partial_n \phi,$$

which indicates the breaking of translation symmetry.

- More generally in such systems

$$\partial^\mu T_{\mu\nu} = n_\nu \delta(x^n) d,$$

and  $d$  is known as the **displacement operator**

- In our case

$$d = \partial_n V_I(\phi, a).$$

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- A famous example is the case of a 1d defect.
- Consider the world-line of a particle coupling with the bulk gauge field.
- We can integrate out the 1d field, ending up with just a Wilson loop.
- The displacement is

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- This introduces a field strength insertion  $F_{n\mu}\dot{x}^\mu$  into the Wilson loop, which is the simplest insertion.
- In  $\mathcal{N} = 4$  SYM, this insertion is a protected operator of dimension 2. Its two point function is known as the Bremsstrahlung function and is known exactly.

- I will study the displacement operator in supersymmetric theories.
- I will focus on 4d theories with  $\mathcal{N} = 1$  SUSY.
- Breaking of translation symmetry is clearly accompanied by breaking of some SUSYs, but we focus on the case of some preserved SUSY.
- The  $\mathcal{N} = 1$  algebra does not allow BPS time-like line operators, but such theories have BPS domain walls and vortices. More generally:
  - 3d defects preserving an algebra isomorphic to  $\mathcal{N} = 1$  in 3d. This is generated by linear combinations of supercharges of opposite chirality.
  - 2d defects preserving an algebra isomorphic to  $\mathcal{N} = (0, 2)$  in 2d. For a defect in the  $(x^0, x^3)$ , we can choose  $Q_+$  and  $\bar{Q}_+$  as the generators of the subalgebra.

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- The displacement operator is part of a multiplet which includes the violation of supercurrent conservation.

## Outline

- The  $\mathcal{S}$ -multiplet.
- Embedding the 3d energy-momentum multiplet in the  $\mathcal{S}$ -multiplet.
- The displacement multiplet.
- Discussion.

## The $\mathcal{S}$ -multiplet

[Komargodski, Seiberg]

- The energy multiplet of a 4d theory with  $\mathcal{N} = 1$  supersymmetry resides in a  $16 + 16$  multiplet defined by the equation

$$\bar{D}^{\dot{\alpha}} \mathcal{S}_{\alpha\dot{\alpha}} = 2(\chi_{\alpha} - D_{\alpha} X),$$

- $\mathcal{S}_{\alpha\dot{\alpha}}$  is a real vector superfield.
  - $X$  and  $\chi_{\alpha}$  are chiral superfields.
  - $D^{\alpha} \chi_{\alpha} = \bar{D}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$ .
- The superspace expansion of the  $\mathcal{S}$ -multiplet is schematically

$$\mathcal{S}_{\mu} = -i\theta^{\alpha} S_{\alpha\mu} + i\bar{\theta}_{\dot{\alpha}} \bar{S}^{\dot{\alpha}}_{\mu} + 2\theta\sigma^{\nu}\bar{\theta}T_{\nu\mu} + \dots$$

- The question we are asking is what new terms can appear on the right of the  $\mathcal{S}$ -multiplet equation in the presence of a defect and where does the displacement operator reside.
- Another useful equation is

$$\bar{D}^2 \mathcal{S}_\mu = 0$$

- In the presence of defects this becomes

$$\bar{D}^2 \mathcal{S}_\mu = \begin{cases} i\delta(\tilde{y}^n) n_\mu \Pi, & \text{3d defect,} \\ i\delta^{(2)}(y^1, y^2) \theta^- n_\mu^a \Pi_{-a}, & \text{2d defect.} \end{cases}$$

And  $\Pi$  is the displacement multiplet, containing the displacement operator.

## 3d in 4d

- We defined the normal vector  $n^\mu$  as a bi-spinor  $n_{\alpha\dot{\alpha}}$ .
- $x^i$  are coordinates along the defect.
- We choose to preserve two supercharges

$$Q_A = \lambda_A^\alpha Q_\alpha + \bar{\lambda}_A^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}$$

for  $A = 1, 2$

- Choosing  $(\lambda_A^\alpha)^* = \bar{\lambda}_A^{\dot{\alpha}}$ , gives real supercharges.
- Define Grassmannian coordinates  $\Theta_A$  transverse to the defect and  $\tilde{\Theta}_A$  normal to it by

$$\Theta_A = \frac{1}{2i}(\lambda_A^\alpha \theta_\alpha - \bar{\lambda}_A^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}), \quad \tilde{\Theta}_A = \frac{1}{2}(\lambda_A^\alpha \theta_\alpha + \bar{\lambda}_A^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}),$$

- A natural superspace basis is  $(x^i, \Theta_A; \tilde{x}^n, \tilde{\Theta}_A)$ , where  $\tilde{x}^n = x^n - 2i\tilde{\Theta}\Theta$  is an invariant coordinate, like  $\tilde{\Theta}_A$ .
- Another useful coordinate is

$$\tilde{y}^n = x^n - 2i\tilde{\Theta}\Theta - 2\tilde{\Theta}^2 = y^n - i\theta^2.$$

which is both invariant and chiral.

- Any 3d theory with  $\mathcal{N} = 1$  SUSY admits an energy-momentum multiplet  $\mathcal{J}_{Ai}$  satisfying

$$\Delta^A \mathcal{J}_{Ai} = -2\partial_i X, \quad \Gamma^i{}_A{}^B \mathcal{J}_{Bi} = i\Delta_A(Y - X),$$

where  $X$  and  $Y$  are real multiplets.

- The component expansion of  $\mathcal{J}_{Ai}$  is (schematically)

$$\mathcal{J}_{Ai} = S_{Ai} + (\Gamma^j \Theta)_A T_{ij} + \dots$$

where  $S_{Ai}$  is the supercurrent and  $T_{ij}$  the energy-momentum tensor.

- We can embed it in an  $\mathcal{S}$ -multiplet by

$$\mathcal{S}_i^{(3)} = \delta(\tilde{x}^n) \tilde{\Theta}^A \mathcal{J}_{Ai} = 2\delta(x^n) \theta \sigma^j \bar{\theta} T_{ji} + \dots \quad \mathcal{S}_n^{(3)} = 0.$$

- A somewhat tedious computation gives

$$\bar{D}^{\dot{\alpha}} \mathcal{S}_{\alpha\dot{\alpha}}^{(3)} = \frac{1}{4} \bar{D}^2 D_\alpha \left( \tilde{\Theta}^2 \delta(\tilde{x}^n) (Y - X) \right) - \sqrt{2} \delta(\tilde{y}^n) (\Gamma^i \tilde{\Theta})_\alpha \partial_i X.$$

- Evaluating the second derivative one finds from the second term

$$\bar{D}^2 \mathcal{S}_\mu^{(3)} \sim i\delta(\tilde{y}^n) \mathcal{P}_\mu{}^i \partial_i X$$

which does not contribute to the displacement operator.

- The first term fits the structure of the  $\chi_\alpha$  term in the usual  $\mathcal{S}$ -multiplet.

- More generally we can expand any new pieces on the rhs in  $\tilde{\Theta}$

$$\bar{D}^{\dot{\alpha}} \mathcal{S}_{\alpha\dot{\alpha}} = \delta(\tilde{y}^n) \mathcal{Z}_{\alpha}, \quad \mathcal{Z}_{\alpha} = \Sigma_{\alpha} + \tilde{\Theta}_{\alpha} \Sigma + (\Gamma^i \tilde{\Theta})_{\alpha} \Sigma_i$$

where all the  $\Sigma$  superfields are chiral.

- Note that to make them 4d chiral they have further  $\tilde{\Theta}$  dependence, but that is fixed fully by their 3d superspace expansion and 4d chirality (they can also have arbitrary  $x^n$  dependence, though it mostly will be a delta function).
- In the  $\tilde{\Theta}$  expansion of  $\mathcal{S}$  one finds at the linear level

$$\tilde{\Delta}_A \mathcal{S}_{\mu} |_{\tilde{\Theta}=0} = -(\lambda_A^{\alpha} \mathcal{S}_{\alpha\mu} + \bar{\lambda}_A^{\dot{\alpha}} \bar{\mathcal{S}}_{\mu\dot{\alpha}}) - 4i(\Gamma^j \Theta)_A T_{j\mu} + \dots$$

which are the terms which should be conserved even in the presence of the defect.

- A straightforward computation then leads to

$$\begin{aligned} \tilde{\Delta}_A \partial^{\mu} \mathcal{S}_{\mu} |_{\tilde{\Theta}=0} &= \delta(x^n) \left( -\frac{i}{4} \Delta^B \Delta_A (\Sigma_B + \bar{\Sigma}_B) - \frac{1}{4} (\Gamma^i \Delta)_A (\Sigma_i + \bar{\Sigma}_i) \right) \Big|_{\tilde{\Theta}=0} \\ &\quad + \frac{i}{2} \partial_n (\delta(x^n) (\Sigma_A - \bar{\Sigma}_A)) \Big|_{\tilde{\Theta}=0}. \end{aligned}$$

Setting the first line to zero imposes extra conditions on the  $\Sigma$  which are consistent.

- The sub-multiplet containing the violations is given by

$$\mathcal{S}_\mu|_{\tilde{\Theta}=0} = -i\Theta^A(\lambda_A^\alpha S_{\alpha\mu} - \bar{\lambda}_{A\dot{\alpha}} \bar{S}_\mu^{\dot{\alpha}}) - 2i\Theta^2 T_{n\mu} + \dots$$

- The derivative is

$$\partial^\mu \mathcal{S}_\mu|_{\tilde{\Theta}=0} = -\frac{1}{4}\delta(x^n)\Delta^A(\mathcal{Z}_A - \bar{\mathcal{Z}}_A)|_{\tilde{\Theta}=0} - \frac{i}{4}\delta(x^n)(\bar{D}\bar{\sigma}_n \mathcal{Z} - D\sigma_n \bar{\mathcal{Z}})|_{\tilde{\Theta}=0}.$$

- Since  $T_{n\mu}$  appears in the  $\Theta^2$  component of  $\mathcal{S}_\mu$  only the second term contributes to the displacement.
- This comes from the scalar  $\Sigma$  piece. We find

$$\partial^\mu T_{n\mu} \sim \delta(x^n)\Delta^2 \Sigma$$

- We finally find that

$$\bar{D}^2 S_n = \sqrt{2}i\delta(\tilde{y}^n)\Sigma$$

as anticipated.

## Example: Chiral-scalar couplings

- 4d chirals  $\Phi^a$  with Kähler potential  $K$  and superpotential  $W$
- 3d real scalar multiplets  $A^I$  with target space metric  $\mathcal{G}_{IJ}$ .
- Interacting potential

$$P(\Phi^a, \bar{\Phi}^{\bar{a}}, A^I)|_{\tilde{\Theta}=0}$$

- The equations of motion are

$$\bar{D}^2 K_a = 4W_a + 2\delta(\tilde{y}^n)\mathcal{P}_a,$$

$$\Delta^A(\mathcal{G}_{IJ}\Delta_A A^J) = \frac{1}{2}\partial_I \mathcal{G}_{JK}\Delta^A A^J \Delta_A A^K + P_I.$$

$\mathcal{P}$  to denote the chiral embedding of  $P$ .

- We define the 4d and 3d parts of the energy-momentum multiplet by

$$\mathcal{S}_{\alpha\dot{\alpha}}^{(4)} = K_{a\bar{a}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{a}}D_{\alpha}\Phi^a,$$

$$\mathcal{S}_{AB}^{(3)} = \delta(\tilde{x}^n)\tilde{\Theta}^C(-2\mathcal{G}_{IJ}\partial_i A^I \Delta_C A^J)\Gamma_{AB}^i.$$

- We find for the 4d part

$$\bar{D}^{\dot{\alpha}} \mathcal{S}_{\alpha\dot{\alpha}}^{(4)} = 2(\chi_{\alpha} - D_{\alpha}W) - \delta(\tilde{y}^n) \mathcal{P}_a D_{\alpha} \Phi^a,$$

where  $\chi_{\alpha} = -\frac{1}{4} \bar{D}^2 D_{\alpha} K$ .

- For the 3d part, after some work gives

$$\mathcal{Z}_{\alpha} = -\frac{1}{2} \left( \mathcal{P}_a D_{\alpha} \Phi^a - \mathcal{P}_{\bar{a}} D_{\alpha} \tilde{\Phi}^{\bar{a}} \right) - \sqrt{2} \tilde{\Theta}_{\alpha} \partial_n \mathcal{P} - \frac{1}{4} D_{\alpha} \mathcal{P} + i(\sigma^n \bar{D})_{\alpha} (\tilde{\Theta}^2 \mathcal{P}).$$

- The displacement operator comes from

$$\frac{i}{4} (\bar{D} \bar{\sigma}_{\mu} \mathcal{Z} - D \sigma_{\mu} \bar{\mathcal{Z}})|_{\tilde{\Theta}=0} = -2n_{\mu} \partial_n P$$

- We find  $2d = \partial_n P$ , which is the obvious supersymmetric generalization of the scalar expression from the start of my talk.

## Summary

- The displacement operator is a natural object that arises in theories with defects.
- It has applications for quantities like the Bremsstrahlung function, entanglement entropy and more.
- A natural generalization exists for supersymmetric theories.
  - The rhs of the  $\mathcal{S}$ -multiplet equation for  $\mathcal{N} = 1$  theories in 4d acquires further terms.
  - The displacement operator resides in the scalar  $\Sigma$
  - It forms a multiplet with  $\Delta^A(\mathcal{Z}_A - \bar{\mathcal{Z}}_A)$
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  - Finding SUSY boundaries or defects on curved space.
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  - This multiplet includes the violation of supercurrent conservation.
- Possible applications:
  - Finding SUSY boundaries or defects on curved space.
  - Anomalies.
  - superbremstrahlung?
- Possible generalizations:
  - $\mathcal{N} = 2$  SUSY.
  - Not Minkowski subspaces.

The end