Spacetime Penrose Inequality for Cohomogeneity One Initial Data

Hari K Kunduri

Department of Mathematics and Statistics
Department of Physics and Astronomy
McMaster University

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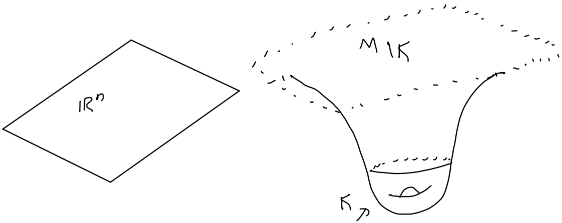
Asymptotically flat manifolds

Initial data describing isolated gravitating systems are modelled in GR by **AF** manifolds approaching $(\mathbb{R}^n, \delta)$ in an asymptotic region.

1. Topology: $M \setminus K \cong \mathbb{R}^n \setminus$ Ball for some compact set $K \subset M$.
2. Geometry: the metric g must approach the flat Euclidean metric δ



Cauchy Problem in General Relativity

- Einstein's equations form a set of coupled, hyperbolic quasilinear PDEs for the **spacetime** metric tensor $\mathbf{g}(t, x)$.

- Initial data: Triple (M, g, k)

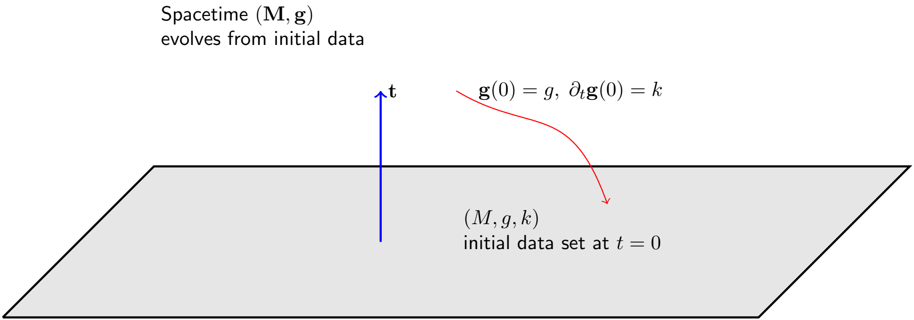
$$\mathbf{g}(0, x) = g, \qquad \frac{\partial}{\partial t}\mathbf{g}(0, x) = k$$

  Basic PDE result guarantees local existence and uniqueness.

- Fundamental theorem: GR admits well posed Cauchy problem.

Cauchy Problem

Einstein eqns $\Rightarrow$ constraint eqns on data + evolution eqns



Initial data consists of a Riemannin manifold (M, g) and rank 2 tensor k

k is the 2nd FF of M in ambient space $\mathbf{M}$:

Well posedness of the Cauchy Problem

Einstein onstraint equations

An initial data set (M, g, k) consists of a Riemannian manifold (M, g) and symmetric 2-tensor k satisfying

$$R_g - |k|_g^2 + \mathrm{Tr}_g\, k = 16\pi\mu \qquad \text{Hamiltonian Constraint}$$

$$\mathrm{div}_g\, k - \nabla(\mathrm{Tr}_g\, k) = 8\pi J \qquad \text{Momentum Constraint}$$

Theorem (Well posedness) Choquet-Bruhat 52, Choquet-Bruhat, Geroch 1969

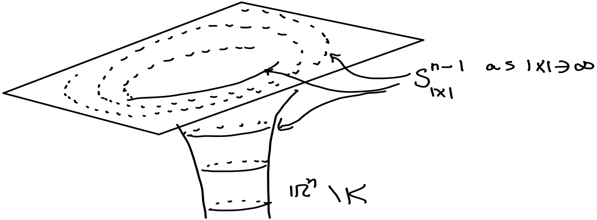
Let (M, g, k) be an initial data set, Then there is a unique* maximal spacetime $(\mathbf{M}, \mathbf{g})$ solving Einstein's equations with this initial data.

The ADM mass

An isolated system (e.g. black hole) evolves from AF initial data. Define

$$m := c_n \int_{S^{n-1}_{\infty}} (\mathrm{div}\, g - \nabla \mathrm{Tr} g) \cdot \nu \, \mathrm{d}S$$

The integral is taken in the limit $|x| \to \infty$ of asymptotic $\mathbb{S}^{n-1}$ 'boundary sphere at infinity' with unit normal ν that encloses M.

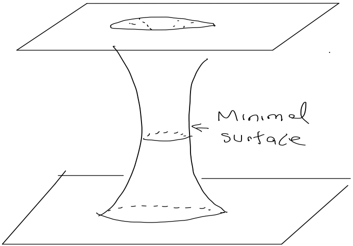


Schwarzschild initial data on $\mathbb{R}^3 \setminus \{0\}$

The simplest non-trivial model is the spherically symmetric metric

$$g_S = \left(1 + \frac{m}{2|x|}\right)^4 \delta_3, \quad k = 0, \qquad m \in \mathbb{R}, \qquad R_{g_S} = 0.$$

- If $m > 0$ the geometry is AF with a *minimal surface* at $|x| = m/2$.
- If $m < 0$, there is a singularity at $|x| = -m/2 > 0$.
- If $m = 0$ the space is flat



Gravitational collapse and geometric inequalities

Standard picture based on two conjectures:

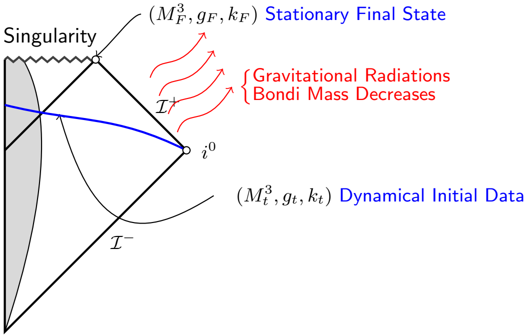
Weak cosmic censorship conjecture (WCC)

The maximal Cauchy development of generic AF initial data for the Einstein equations possesses a complete future null infinity $\mathcal{I}^+$
(No visible singularities!)

Final State Conjecture (FSC)

The maximal Cauchy development of generic AF initial data for the vacuum Einstein equations can be described asymptotically in time as a equilibrium black hole solution.

Gravitational collapse



- WCC implies formation of BH when we have singularity and FSC implies stationary final state
- energy lost by gravitational radiation $\Rightarrow m_i \geq m_F$.

Penrose recast these conjectures about the evolution of an initial data set to produce a conclusion relating geometric properties of (M, g, k).

Conjecture (Penrose)

Let (M, g, k) be a complete AF initial data set containing an **outermost apparent horizon** S. Then its mass m satisfies

$$m \geq \sqrt{\frac{A}{16\pi}}$$

where A is the smallest area required to enclose S. Equality iff (M, g, k) lies inside the Schwarzschild spaceitme.

Here *apparent horizon* is a hypersurface $S \subset M$ satisfying

$$\theta_+ := H_S + \mathrm{Tr}_S k = 0.$$

where H_S is the mean curvature of $S \subset M$.

The Riemannian Penrose inequality

Riemannian Penrose Inequality Huisken & Illmanen, Bray, Bray & Lee

Let $(M, g, 0)$ be a complete $n = 3$ AF initial data set containing an minimal hypersurface S with area A. Then

$$m \geq \sqrt{\frac{A}{16\pi}}$$

with equality iff the part of (M, g) outside S is isometric to the Schwarzschild initial data (M_S, g_S). Extends to $n < 8$.

- proved by geometric flow techniques:
  - inverse mean curvature flow on $S \subset M$ for $n = 3$
  - conformal flow on (M, g) for $3 \leq n < 8$.

Spacetime Penrose Inequality

- Bray-Khuri (2010) introduce a quasilinaer elliptic PDE on (M, g, k) - the **generalized Jang equation**, which, when coupled to an inverse mean curvature flow, would prove the SPI by reducing it to the RPI.

- Reduces problem to proving existence and uniqueness of solutions of a coupled system of quasilinear elliptic PDEs.
  Works in **spherical symmetry**.

Cohomogeneity-one initial data

- (M, g) is cohomogeneity-one if a Lie group G acts transitively by isometries on M with principal orbits G/K where K is a principal isotropy subgroup $\Rightarrow M \cong [0, \infty) \times G/K$.
- AF requirement $\Rightarrow G/K$ are spheres (Ziller):
  - $\mathbb{S}^d = SO(d+1)/SO(d)$
  - $\mathbb{S}^{2n+1} = SU(n+1)/SU(n)$
  - $\mathbb{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n), \mathbb{S}^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$
- Can represent (M, g, k) in concise form, e.g.,

$$g = \mathrm{d}s^2 + r(s)^2 \bar{g}$$

where $\bar{g}$ is a homogeneous metric.

Spacetime Penrose inequality with symmetry

Theorem Khuri & HK 2024

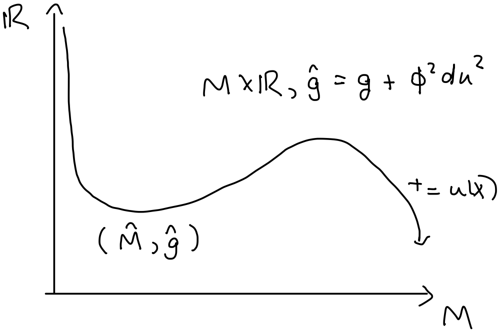
Let (M, g, k) and $\dim(M) = 2(n+1)$ be an AF $SU(n+1)$-invariant initial data set with outermost apparent horizon boundary of area A. If $\mu \geq |J|$ holds, then

$$m \geq \frac{1}{2}\left(\frac{A}{\omega_{2n+1}}\right)^{\frac{2n}{2n+1}}.$$

with equality iff (M, g, k) can be isometrically embedded into Schwarzschild spacetime.

Extends to **asymptotically hyperbolic** setting (relevant for asymptotically Anti-de Sitter spacetimes) + other homogeneous metrics on $\mathbb{S}^d$.

Sketch of Proof



Given initial data (M, g, k):

- **Jang deformation** $(\hat{M}, \hat{g})$:
  $\hat{g}$ =induced metric on the graph $\hat{M} = (x, u(x))$ in the product space

$$(\mathbb{R} \times M, g + \phi^2 \mathrm{d}u^2)$$

Sketch of Proof

Consider generalized Jang equation:

$$H_{\hat{M}} = \mathsf{Tr}_{\hat{M}} k$$

quasilinear 2nd order elliptic PDEs for (u, ϕ) .

- Bray-Khuri approach: obtain auxiliary PDE for ϕ using IMCF
- for cohomgeneity-1 data, can reduce Jang PDE to nonlinear ODE for a single function $v = v(u, \phi)$.
- prove existence and uniqueness of solutions v with appropriately decay $\Rightarrow (\hat{M}, \hat{g})$ is AF, ∂M =minimal surface, and $\hat{m} = m$.
- **generalized Schoen-Yau identity** (Bray-Khuri)

$$\hat{R} = [\dots]^2 + [\mu - J] + \text{divergence term involving } \phi$$

Sketch of Proof

- Define Hawking mass on surfaces of homogeneity:

$$\mathbf{m}(\Sigma_t) := \frac{1}{2}\left(\frac{[A_t]}{\omega_{d-1}}\right)^{\frac{d-2}{d-1}}\left[1 - f_1(\Sigma_t)\int_{\Sigma} H^2 \,\mathrm{d}V\right]$$

- The Jang surface has a minimal surface boundary: apply IMCF

$$\frac{\mathrm{d}\mathbf{m}(\Sigma_t)}{\mathrm{d}t} = \frac{1}{2}\left(\frac{[A_t]}{\omega_{d-1}}\right)^{\frac{d-2}{d-1}}\left[\frac{d-2}{d-1} - f_1(A_t)\int_{\Sigma_t} R_{\Sigma_t}\,\mathrm{d}t\right] + f_2(A_t)\int_{\Sigma_t}\left[\frac{|\nabla H|^2}{H^2} + |\Pi|^2 - \frac{H^2}{n-1} + \hat{R}\right]\mathrm{d}V$$

For our data this is **non-negative** up to divergence term

- Integrate from apparent horizon to AF region and show that

$$\hat{m} = m = \lim_{t \to \infty} \mathbf{m}(\Sigma_t)$$

$$\mathbf{m}(\Sigma_\infty) - \frac{1}{2}\left(\frac{A}{\omega_{2n+1}}\right)^{\frac{2n}{2n+1}} = \int_0^\infty \frac{\mathsf{d}\mathbf{m}(\Sigma_t)}{\mathsf{d}t} \mathsf{d}t \geq 0$$

after careful analysis of boundary terms.

- Technical difficulty: prove the Jang deformation preserves the area of the apparent horizon and the total mass.

Summary

- Existence of black holes poses many open problems in general relativity (e.g. can gravitational collapse lead to naked singularities?)

- $\Rightarrow$ Physical expectations can be reformulated into geometric statements which can be proved or refuted

- proving the general spacetime Penrose inequality remains a fundamental problem in mathematical relativity.