Non-geodesic timelike observers and the ultralocal limit

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Outline

1. Ultralocal (UL) limit along timelike geodesics
2. Does it extends to non-geodesic curves? → **NO**
3. ... but there are exceptions:
   - particles with variable mass;
   - fluid elements in FLRW and Bianchi cosmology;
   - test particles in Einstein frame scalar-tensor gravity;
   - self-interacting dark matter.

In an ultralocal limit, one restricts to looking at 3-space in an infinitesimally small neighborhood of a null or timelike geodesic. Light cones close and collapse on the observer's wordline, nothing propagates in its 3-space (Carrollian limit of GR, opposite situation to Newtonian limit, is a formal limit of the Poincaré algebra associated with quantum effects in strong gravity and with the AdS/CFT correspondence).

Penrose 1976: "Any space-time has a plane wave as a limit". Along *null* geodesics, *every* spacetime metric reduces to a plane wave in the ultralocal limit. For a freely falling observer reaching asymptotically the speed of light, *any* gravitational field looks like that of an exact plane gravitational wave passing by at light speed (*universality*).

THE CROPP-VISSER ULTRALOCAL LIMIT

Cropp & Visser 2011: freely falling **timelike** observers see *any* gravitational field as a Bianchi I model in the UL limit.

Begin by assuming a freely falling observer along a timelike geodesic γ of the metric $g_{\mu\nu}(x^\alpha)$.

1) Adopt synchronous coordinates (always defined locally in the neighborhood of a timelike geodesic); locate the origin of the spatial coordinates x^i at γ; the line element becomes

$$ds^2 = -c^2 dt^2 + g_{ij}\left(t, x^k\right) dx^i dx^j$$

2) Rescale the spatial coordinates $x^i \rightarrow x^{i'}(x^j) = \epsilon\, x^i$, where ϵ is a positive constant. $\epsilon \rightarrow 0$ corresponds to shrinking the 3-space around the worldline onto the worldline itself. The line element has now the form

$$ds^2 = -c^2 dt^2 + \epsilon^2 g_{ij}\left(t, \epsilon\, x^k\right) dx^i dx^j \,;$$

3) now rescale the speed of light as $c \rightarrow \epsilon\, c$, obtaining

THE CROPP-VISSER ULTRALOCAL LIMIT

$$ds^2 = \epsilon^2 \left[-c^2 dt^2 + g_{ij}\left(t, \epsilon\, x^k\right) dx^i dx^j \right] ,$$

and perform the conformal transformation $ds^2 \to d\tilde{s}^2 = \epsilon^{-2} ds^2$ (as in the Penrose limit along null geodesics) $\to$

$$ds^2 = -c^2 dt^2 + g_{ij}\left(t, \epsilon\, x^k\right) dx^i dx^j . \tag{1}$$

4) Take the limit $\epsilon \to 0$, obtaining

$$\boxed{ds^2 = -c^2 dt^2 + g_{ij}\left(t, \vec{0}\right) dx^i dx^j}$$

which describes a **Bianchi I** universe

ACCELERATED TIMELIKE OBSERVERS

But for timelike **non-geodesic** observers, the derivation breaks down at its very first step: synchronous coordinates cannot be introduced

Standard derivation of Gaussian normal coordinates (e.g., Wald) : Consider a massive particle subject to a 4-force per unit mass f^μ, which follows the spacetime trajectory γ' with 4-tangent u^μ:

$$u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\, u^\alpha u^\beta = f^\mu \,,$$

where $\tau =$ proper time. To introduce synchronous coordinates $\left(\tau, x^i\right)$, the wordline γ' of the particle must be normal to every hypersurface of 3-space Σ_τ of constant time τ, in which there are spatial coordinates x^i with 3 purely spatial coordinate vectors X^ν according to the observer u^μ. For all Σ_τ, we must have $u^\mu X_\mu = 0$ for each of the vectors X^α. Or, defining the spatial coordinates so that $X^\mu u_\mu = 0$ *initially*, this condition is preserved along γ', i.e.,

$$\frac{D}{D\tau}\left(X^\mu u_\mu\right) = u^\nu \nabla_\nu \left(X_\mu u^\mu\right) = 0 \,,$$

but this property fails to hold because it is instead

$$u^{\nu}\nabla_{\nu}\left(X_{\mu}u^{\mu}\right)=X_{\alpha}f^{\alpha}\,.$$

To wit,

$$\begin{aligned} u^{\nu}\nabla_{\nu}\left(X_{\alpha}u^{\alpha}\right) &= u_{\alpha}u^{\beta}\nabla_{\beta}X^{\alpha}+X^{\alpha}u^{\beta}\nabla_{\beta}u^{\alpha} \\ &= u_{\alpha}u^{\beta}\nabla_{\beta}X^{\alpha}+X_{\alpha}f^{\alpha}\,. \end{aligned}$$

Since u^{μ} and $X^{\mu}$are the vectors of a coordinate basis on the spacetime manifold their commutator vanishes, yielding

$$u^{\beta}\nabla_{\beta}\left(X_{\alpha}u^{\alpha}\right)=\cancel{u_{\alpha}X^{\beta}\nabla_{\beta}u^{\alpha}}+X_{\alpha}f^{\alpha}=X_{\alpha}f^{\alpha}$$

because $u^{\mu}u_{\mu}=-1 \implies u^{\alpha}\nabla_{\beta}u_{\alpha}=0$.
The rhs vanishes if and only if the particle is free or is subject to a 4-force perpendicular to X^{μ},a clear obstruction to constructing synchronous coordinates along non-geodesic timelike curves.

EXCEPTIONS

If the 4-force f^μ is parallel or antiparallel to the 4-velocity u^α, the product $X_\alpha f^\alpha$ vanishes, the eq. of motion reduces to the affinely parametrized geodesic equation, and the obstruction to constructing synchronous coordinates is removed.
Situation not contemplated in most textbooks but far from unphysical, includes:

- non-constant particle mass;
- particles subject only to gravity in the Einstein frame description of scalar-tensor gravity;
- fluid elements of perfect or imperfect fluids in FLRW or Bianchi cosmology;
- mass-changing particles in cosmology and in scalar-tensor gravity (Mbelek '98, '04; Damour '90; Casas '91; Garcia-Bellido '92; Anderson & Carroll '97)
- certain scenarios in which self-interacting dark matter is subject to a sort of anti-friction (Zimdahl '00).

VARIABLE MASS OF TEST PARTICLES

When the mass *m* of a particle following a timelike worldline γ' with 4-tangent u^μ is constant, the 4-force acting on it is simply $f^\alpha = ma^\alpha$, where $a^\beta \equiv \dot{u}^\beta \equiv u^\mu \nabla_\mu u^\beta =$ 4-acceleration.

Rockets are prototypical systems with variable mass and exact solutions describing rockets (Kinnersley '69, '70; Bonnor'94, '97; Damour 94; Dain '96; Podolsky '00, '10; Ge '11; Hogan '20; Stephani +) and **solar sails** (Forward '84; Fuzfa '19, '20) abound in GR.

Cosmological scenarios: Quantum processes in the early universe lead to particle production and negative bulk pressure (Zel'dovich '70; Hu +). This mechanism could potentially drive inflation (Zimdahl '96-'98, '00; Schwarz '01; Zimdahl & Balakin '98).

Likewise, the self-interaction of dark matter can cause negative bulk stresses, a mechanism investigated as a possible cause of the present acceleration of the universe (Zimdahl '00).

Self-interaction could be responsible for a cosmic "antifriction" on the dark matter fluid → a force antiparallel to the 4-trajectories of dark matter particles.

EINSTEIN frame scalar-tensor gravity

In (Jordan frame) scalar-tensor gravity, a scalar degree of freedom ϕ appears together with the two massless spin 2 modes contained in the metric tensor. The action is

$$S_{ST} = \int \frac{d^4x \sqrt{-g}}{16\pi} \left[\phi R - \frac{\omega(\phi)}{\phi} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) + \mathcal{L}^{(m)} \right]$$

The Jordan frame variables are $(g_{\mu\nu}, \phi)$, Newton's constant $G \to G_{eff} \simeq G/\phi$. The Einstein frame variables are

$$\tilde{g}_{\mu\nu} = \phi \, g_{\mu\nu} \, , \qquad d\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi}} \frac{d\phi}{\phi}$$

In this frame $\tilde{\phi}$ couples to matter instead:

$$S_{ST} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - \frac{V(\phi)}{\phi^2} + \frac{\mathcal{L}^{(m)}}{\phi^2(\tilde{\phi})} \right]$$

As a consequence, particles subject only to gravity deviate from geodesics in the Einstein frame:

$$\frac{d^2 x^\mu}{d\tau^2} + \tilde{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = \sqrt{\frac{4\pi}{2\omega + 3}}\, \tilde{\nabla}^\mu \tilde{\phi}$$

Interpretation: the mass of a test particle depends on $\tilde{\phi}$ in the Einstein frame and is no longer geodesic. The gradient of $\tilde{\phi}$ translates in the dependence of the particle mass m on the spacetime position and in a fifth force proportional to $\tilde{\nabla}^\mu \tilde{\phi}$. Once the particle trajectory γ' is fixed by initial conditions, the particle mass depends only on the proper time τ along this trajectory. In the UL limit the spatial dependence of ϕ is killed anyway, leaving $\phi\left(t, \vec{0}\right)$ instead of $\phi\left(t, \vec{x}\right)$ and $m = m(\tau)$. One effectively has a time-dependent mass along the trajectory and a 4-force f^μ parallel to the 4-tangent u^μ. Then, test particles "see" ultra-locally any spacetime geometry as Bianchi I.

Another GR situation in which a 4-force parallel to massive particle worldlines occurs in cosmology. Consider a FLRW universe sourced by a fluid, with

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right]$$

Except for the case in which the matter fluid is dust or a cosmological constant Λ, this fluid has pressure $P(t)$ and pressure gradient $\nabla_\mu P \neq 0$, which generates a 4-force pointing in the (comoving) time direction u^μ. The 4-force and 4-acceleration must have vanishing spatial components to respect spatial isotropy $\rightarrow$ fluid particles deviate from geodesics and obey

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = B(\tau)\frac{dx^\mu}{dt},$$

the non-affinely parametrized geodesic equation

proper time t of comoving observers $\neq$ affine parameter

FLRW cosmology

It is always possible to switch from t to an affine parameter: then the rhs of the geodesic equation vanishes. This 4-acceleration is somehow trivial, but the reparametrization is not. If s is an affine parameter, then

$$B(\tau) = \frac{dt}{ds}\frac{d^2s}{dt^2}$$

The equation describing the worldlines of the fluid elements cannot be affinely parameterized by the fluid's proper time, which causes a 4-force parallel to the 4-velocity u^μ. Immaterial from the mathematical point of view, but the difference between proper time of the comoving observers and an affine parameter matters for physics (FLRW cosmology is usually formulated in the frame of comoving observers).

BIANCHI COSMOLOGY

This discussion extends to spatially anisotropic **Bianchi cosmologies**: gradients of isotropic pressure and anisotropic stresses generate 4-forces parallel to the fluid worldlines, therefore the UL limit applies.

Perhaps this is a consistency check of the UL limit for non-geodesic fluids, but is almost trivial from the geometric point of view: it states that a FLRW or Bianchi universe (possibly with spatial curvature) looks locally like a Bianchi I spacetime! FLRW is a special case of Bianchi I, so a Bianchi model reduces to Bianchi I when the spatial curvature is neglected in the UL limit.

In general, one cannot introduce synchronous coordinates along the wordline γ' of an accelerated observer, the best on can do is the following. Given the force f^μ acting on an accelerated timelike observer and it worldline γ' satisfying Eq. (6), one can perform a rescaling of the spatial coordinates in the 3-space with Riemannian metric $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and then take the limit $\epsilon \to 0$, obtaining

$$\begin{aligned} ds^2 &= g_{00}\left(t(\tau),\vec{0}\right) dt^2 + 2g_{0i}\left(t(\tau),\vec{0}\right) dt dx^i \\ &\quad + g_{ij}\left(t(\tau),\vec{0}\right) dx^i dx^j \,, \end{aligned}$$

where $t(\tau) = x^0(\tau)$ is the time component of the solution $x^\mu(\tau)$ of the non-geodesic equation). Redefine time by

$$-g_{00}\left(t(\tau),\vec{0}\right) dt^2 \equiv d\bar{t}^2 \iff \bar{t} = \int d\tau \, \frac{dt}{d\tau} \sqrt{-g_{00}\left(t(\tau),\vec{0}\right)}$$

$d\bar{t}$ is an exact differential.

General accelerated observer

We are left with

$$ds^2 = -d\bar{t}^2 + 2\bar{g}_{0i}\left(\bar{t}\right) d\bar{t} dx^i + g_{ij}\left(\bar{t}\right) dx^i dx^j ,$$

where $\bar{g}_{0i}\left(\bar{t}\right) = g_{0i}\left(t(\tau(\bar{t})), \vec{0}\right)$, $\bar{g}_{ii}\left(\bar{t}\right) = g_{ij}\left(t(\tau(\bar{t})), \vec{0}\right)$ are obtained by inverting the relation $\bar{t}(\tau)$. This geometry is still very general and the universality of the ultralocal limit for geodesic observers (in which *any* geometry looks like Bianchi I) is completely lost.

CONCLUSIONS

- The UL limit does not extend to non-geodesic timelike observers ...
- ... but there are **exceptions**: variable mass test particles; E-frame test particles; fluid elements in FLRW and Bianchi cosmologies; self-interacting dark matter
- Trying to further extend the UL limit to non-geodesic timelike observers leads to mathematically ill-defined problems and the universality of the UL limit is lost.

THANK YOU