General covariance and dynamics with a Gauss law

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work in collaboration with Viqar Husain (arXiv: 2312.06079)

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The Tragic Tale of Sohni Mahiwal[1]

[1] Schimmel, Annemarie. 1976. *Pain and Grace : A Study of Two Mystical Writers of Eighteenth-Century Muslim India*. Leiden: E.J. Brill.

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> Ephemeral this name of mine,
> And unbaked my clay:
> Fall, oh, I shall fall!
> For souls such as mine
> Are condemned to perish;
> 'Tis a truth, known to all!

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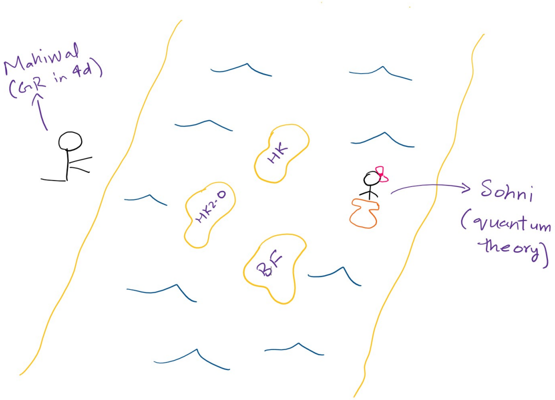
Lesson for quantum gravity



Prelude: BF theory

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Take a Lie group G and its associated Lie algebra $\mathfrak{g}$ and a spacetime manifold M. Consider a $\mathfrak{g}$-valued spacetime 2-form B and a G-connection A, and construct the action

$$S = \frac{1}{2}\int_M d^4x\,\mathrm{Tr}(B \wedge F),$$

where $F = dA + \frac{1}{2}A \wedge A$ is the curvature of A.

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- Equation of motion for B implies $F = 0$, i.e. A is flat – the theory is topological.
- The action is diffeomorphism invariant, but the first class constraints of the theory generate G-valued gauge transformations of A and B only. Hence, canonical quantization of the theory is easy.

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Contrast also with BF theory, where $e \wedge e$ is replaced with an $\mathfrak{su}(2)$-valued 2-form B.

Husain and Kuchar, Phys. Rev. D 42, 4070 (1990).

Canonical HK

Assuming $M = \mathbb{R} \times \Sigma$, the canonical decomposition of the action is straightforward:

$$S = \int dt \int_\Sigma d^3x (\tilde{E}^a_i \dot{A}^i_a - A^i_0 \tilde{G}_i - (e^i_0 E^a_i) \tilde{C}_a)$$

where $\tilde{E}^a_i = \det(e) E^a_i = \tilde{\epsilon}^{0abc} \epsilon_{ijk} e^j_b e^k_c$, and

$$\tilde{G}_i = -D_a \tilde{E}^a_i \approx 0 \quad \text{(Gauss)}$$
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- There's an invertible spatial metric $g_{ab} = \delta_{ij} e^i_a e^j_b$, $a, b \in \{1, 2, 3\}$. Thus interesting three-geometries exist.

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Consider the replacement

$$e_a^i \to D_a \phi^i$$

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The action now decomposes as

$$S = \int dt d^3x (\tilde{E}_i^a \dot{A}_a^i + \tilde{p}_i \dot{\phi}^i - A_0^i \tilde{G}_i);$$

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with only one constraint, a modified Gauss law with a source:

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No Hamiltonian and diffeomorphism constraints!

But whither the constraints?

The theory is generally covariant. But the first-class constraints of the theory (namely, the Gauss law) generate only $SU(2)$ transformations of the gauge field A. Where does the remaining gauge redundancy go?

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In these theories, for any generator of diffeomorphisms v,

$$\mathcal{L}_v A = G\text{-transformations} + \text{equations-of-motion terms}$$

where G is the gauge group of the connection A.

Horowitz, Commun. Math. Phys. 125, 417-437 (1989). Witten, Nuclear Physics B311 (1988/89) 46-78. Henneaux and Teitelboim, *Quantization of Gauge Systems*. Princeton University Press, 1992.

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Though the full story is a bit more subtle, essentially, if the first equation is required to hold, it can be shown that diffeomorphisms are equivalent to $SU(2)$ gauge transformations.

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  Thus, by appropriately choosing A and ϕ, one can construct any 3-metric.

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- Would be interesting to look at the spinfoam and group field theory models of the theory (work currently underway).

Thank you!