On Form-Preserving Wave Functions

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Outline

Accelerating free particles?

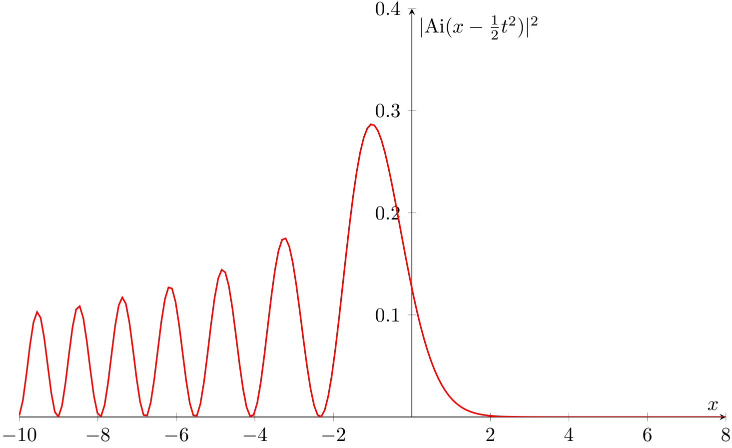
- Airy beam: free space solution to TDSE (Berry and Balazs, 1979)
- No forces, yet accelerates $a = B^3/(2m^2)$

$$\psi(x,t) = \mathrm{Ai}\left[\frac{B}{\hbar^{2/3}}\left(\underbrace{x - \frac{B^3 t^2}{4m^2}}_{\text{Acceleration!}}\right)\right]e^{i\phi(x,t)}$$

$$\phi(x,t) = \frac{B^3 t}{2m\hbar}\left(x - \frac{B^3 t^2}{6m^2}\right)$$

- Airy function $\longleftrightarrow$ linear potential

Airy beam probability density



Coherent excited states

- Coherent ground state — Gaussian wave packets classical motion, arbitrary amplitude, same ω (Schrödinger, 1926)
- Excited states also exhibit this (Senitzky, 1954)

$$\psi_n(x,t) = \psi_n^{\mathsf{SHO}}(x - u(t))e^{i\phi(x,t)}$$
$$\ddot{u} = -\omega^2 u$$

$$\hbar\phi(x,t) = -\hbar\omega\left(n+\frac{1}{2}\right)t + mx\dot{u} + \frac{1}{2}mu\dot{u}$$

Why are they interesting

- Airy beam — beam of particles with collective behavior looks like acceleration (Berry and Balazs, 1979)
- CES — localized particle with quantum and classical properties
- Obvious differences but there seems to be deeper similarities connecting them

Form of solutions

- Firstly, both have the form

$$\psi(x,t) = \psi_0(x - u(t))e^{i\phi(x,t)}$$

- Probability density $|\psi|^2$ form preservation

- $\psi_0(x')$ solution to another Schrödinger equation

- $u(t)$ obeys classical eq. of motion

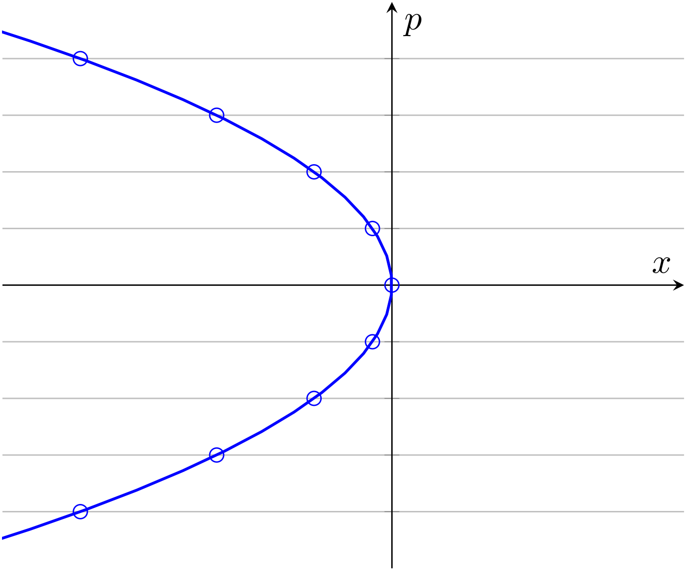
Rigid Hamiltonian flow

- Special curves remain rigid under Hamiltonian flow
- Free space (force-free) flow:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = 0,$$

- Rigid curves are parabolas

Free space Hamiltonian Flow



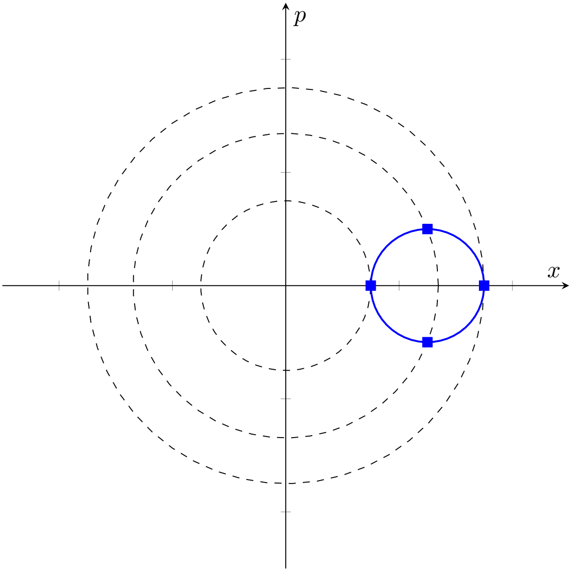
Harmonic potential rigidity

- Harmonic Hamiltonian flow:

$$\dot{x} = \frac{p}{m} \qquad \dot{p} = -m\omega^2 x,$$

- Rigid curves are ellipses / circles

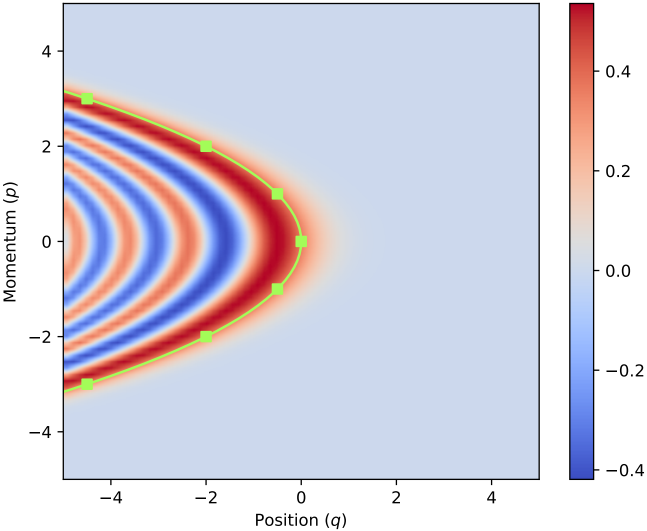
Harmonic potential rigidity



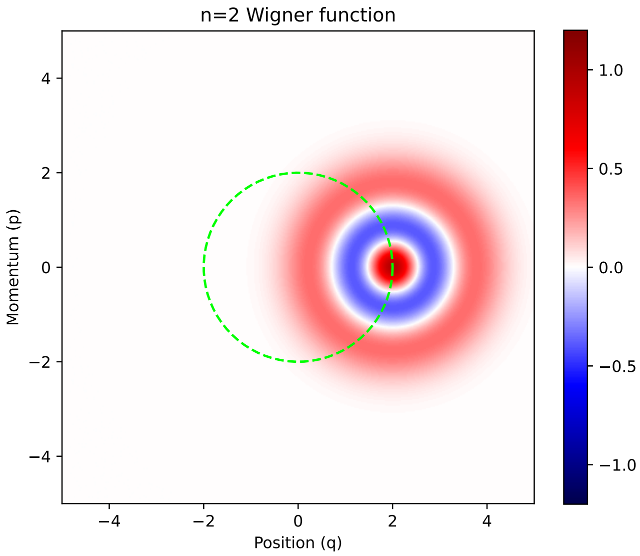
Phase space quantum mechanics

- QM possible in phase space $(\mathbf{x}, \mathbf{p})$
- Performed on Wigner functions $W(\mathbf{x}, \mathbf{p}; t)$
- Wigner function $\longleftrightarrow$ wave function

Airy Beam Wigner function



CES Wigner function



Moyal Equation

- Time evolution in phase space

$$\frac{\partial W}{\partial t} = -\frac{1}{i\hbar}(W \star H - H \star W)$$

- For quadratic potentials:

$$\frac{\partial W}{\partial t} = -\{W, H\}$$

Form preserving wave functions

- Both solutions share several mathematical similarities
- Simplest examples of form preserving wave functions
- Maps solutions of one potential to another $U(x', t')$ to $V(x, t)$

- Point transformations

$$x' = x + \beta \qquad t' = t$$

$$\Psi(x,t) = \psi_0(x',t') \exp\left[-\frac{im\dot{\beta}}{\hbar}x + i\alpha\right]$$

$$V(x,t) = U(x',t') + m\ddot{\beta}x - \frac{m}{2}\dot{\beta}^2 - \hbar\alpha.$$

Form preserving wave functions

- Extend the transform to "stretching"

$$x' = \frac{x}{\gamma} + \beta \qquad\qquad t' = \int_0^t \frac{d\tau}{[\gamma(\tau)]^2}$$

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{\gamma}}\psi_0(x',t')\exp\left[\frac{im}{2\hbar}\left(\frac{\dot{\gamma}}{\gamma}x^2 - 2\gamma\dot{\beta}x\right) + i\alpha\right]}$$

$$V(x,t) = \frac{U(x',t')}{\gamma^2} - \frac{m\ddot{\gamma}}{2\gamma}x^2 + m(2\dot{\gamma}\dot{\beta} + \gamma\ddot{\beta})x - \frac{m}{2}\gamma^2\dot{\beta}^2 - \hbar\alpha.$$

- The form preserving Wigner function is simplified
- Introduce $p'(x,p,t)$

$$p' = p - mx\dot{\gamma}/\gamma + m\gamma\dot{\beta}$$

$$\boxed{W(x,p;t) = \frac{1}{|\gamma|} W_0(x',p';t')}$$

Frame Transformations

- $\gamma = 1$ special case, frame transformations
- Einstein's equivalence principle (Nauenberg, 2016)
- Neutron interferometry (Collela et al., 1975, COW)
- Time dep. SHO $\rightarrow$ SHO, SHO $\rightarrow$ free, Cubic $\rightarrow$ Cubic

Summary

- Airy beam and CES
- Share several similarities
- Form preservation, amplitude is a solution to a different TDSE, and they correspond to rigid curves in classical phase space
- General class of so-called form-preserving wave functions
- Applications for quantum gravity and other experiments
- Plan to continue exploring the physics in phase space

Thank you!

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