// Xanadu Presents

Weight Reduced Stabilizer Codes with Lower Overhead

CAP Congress 2024, Western University, London Ontario

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Motivation

- Noise severely limits the performance of today's quantum computers
- Quantum error correction enables scalable quantum computation for applications
  - Factoring [Sho96], chemical simulation [BBM+20], optimisation [OML19], ...
  - But space & time overheads are currently prohibitive
- Big focus for academia, industry, and government
- Surprising connections to other fields in physics
  - Topological codes ↔ topological phases of matter [Kit03, Kit06]
  - Tensor network codes ↔ AdS/CFT correspondence [PYH+15]

Contents

XANADU

01

A bit on Xanadu

A bit on Xanadu

- A photonic quantum computing company with eyes set on fault-tolerance
- Full-stack:
    - Hardware, including Architecture
    - Software and Algorithms
- Located in downtown Toronto, Canada:

Premise of Xanadu's architecture

need low-depth physical circuits

photonics → MBQC

building blocks

nonlocal connectivity

path to scaling up

benefit of existing R&D

Probabilistic sources (with cryostats)

Deterministic entanglement generation (offline) and gates (via adaptive local measurements), at room temperature

Silicon chips, waveguides, fibers
Phase shifters, squeezers
Beamsplitters
Homodyne detectors
Photon **N**umber **R**esolving detectors

want fast, room-temperature, deterministic, accessible gates

GKP qubits

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Gaussian ~ "easy"
Non-Gaussian ~ hard

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MBQC

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MBQC

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blocks

nonlocal
connectivity

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Probabilistic sources (partly cold)

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photonics → MBQC

Probabilistic sources

Deterministic entanglement generation and gates

GKP qubits

02

Quantum error correction

For photonics

Quantum error correction 101

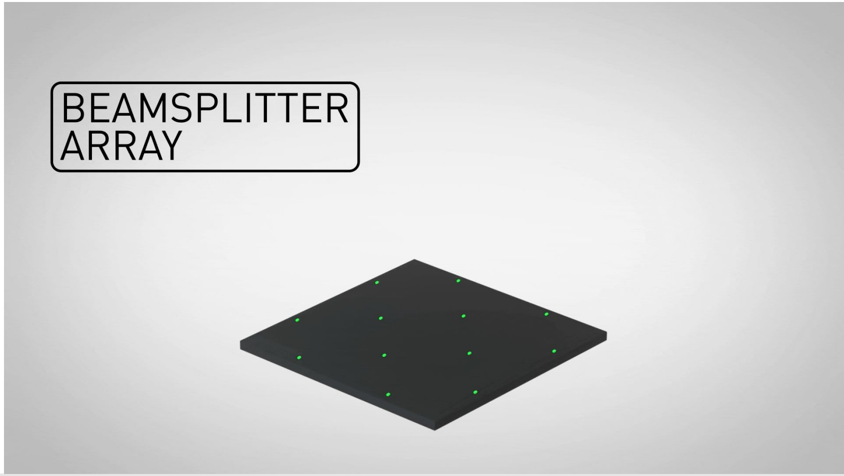
- n-qubit Hilbert space $(\mathbb{C}^2)^{\otimes n}$
- Quantum error correcting code: subspace of $(\mathbb{C}^2)^{\otimes n}$
- Code parameters $[\![n,k,d]\!]$
  - Number of physical qubits n
  - Number of encoded qubits k
  - Code distance d

Quantum error correction 101

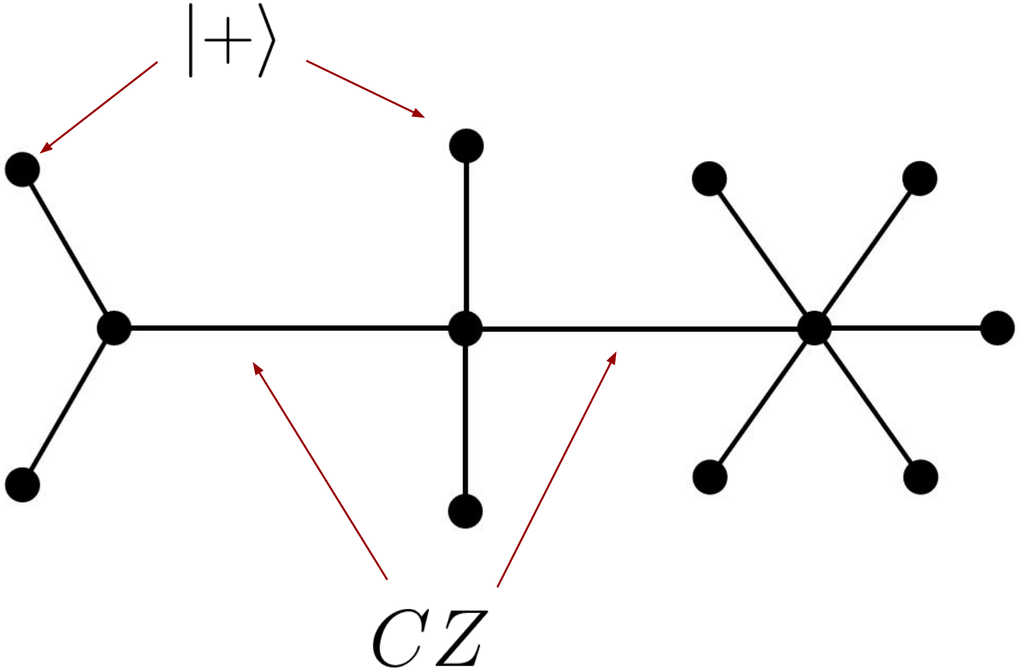
- Pauli matrices X, Y, and Z
- n-qubit Pauli group generated by tensor products of Paulis and I
- Stabilizer codes [Got97]
  - Abelian subgroup S of Pauli group
  - Code (sub)space $\{ |\Psi\rangle \text{ such that } g|\Psi\rangle = |\Psi\rangle \text{ for all } g \in S \}$
- Measure stabilizers to diagnose errors

Quantum error correction for photonics

- Measurement-based quantum computing [RBB03] natural for photonics
  - Start with entangled resource state (cluster state)
  - Computation proceeds via single-qubit measurements
- Foliation [BD-CP+16]
  - Input: quantum error-correcting code
  - Output: fault-tolerant cluster state



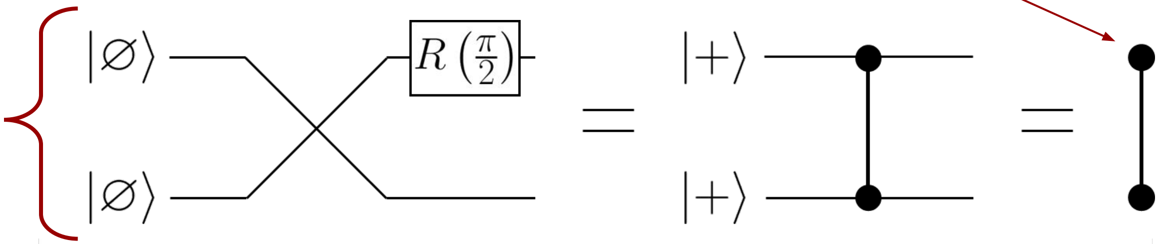
The target: a cluster state—a resource for FT MBQC [RHG06]

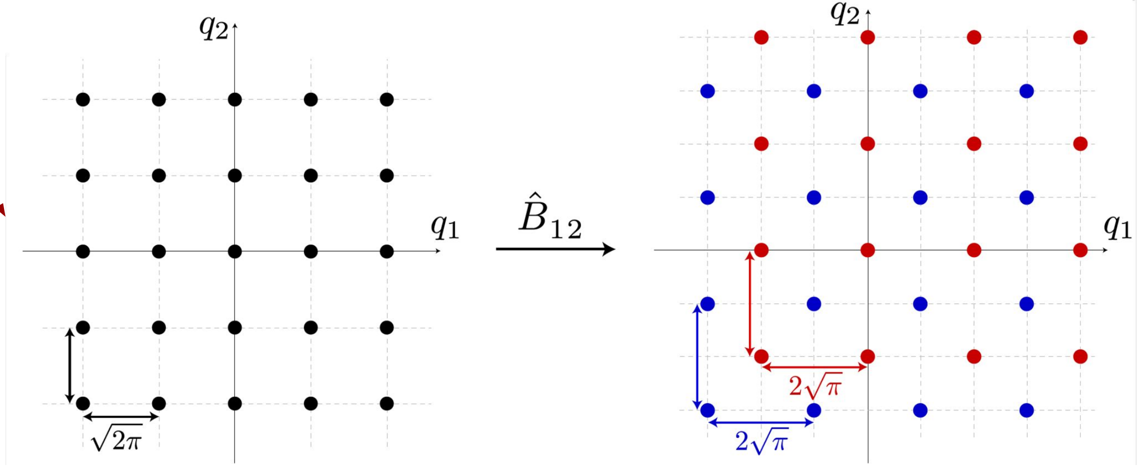


Entanglement generation with arbitrary connectivity [WBA+20,TMA+21]

Step 1: Create entangled GKP pairs ("dumbbells")

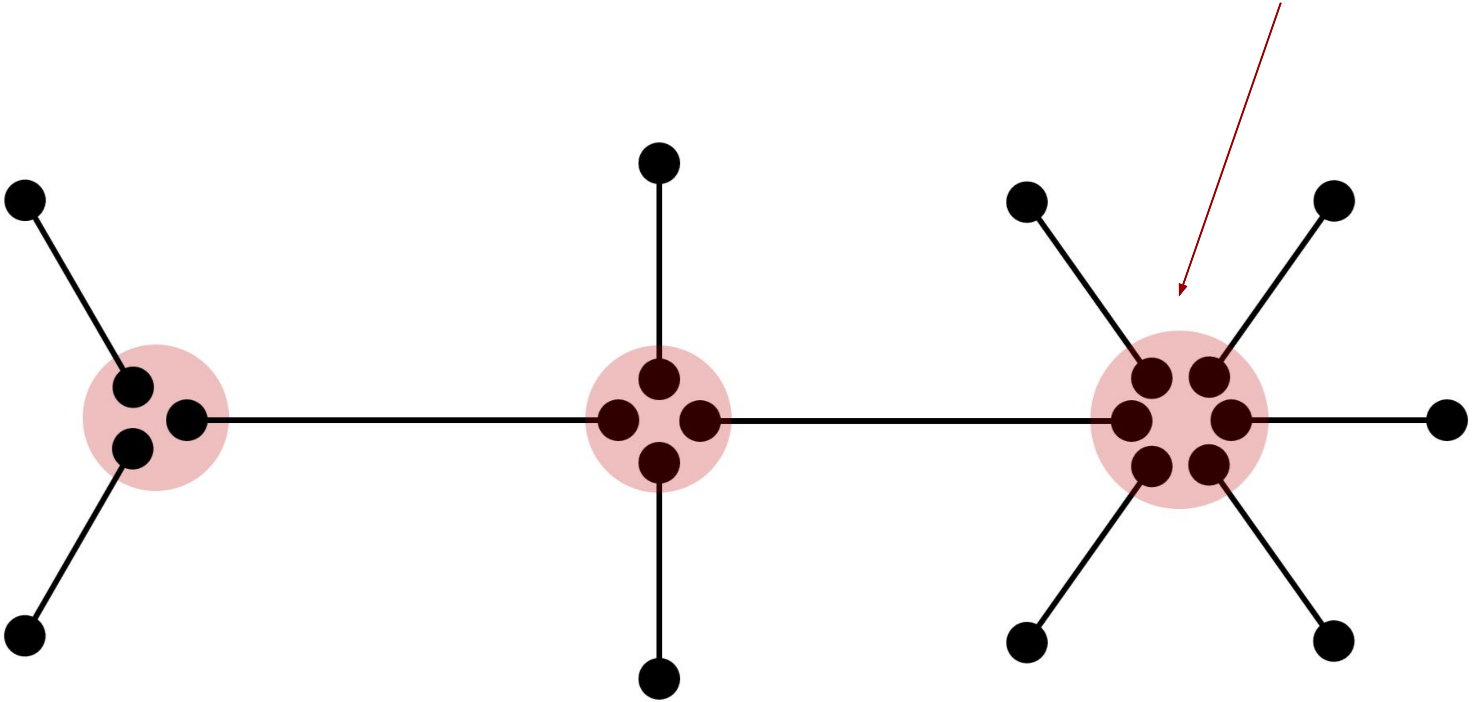
GKP sensor states
(q-naught states)





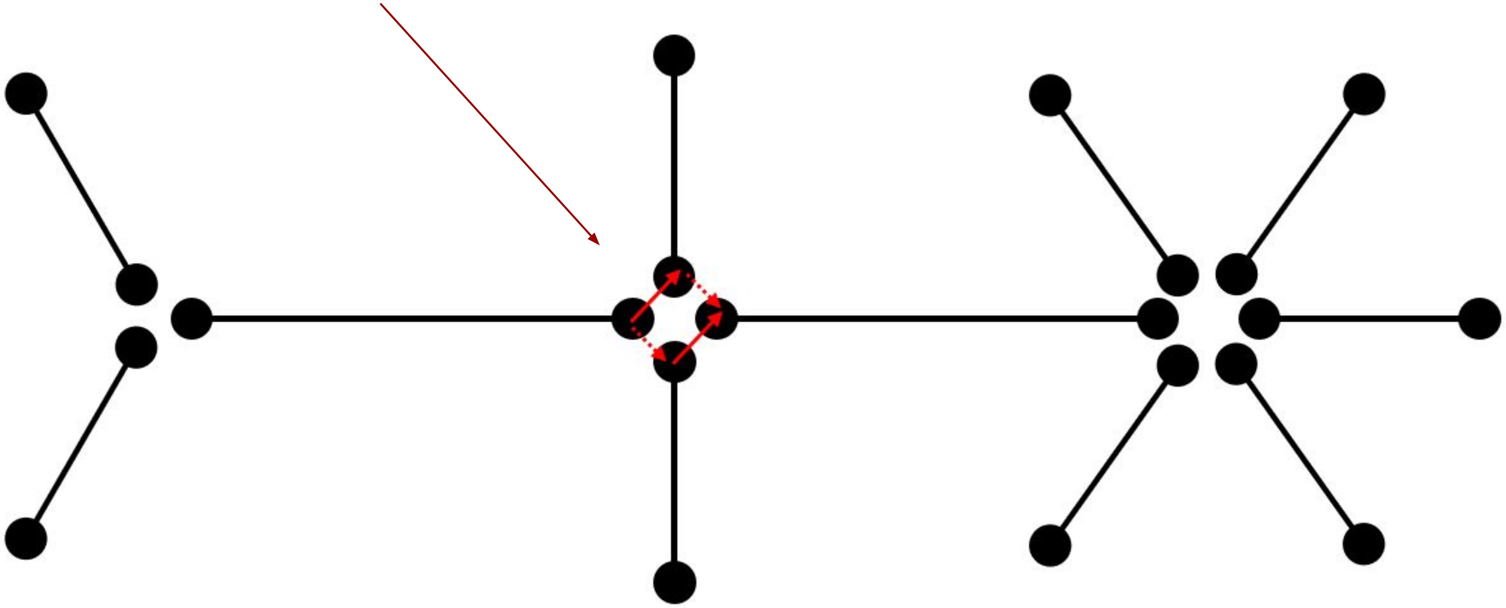
Entanglement generation with arbitrary connectivity

Step 2: Route modes from dumbbells to meet at macronodes



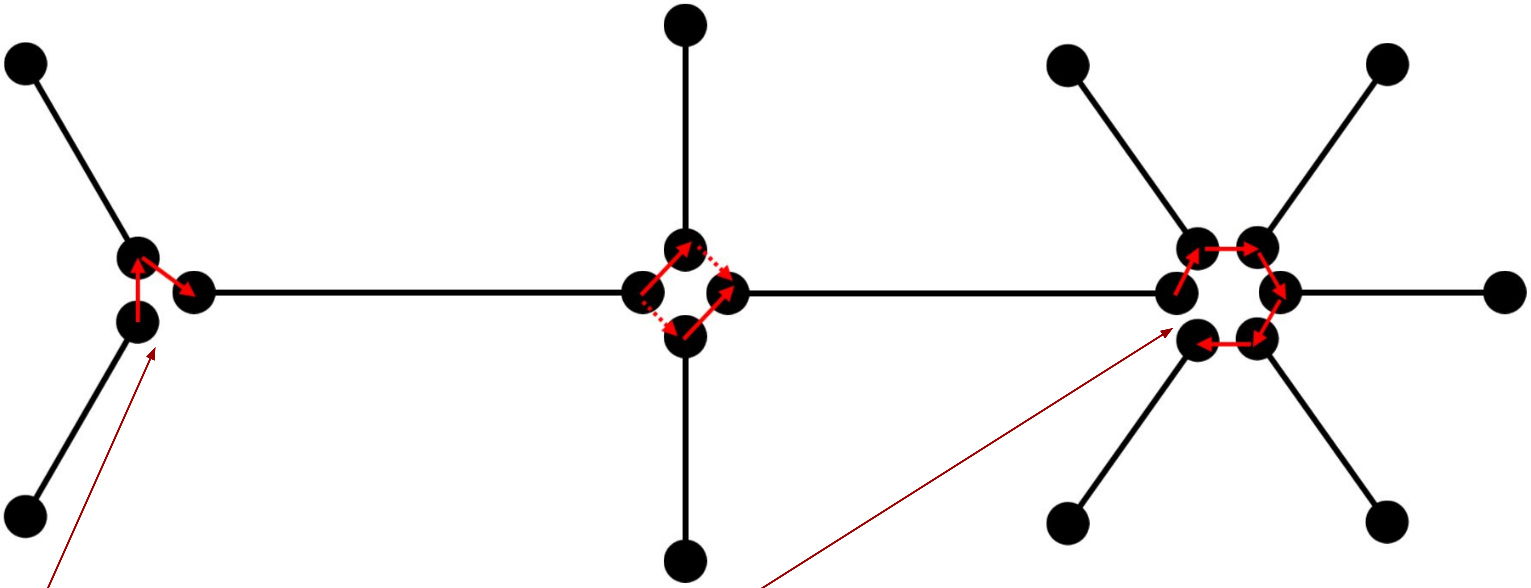
Step 3: Entangle modes within macronodes

balanced foursplitter: static array of 50:50 beamsplitters



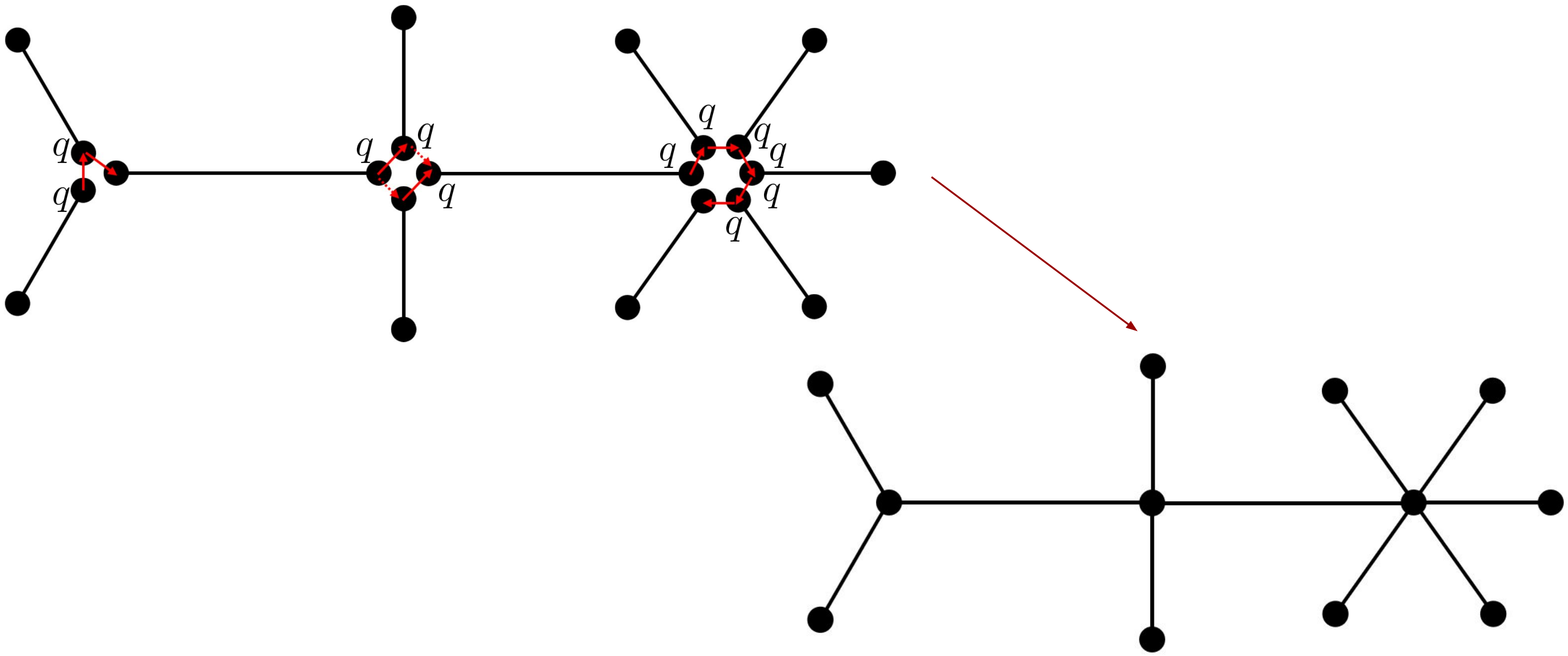
Entanglement generation with arbitrary connectivity

Step 3: Entangle modes within macronodes

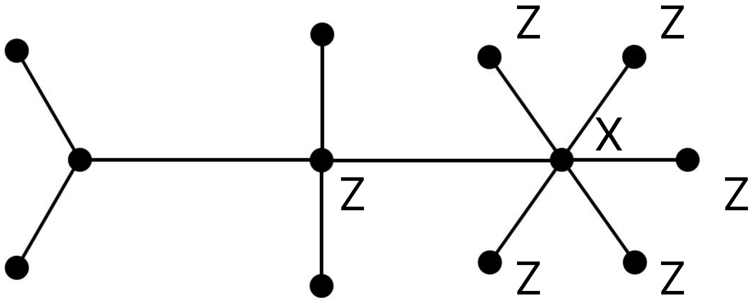


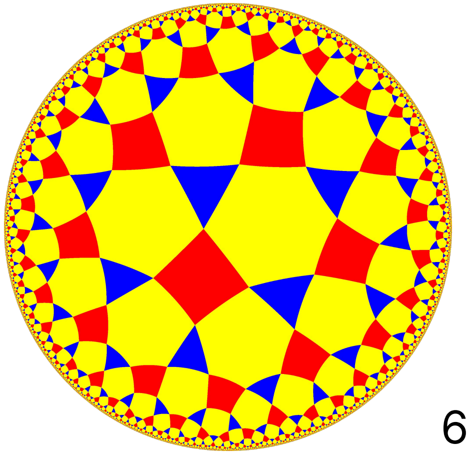
generalized splitters, e.g. linear cascade

Step 4: Finish reduction via homodyne measurements



Step 5: Implement QEC code cluster state of any valence





6.6.4.3 tiling

Implement checks corresponding to, e.g., a quantum code with arbitrary, potentially non-local, connectivity—at almost no cost to hardware

03

Weight reduction

For stabilizer codes

// Problem

The effective error in a stabilizer measurement usually scales with the weight of the stabilizer.

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The effective error in a stabilizer measurement usually scales with the weight of the stabilizer.

// Question

Given a code, can we reduce the weights of its stabilizers while retaining the desirable properties of the code?

Notation

- $[\![n,k,d]\!]$ stabilizer codes
  - Maximum stabilizer weight w
  - Maximum qubit degree q
- CSS codes
  - Parity-check matrices H_X and H_Z
  - Maximum stabilizer weights w_X and w_Z
  - Maximum qubit degrees q_X and q_Z

Weight reduction review

- Stabilizer code → subsystem / Floquet code [BD-CP+13, BFH+15, HH21, ...]
- Layer codes [WB23]
- Hastings's weight reduction [H16]
  - Works for any stabilizer code
  - Output code has stabilizer weights ≤ 5
  - Constant factor increase in n, constant factor decrease in d
- Hastings's procedure is complex
  - Essentially because of the requirement of preserving stabilizer commutation

Weight reduction review

- Weight reduction for classical codes
  - No need to worry about commutation
  - Can achieve check weights ≤ 3 [HHO21]
  - Checks can also be made geometrically local in two dimensions [B23]
- Many quantum code constructions use classical codes as input [TZ14, PK21, BE21, ...]

Classical weight reduction

- Let H be the parity-check matrix of an $[\,n,k,d\,]$ (classical) linear code
- Let h be a row of H with weight w
- Replace $h \mapsto [\,I_w\; 0\; H_w^{\mathsf{T}}\,]$ (H_w is the parity-check matrix of the w-bit repetition code)

$$
\begin{array}{cccccc}
v_1 & \cdots & v_w & & & \\
\end{array}
\!
\begin{array}{c}
\\
\begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}
\end{array}
\mapsto
\begin{array}{c}
\begin{array}{ccccccccccc}
v_1 & v_2 & \cdots & v_{w-1} & v_w & & v'_1 & v'_2 & \cdots & v'_{w-2} & v'_{w-1}
\end{array} \\
\begin{pmatrix}
1 & & & & & \cdots & 1 & & & & \\
 & 1 & & & & \cdots & 1 & 1 & & & \\
 & & \ddots & & & \cdots & & & \ddots & & \\
 & & & 1 & & \cdots & & & & 1 & 1 \\
 & & & & 1 & \cdots & & & & & 1
\end{pmatrix}
\end{array}.
$$

Classical weight reduction: Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Classical weight reduction

- Algorithm
  - Input: parity-check matrix H
    1. Apply weight reduction to each row with weight ≥ 4
    2. Transpose the output and repeat 1
    3. Undo the transpose
  - Output: new parity-check matrix H' with row and column weights ≤ 3

Examples: Hypergraph product codes

- For an input linear code with parity-check matrix H and parameters $[\ n,k,d\ ]$, the hypergraph product code HGP(H) has
  - parameters $[\![\ \Theta(n^2), \Theta(k^2), \Theta(d)\ ]\!]$
  - parity-check matrices $H_X = (\ H \otimes I \ I \otimes H^T\ )$ and $H_Z = (\ I \otimes H^T \ H \otimes I\ )$
- Let r and c be the row and column weights of H
- HGP(H) has $w = r + c$ and $q = \max(\ r,c\ )$
- Classical weight reduction: HGP(H) $\mapsto$ HGP(H') with $w' = 6$ and $q' = 3$

Examples: Hypergraph product codes

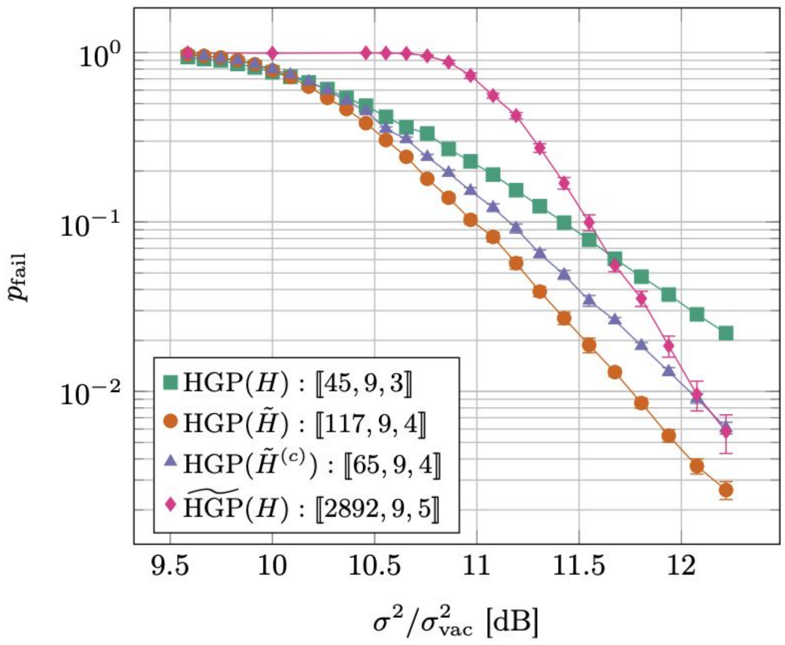
- *H* is the parity-check matrix of a [6,3,3] code, where *r* = 4 and *c* = 3
- HGP(*H*) has *w* = 7 and *q* = 4

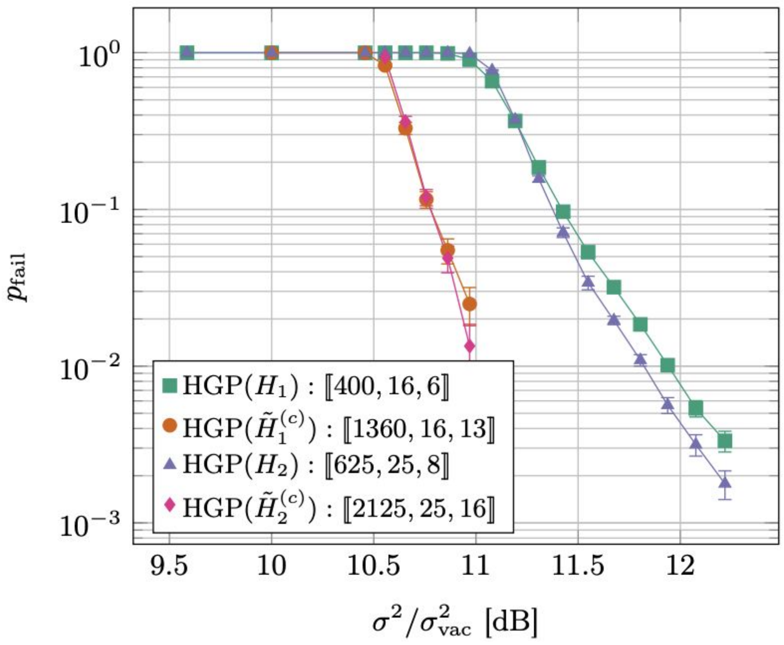
$\mathcal{C}(H)$	$\mathrm{HGP}(H)$	R	$\mathrm{HGP}(\tilde{H})$	R	$\mathrm{HGP}(\tilde{H}^{(c)})$	R
$[6,3,3]$	$[\![45,9,3]\!]$	0.200	$[\![117,9,4]\!]$	0.077	$[\![65,9,4]\!]$	0.138

- Compare with Hastings's weight reduction

$\mathcal{C}(H)$	$\mathrm{HGP}(H)$	R	(w_X, q_X, w_Z, q_Z)	$\widetilde{\mathrm{HGP}}(H)$	R	$(\tilde{w}_X, \tilde{q}_X, \tilde{w}_Z, \tilde{q}_Z)$
$[6,3,3]$	$[\![45,9,3]\!]$	0.200	(7, 4, 7, 4)	$[\![2892,9,5]\!]$	0.003	(6, 6, 6, 3)

Simulations: Xanadu architecture





Conclusion

Summary

- Classical weight reduction can reduce the stabilizer weights of quantum product codes while retaining competitive parameters
- The weight reduced codes have superior performance in Xanadu’s architecture for a fault-tolerant photonic quantum computer
- Paper on the arXiv [SGI+24]
  - Also contains a self-contained explanation of Hastings’s weight reduction algorithm with lots of examples (and some optimizations)
  - Both algorithms at https://github.com/esabo/CodingTheory

Open questions

- Can Hastings's algorithm be improved further?
- How does classical weight reduction compare with the layer codes approach?
- How to compare with the stabilizer → subsystem code approach?
- What is the lowest stabilizer weight compatible with e.g. constant encoding rate?

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Thank you

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Classical weight reduction

Theorem

Let H be the parity-check matrix of an $[\, n,k,d \,]$ binary, linear code, w be the maximum row weight of H, and q be the maximum column weight of H. Then Algorithm 1 outputs a parity-check matrix H' with $w' = q' = 3$ whose code has parameters $[\, N,k,D \,]$, where N = O(max(w, q) n) and $D \geq d$.

Proof

In the worst case, all checks have weight ≥ 4, which gives the bound on N.

Weight reducing the rows does not change the column weights and vice versa, so $w' = q' = 3$.

Classical weight reduction

$$f\big|_{\mathrm{supp}\,h_i} = I_{w_i}$$

$$\left(\begin{array}{c|c} h_1 & \\ \vdots & 0 \\ h_{i-1} & \\ \hline f & H^{\mathrm{T}}_{w_i\text{-}1} \\ \hline h_{i+1} & \\ \vdots & 0 \\ h_{n-k} & \end{array}\right),$$

Bits to the left = old, bits to the right = new.

Suppose c is a codeword of this parity-check matrix.

$$0 = \begin{pmatrix} f & H^{\mathrm{T}}_{w_i-1} \end{pmatrix} c = \begin{pmatrix} f & H^{\mathrm{T}}_{w_i-1} \end{pmatrix} \begin{pmatrix} c\big|_{\mathrm{old}} \\ c\big|_{\mathrm{new}} \end{pmatrix} = fc\big|_{\mathrm{old}} + H^{\mathrm{T}}_{w_i-1} c\big|_{\mathrm{new}}$$

$$H^{\mathrm{T}}_{w_i-1} c\big|_{\mathrm{new}} = fc\big|_{\mathrm{old}} = c\big|_{\mathrm{supp}\,h_i}$$

The image of $H_w{}^T$ contains only even weight strings.

Therefore $c|_{\mathrm{old}}$ is a codeword of the original code.

If $c|_{\mathrm{old}} = 0$ then $c|_{\mathrm{new}} = 0$ as $\ker H_w{}^T = 0$.

Hence $K = k$ and $D \geq d$. ∎

Examples: Lifted product codes

- The lifted product is a generalization of the hypergraph product to (amongst other things) quasi-cyclic code inputs.
- Quasi-cyclic codes are defined using matrices whose entries are elements of a polynomial quotient ring. Classical weight reduction generalizes straightforwardly to this case.
- Lifted product codes often have superior parameters to hypergraph product codes of similar size.

Examples: Lifted product codes

- Each weight reduction step can be randomized: $h \mapsto [\ \Pi\ I_w\ 0\ H_w^{\top}\ ]$.
- We find empirically that this can give substantial increases in the distance.

$\mathcal{C}(A)$	$\mathrm{LP}(A)$	R	$\mathcal{C}(\tilde{A})$	$\mathrm{LP}(\tilde{A})$	R	$\mathcal{C}(\tilde{A}^{(c)})$	$\mathrm{LP}(\tilde{A}^{(c)})$	R
$[52, 27, 6]$	$[[260, 58, \leq 6]]$	0.223	$[130, 27, 12 \to 14]$	$[[2132, 58, \leq 14]]$	0.027	$[78, 27, 6 \to 8]$	$[[676, 58, \leq 8]]$	0.086
$[28, 9, 10]$	$[[175, 19, \leq 10]]$	0.109	$[91, 9, 28 \to 33]$	$[[2191, 19, \leq 39]]$	0.009	$[49, 9, 14 \to 18]$	$[[595, 19, \leq 18]]$	0.032
$[36, 11, 12]$	$[[225, 21, \leq 12]]$	0.093	$[117, 11, 36 \to 40]$	$[[2817, 21, \leq 48]]$	0.007	$[63, 11, 18 \to 22]$	$[[765, 21, \leq 22]]$	0.027
$[68, 19, 18]$	$[[425, 29, \leq 18]]$	0.068	$[221, 19, 54 \to 62]$	$[[5321, 29, \leq 74]]$	0.005	$[119, 19, 32 \to 34]$	$[[1445, 29, \leq 34]]$	0.020
$[76, 21, 20]$	$[[475, 31, \leq 20]]$	0.065	$[247, 21, 60 \to 68]$	$[[5947, 31, \leq 93]]$	0.005	$[133, 21, 37 \to 38]$	$[[1615, 31, \leq 38]]$	0.019
$[124, 33, 24]$	$[[775, 43, \leq 24]]$	0.055	$[403, 33, 71 \to 84]$	$[[9703, 43, \leq 115]]$	0.004	$[217, 33, 44 \to 48]$	$[[2635, 43, \leq 48]]$	0.016