TRIUMF

An ab initio recipe for the taming nuclear-structure dependence of V_{ud}

Michael Gennari

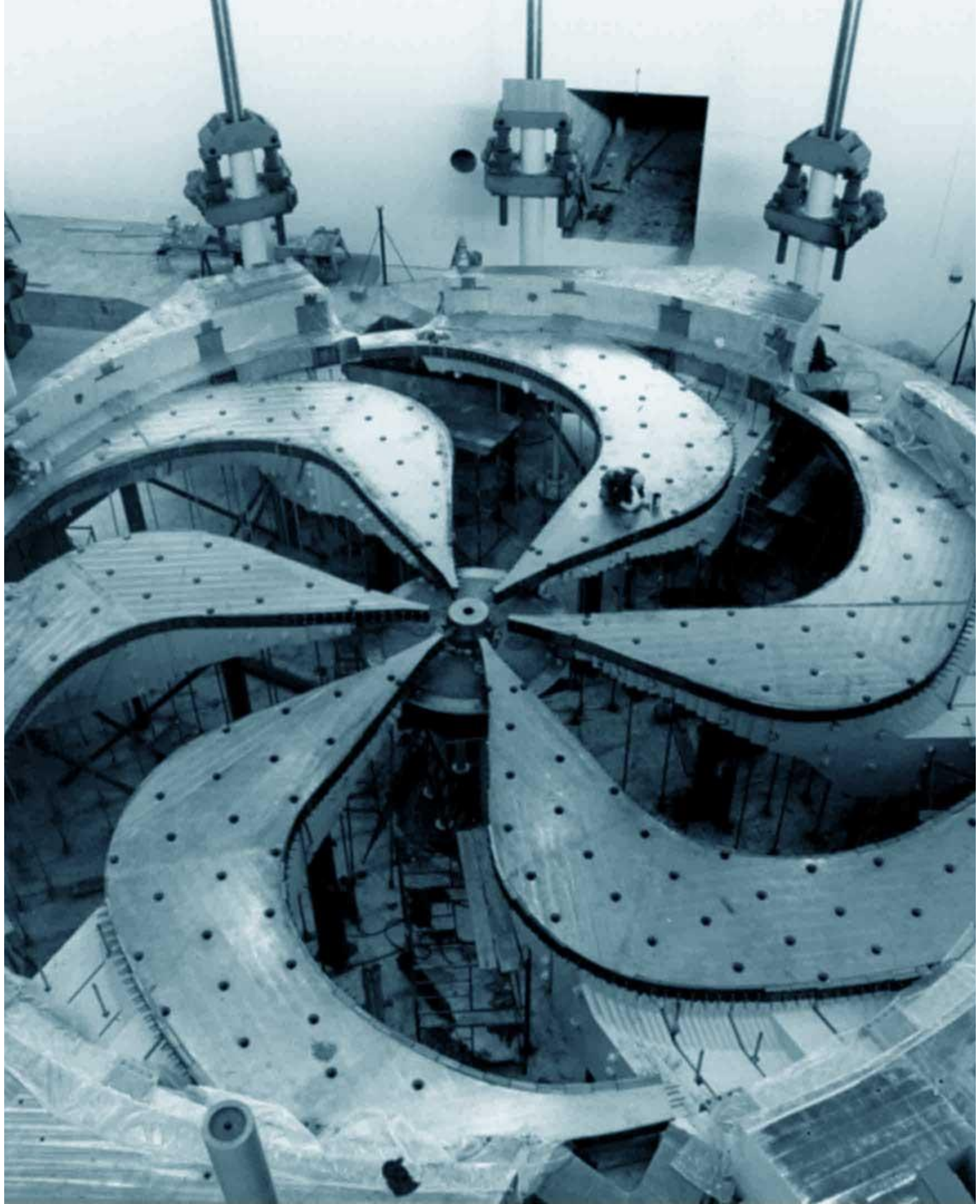
TRIUMF and University of Victoria

Collaborators: Mehdi Drissi, Mack Atkinson, Chien-Yeah Seng, Misha Gorchtein, Petr Navrátil

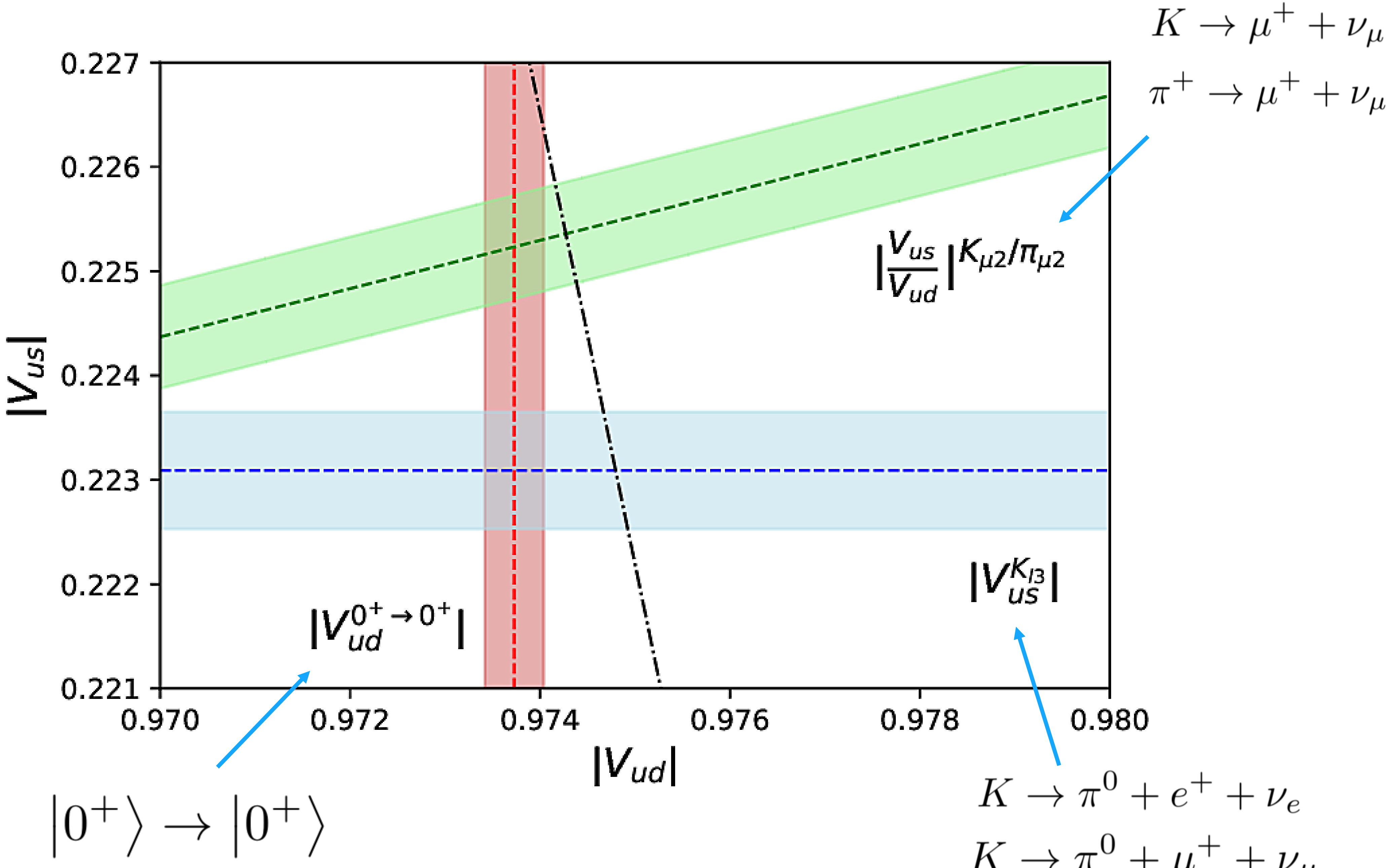
University of Victoria

FRIB

JOHANNES GUTENBERG UNIVERSITÄT MAINZ



Discovery, accelerated


$K \to \mu^+ + \nu_\mu$
$\pi^+ \to \mu^+ + \nu_\mu$
0.227
0.226
0.225
0.224
0.223
0.222
0.221
$|V_{us}|$
$\left|\frac{V_{us}}{V_{ud}}\right|^{K_{\mu2}/\pi_{\mu2}}$
$|V_{us}^{K_{l3}}|$
$|V_{ud}^{0^+\to 0^+}|$
0.970
0.972
0.974
0.976
0.978
0.980
$|V_{ud}|$
$|0^+\rangle \to |0^+\rangle$
$K \to \pi^0 + e^+ + \nu_e$
$K \to \pi^0 + \mu^+ + \nu_\mu$

Seng (2022) and references therein.

V_{ud} element of CKM matrix

	$\lvert V_{ud}\rvert$
superallowed	0.97373(31)[19]
n	0.97377(90)[20]
nuclear mirror	0.9739(10)[21]
π_{e3}	0.9740(28)[22]

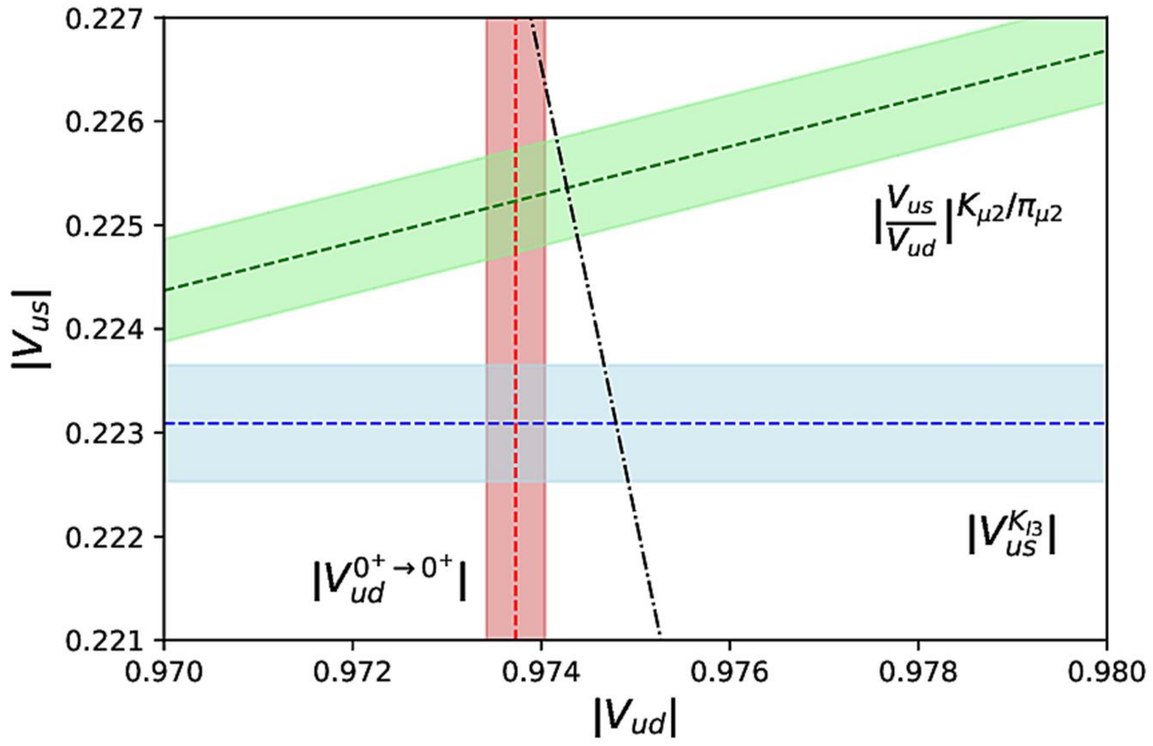
	$\lvert V_{us}\rvert$
$K_{\ell 3}$	0.22309(56)[23]
τ	0.2221(13)[24]
Hyperon	0.2250(27)[25]

	$\lvert V_{us}/V_{ud}\rvert$
$K_{\mu 2}/\pi_{\mu 2}$	0.23131(51)[23]
$K_{\ell 3}/\pi_{e3}$	0.22908(87)[23]

$$|0^+\rangle \to |0^+\rangle$$

$$n \to p e^- \bar{\nu}_e$$

$$\pi^+ \to \pi^0 e^+ \nu_e(\gamma)$$



Beta decay in the Standard Model

$$\mathcal{L}_{\mathrm{CC}} = -\frac{G_F}{\sqrt{2}} \begin{pmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{pmatrix} \gamma^{\mu} W_{\mu}^{+} V_{\mathrm{CKM}} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + \mathrm{h.c.}$$

$$|V_{ud}|^2 = \frac{\hbar^7}{G_F^2 m_e^5 c^4} \frac{\pi^3 \ln(2)}{\mathcal{F}t(1+\Delta_R^V)}$$

$$|0^+\rangle \rightarrow |0^+\rangle$$

$G_F \equiv$ Fermi coupling constant
determined from muon β decay

[2] Zyla et al. (2020)

V_{ud} element of CKM matrix

- Precise V_{ud} from superallowed Fermi transitions

$$|V_{ud}|^2 = \frac{\hbar^7}{G_F^2 m_e^5 c^4} \frac{\pi^3 \ln(2)}{\mathcal{F}t(1+\Delta_R^V)}$$

$G_F \equiv$ Fermi coupling constant
determined from muon β decay

  - hadronic matrix elements modified by nuclear environment
  - Fermi matrix element renormalized by isospin non-conserving forces

$$\mathcal{F}t = ft(1+\delta_R')\underline{(1-\delta_C+\delta_{NS})}$$

$$\mathcal{F}t = \frac{K}{G_V^2 |M_{F0}|^2 (1+\Delta_R^V)}$$

[2] Zyla et al. (2020)

Historical treatment

$$\delta_{NS,A} = \frac{\alpha}{\pi}\left[q_A q_S^{(0)} - 1\right] C_{\mathrm{Born}}^{\mathrm{free}}$$

Pre-2018 (for almost 30 years)

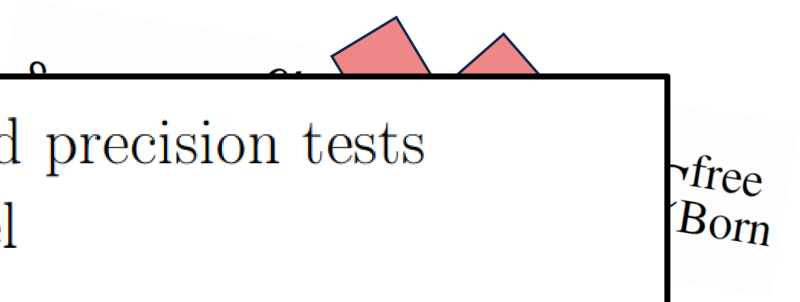
- δ_{NS} from shell model and approximate single-nucleon currents
- δ_C from shell model with Woods-Saxon potential

Since 2018

- Data-driven dispersion integral approach for Δ_R^V **[3-4]** which reduced radiative correction uncertainty by factor of ~ 2
- Ongoing nuclear theory [this work] and lattice QCD calculations of electroweak box diagrams

[3] Seng et al. (2018)
[4] Gorchtein et al. (2019)
[5] Hardy et al. (2020)

Historical treatment



Pre-2

- δ_{NS}
- δ_{C} f

Since

- Dat
  redu
- Ong
  of e

Superallowed nuclear beta decays and precision tests of the Standard Model

Mikhail Gorchtein[1,2] and Chien-Yeah Seng[3,4]

November 20, 2023

Abstract

For many decades, the main source of information on the top-left corner element of the Cabibbo-Kobayashi-Maskawa quark mixing matrix V_{ud} were superallowed nuclear beta decays with an impressive 0.01% precision. This precision, apart from experimental data, relies on theoretical calculations in which nuclear structure-dependent effects and uncertainties play a prime role. This review is dedicated to a thorough reassessment of all ingredients that enter the extraction of the value of V_{ud} from experimental data. We tried to keep balance between historical retrospect and new developments, many of which occurred in just five past years. They have not yet been reviewed in a complete manner, not least because new results are a-coming. This review aims at filling this gap and offers an in-depth yet accessible summary of all recent developments.

[3] Seng et al. (2018)
[4] Gorchtein et al. (2019)
[5] Hardy et al. (2020)

NCSM

Parent Nucleus	$\delta_{NS}(\%)$
^{10}C	-0.400(50)
^{14}O	-0.283(64)
^{18}Ne	-0.326(55)
^{22}Mg	-0.250(50)
^{26}Si	-0.234(54)
^{30}S	-0.195(56)
^{34}Ar	-0.181(60)
^{38}Ca	-0.167(64)
^{42}Ti	-0.233(68)
^{46}Cr	-0.164(72)
^{50}Fe	-0.140(75)
^{54}Ni	-0.143(79)

Parent Nucleus	$\delta_{NS}(\%)$
^{26m}Al	-0.019(51)
^{34}Cl	-0.093(57)
^{34m}K	-0.098(60)
^{42}Sc	0.033(64)
^{46}V	-0.031(65)
^{50}Mn	-0.029(69)
^{54}Co	-0.017(74)
^{62}Ga	-0.016(82)
^{66}As	-0.030(85)
^{70}Br	-0.049(89)
^{74}Rb	-0.032(94)

[5] Hardy et al. (2020)

NCSM

Parent Nucleus	$\delta_{NS}(\%)$
^{10}C	-0.400(50)
^{14}O	-0.283(64)
^{18}Ne	-0.326(55)
^{22}Mg	-0.250(50)
^{26}Si	-0.234(54)
^{30}S	-0.195(56)
^{34}Ar	-0.181(60)
^{38}Ca	-0.167(64)
^{42}Ti	-0.233(68)
^{46}Cr	-0.164(72)
^{50}Fe	-0.140(75)
^{54}Ni	-0.143(79)

Parent Nucleus	$\delta_{NS}(\%)$
^{26m}Al	-0.019(51)
^{34}Cl	-0.093(57)
^{34m}K	-0.098(60)
^{42}Sc	0.033(64)
^{46}V	-0.031(65)
^{50}Mn	-0.029(69)
^{54}Co	-0.017(74)
^{62}Ga	-0.016(82)
^{66}As	-0.030(85)
^{70}Br	-0.049(89)
^{74}Rb	-0.032(94)

CC

[5] Hardy et al. (2020)

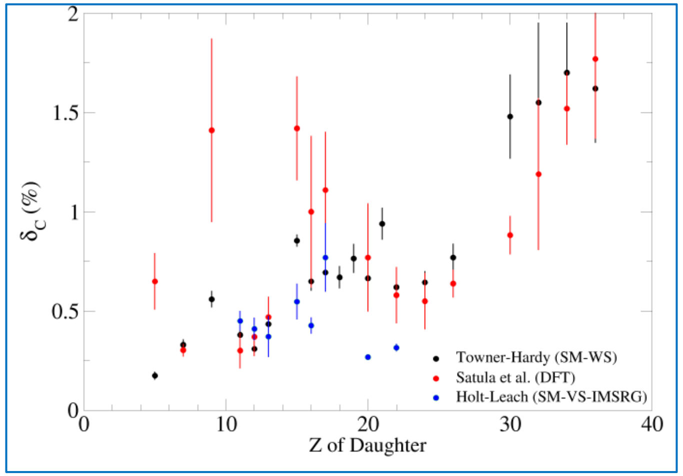
NCSM

Parent Nucleus	$\delta_{NS}(\%)$
^{10}C	-0.400(50)
^{14}O	-0.283(64)
^{18}Ne	-0.326(55)
^{22}Mg	-0.250(50)
^{26}Si	-0.234(54)
^{30}S	-0.195(56)
^{34}Ar	-0.181(60)
^{38}Ca	-0.167(64)
^{42}Ti	-0.233(68)
^{46}Cr	-0.164(72)
^{50}Fe	-0.140(75)
^{54}Ni	-0.143(79)

Parent Nucleus	$\delta_{NS}(\%)$
^{26m}Al	-0.019(51)
^{34}Cl	-0.093(57)
^{34m}K	-0.098(60)
^{42}Sc	0.033(64)
^{46}V	-0.031(65)
^{50}Mn	-0.029(69)
^{54}Co	-0.017(74)
^{62}Ga	-0.016(82)
^{66}As	-0.030(85)
^{70}Br	-0.049(89)
^{74}Rb	-0.032(94)

CC

VS-IMSRG



[5] Hardy et al. (2020)

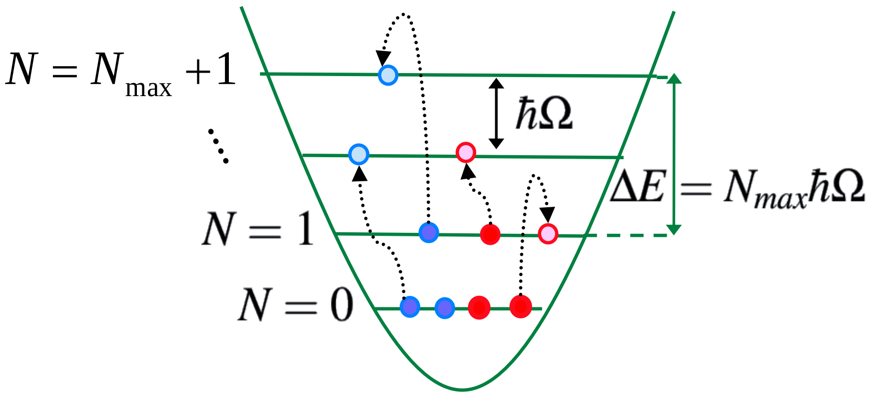
[6] Entem et al. (2017)
[7] Somà et al. (2020)

No-core shell model (NCSM)

- *Ab initio* approach to solving many-body Schrödinger equation
- Sole input are nuclear interactions from chiral effective field theory
  - NN-N[4]LO(500) **[6]**
  - 3N(lnl)-N[2]LO(650) **[7]**

$$H\left|\Psi_A^{J^\pi T}\right\rangle = E^{J^\pi T}\left|\Psi_A^{J^\pi T}\right\rangle$$

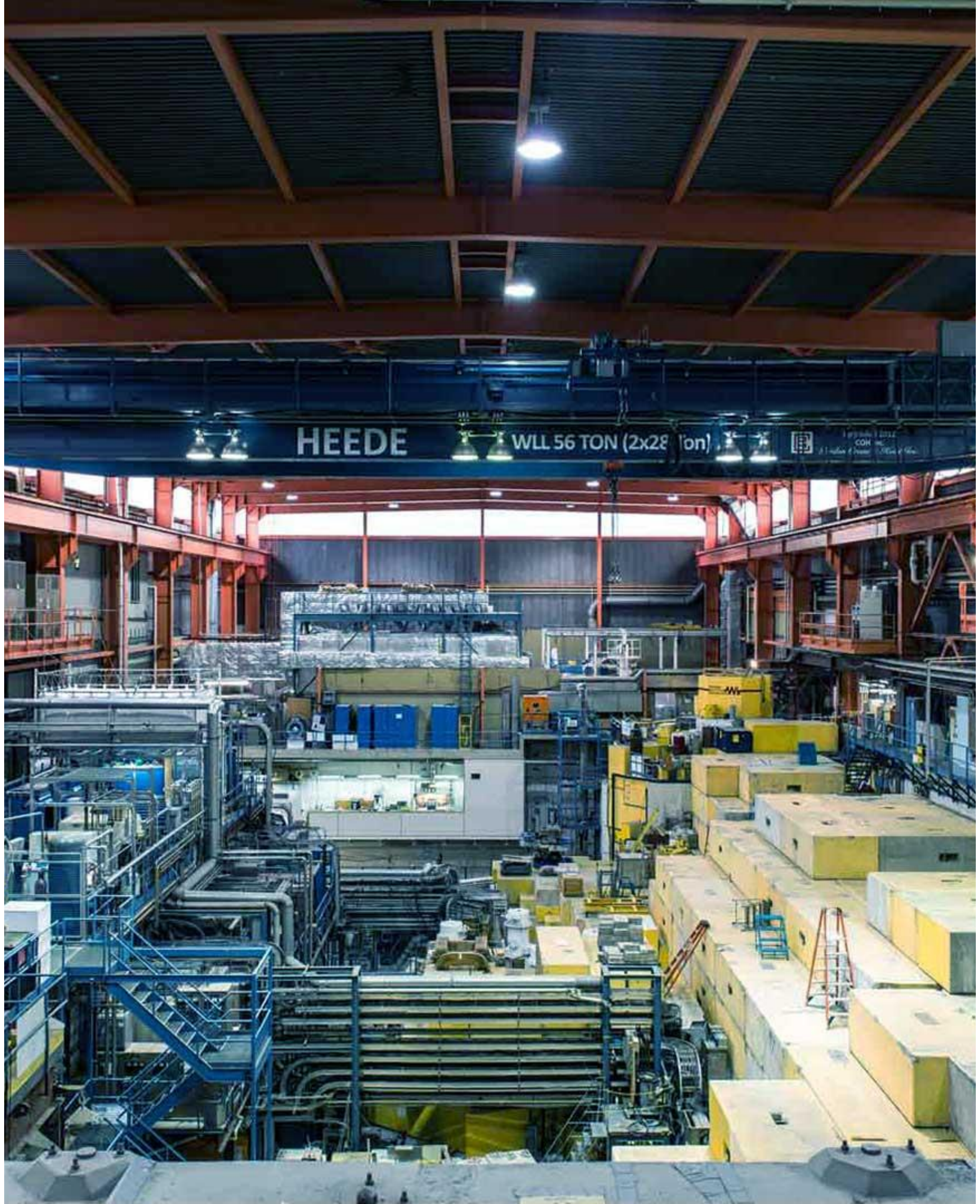
Anti-symmetrized products of many-body HO states



$$\left|\Psi_A^{J^\pi T}\right\rangle = \sum_{N=0}^{N_{max}} \sum_{\alpha} c_{N\alpha}^{J^\pi T} \left|\Phi_{N\alpha}^{J^\pi T}\right\rangle$$

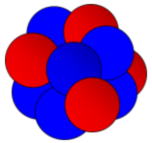
$$\left|\Psi_A^{J^\pi T}\right\rangle_{\mathrm{SD}} = \sum_{N=0}^{N_{max}} \sum_{\alpha} [c^{(\mathrm{SD})}]_{N\alpha}^{J^\pi T} \left|\Phi_{N\alpha}^{J^\pi T}\right\rangle_{\mathrm{SD}} = \left|\Psi_A^{J^\pi T}\right\rangle \otimes \left|\Phi_{000}\right\rangle$$

Electroweak radiative correction δ_{NS}



Discovery, accelerated

Standard Model



$SU(3)_C$

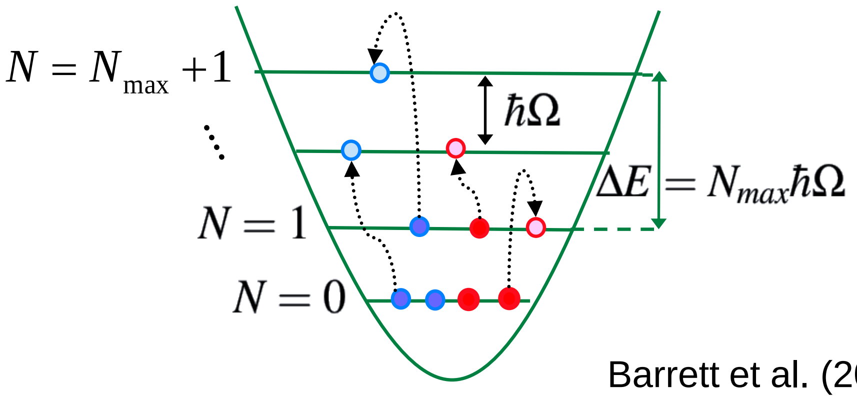
$SU(2)_L \times U(1)_Y$

Chiral Effective Field Theory

Entem et al. (2017)
Somà et al. (2020)

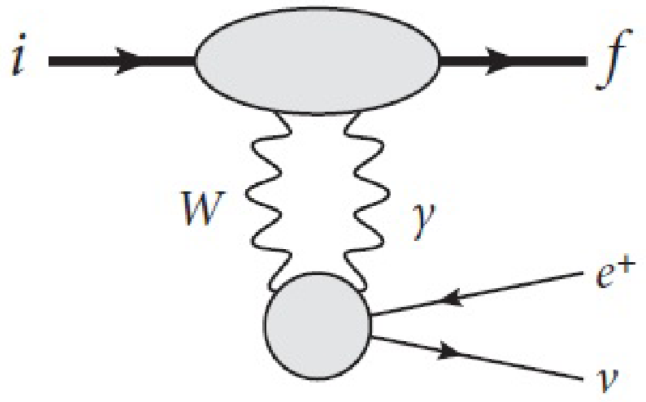
Weinberg (1991)
Epelbaum (2009)

$$H\left|\Psi_A^{J^\pi T}\right\rangle = E^{J^\pi T}\left|\Psi_A^{J^\pi T}\right\rangle$$



Barrett et al. (2013)
Haydock (1974)

E



- **Ultimate goal –** consistent chiral expansion for electroweak currents
- **For now –** leading multipole expansion

Haxton et al. (2007)
Seng et al. (2023)

Δ_R^V and δ_{NS}

Leptonic current

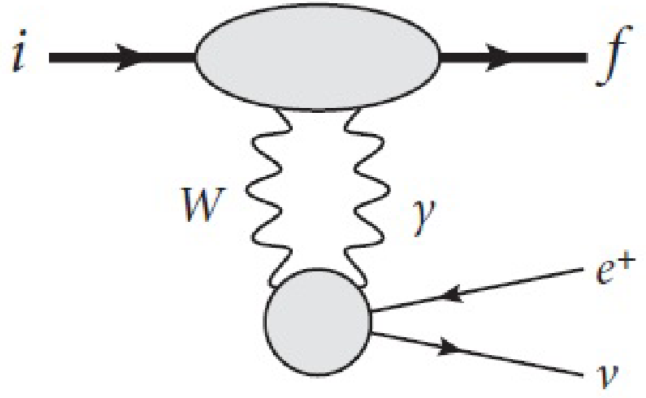
NME of charged weak current

- Tree level beta decay amplitude $$M_{tree} = -\frac{G_F}{\sqrt{2}} L_\lambda F^\lambda(p', p)$$

- Hadronic correction in forward scattering limit

$$\delta M = -i\sqrt{2} G_F e^2 L_\lambda \int \frac{d^4q}{(2\pi)^4} \frac{M_W^2}{M_W^2 - q^2} \frac{\epsilon^{\mu\nu\alpha\lambda} q_\alpha}{[(p_e - q)^2 - m_e^2] q^2} T_{\mu\nu}(p', p, q)$$

$$\delta M = \Box_{\gamma W}(E_e) M_{tree}$$



[8] Seng et al. (2023)

Δ_R^V and δ_{NS}

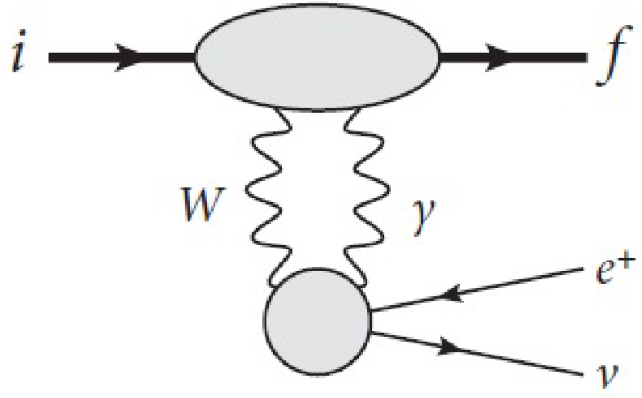
Leptonic current

NME of charged weak current

- Tree level beta decay amplitude

$$M_{tree} = -\frac{G_F}{\sqrt{2}} L_\lambda F^\lambda(p', p)$$

- Hadronic correction in forward scattering limit



$$\delta M = \Box_{\gamma W}(E_e) M_{tree}$$

$$\Box_{\gamma W}^b(E_e) = \frac{e^2}{M} \int \frac{d^4 q}{(2\pi)^4} \frac{M_W^2}{M_W^2 - q^2} \frac{1}{q^2 + i\epsilon} \frac{1}{(p_e - q)^2 + i\epsilon'} \frac{M\nu\left(\frac{p_e \cdot q}{p \cdot p_e}\right) - q^2}{\nu} \frac{T_3(\nu, |\vec{q}|)}{f_+(0)}$$

[8] Seng et al. (2023)

Δ_R^V and δ_{NS}

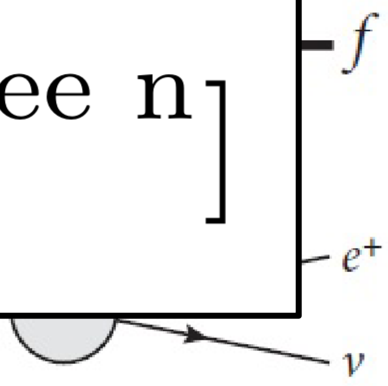
- Tree level beta decay amplitude $$M_{tree} = -\frac{G_F}{\sqrt{2}} L_\lambda F^\lambda(p', p)$$

Leptonic current

NME of charged weak current

- Hadronic correction in forward scattering limit

$$\delta_{NS} = 2[\Box^{\text{b,nuc}}_{\gamma W} - \Box^{\text{b,free n}}_{\gamma W}]$$



$$\Box^b_{\gamma W}(E_e) = \frac{e^2}{M} \int \frac{d^4 q}{(2\pi)^4} \frac{M_W^2}{M_W^2 - q^2} \frac{1}{q^2 + i\epsilon} \frac{1}{(p_e - q)^2 + i\epsilon'} \frac{M\nu\left(\frac{p_e \cdot q}{p \cdot p_e}\right) - q^2}{\nu} \frac{T_3(\nu, |\vec{q}|)}{f_+(0)}$$

[8] Seng et al. (2023)

Lanczos strength method

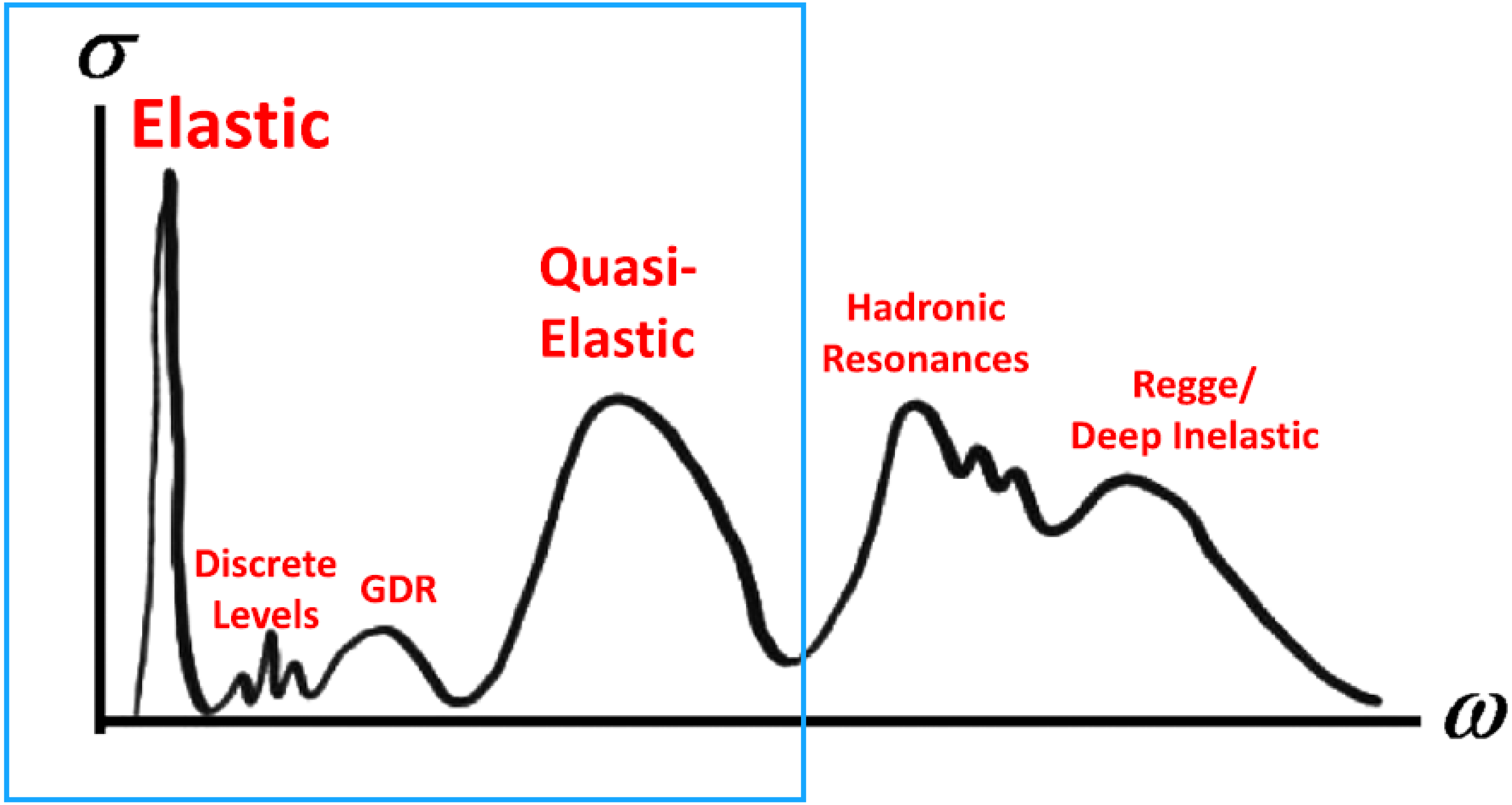
- Reformulate resolvent operator as inhomogeneous Schrödinger equation

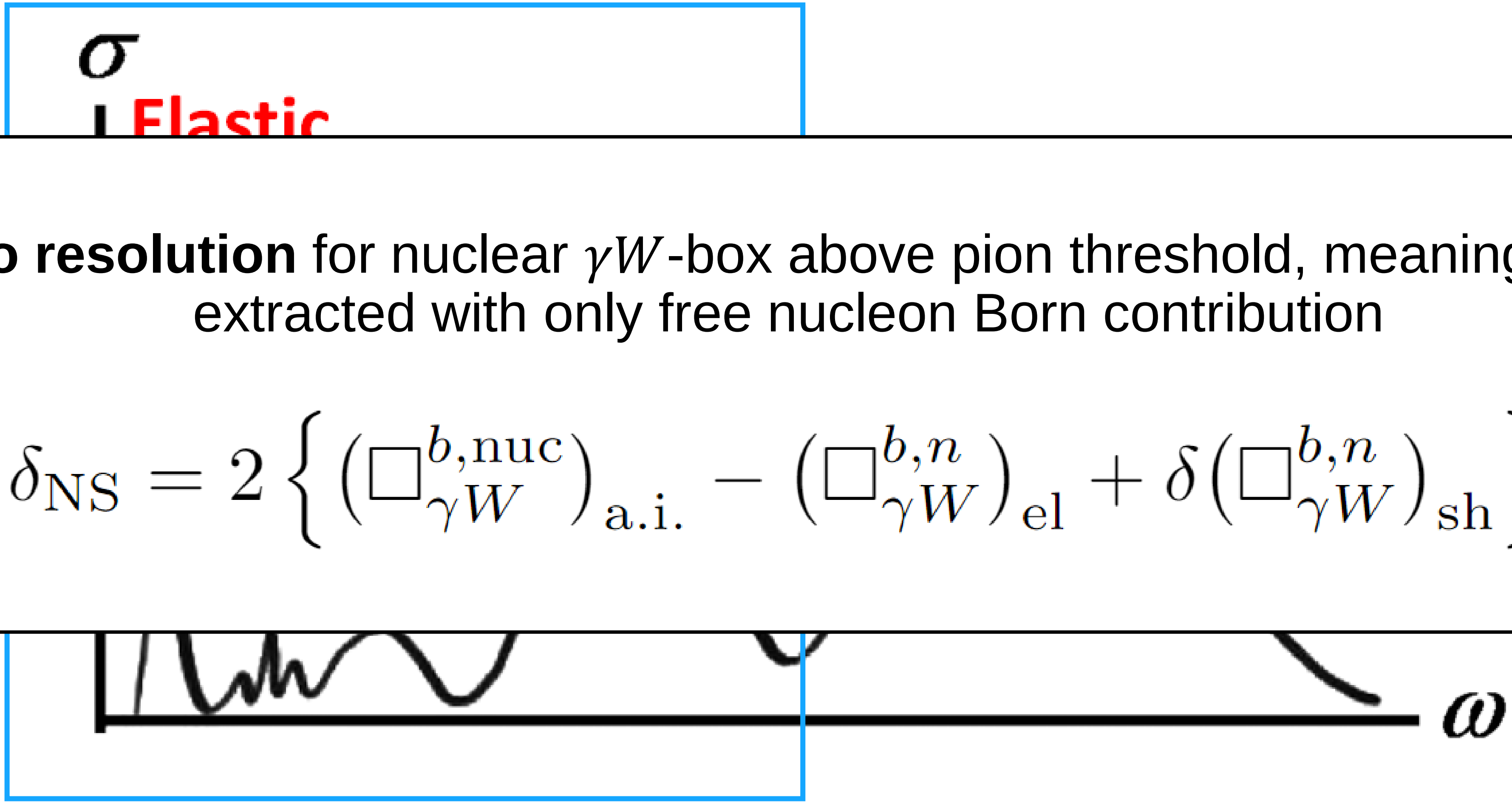
$$(H - E\mathbb{1})|\Phi\rangle = \hat{O}|\Psi\rangle$$

$$\begin{aligned}
H\mathbf{v}_1 &= \alpha_1\mathbf{v}_1 + \beta_1\mathbf{v}_2 \\
H\mathbf{v}_2 &= \beta_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \beta_2\mathbf{v}_3 \\
H\mathbf{v}_3 &= \beta_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \beta_3\mathbf{v}_4 \\
H\mathbf{v}_4 &= \beta_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \beta_4\mathbf{v}_5
\end{aligned}$$

- Resolvent amplitudes reconstructed via Lanczos basis
- Avoids (total) brute force calculation of intermediate states

[9] Haydock (1974)
[10] Dagotto (1994)


σ
Elastic
Discrete
Levels
GDR
Quasi-
Elastic
Hadronic
Resonances
Regge/
Deep Inelastic
ω



No resolution for nuclear γW-box above pion threshold, meaning δ_{NS} extracted with only free nucleon Born contribution

$$\delta_{\mathrm{NS}} = 2\left\{ \left(\Box_{\gamma W}^{b,\mathrm{nuc}}\right)_{\mathrm{a.i.}} - \left(\Box_{\gamma W}^{b,n}\right)_{\mathrm{el}} + \delta\left(\Box_{\gamma W}^{b,n}\right)_{\mathrm{sh}} \right\}$$

Nuclear poles

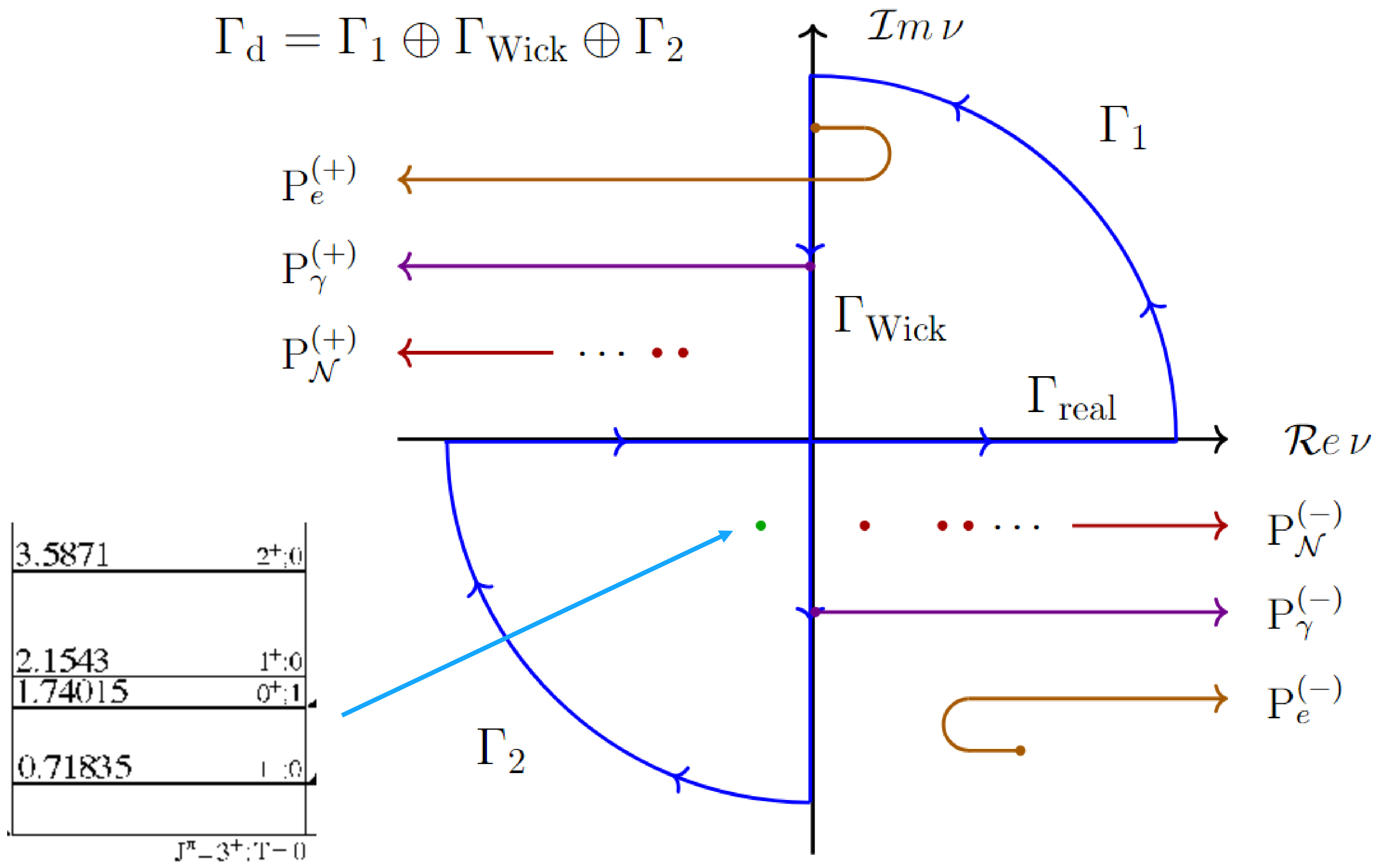
$$G(\nu + M_f + i\epsilon) = \sum_n \frac{|n\rangle\langle n|}{[\nu + M_f + i\epsilon] - M_n} \quad : \quad P_- = \{M_n - M_f - i\epsilon\}$$

$$G(-\nu + M_i + i\epsilon) = \sum_n \frac{|n\rangle\langle n|}{[-\nu + M_i + i\epsilon] - M_n} \quad : \quad P_+ = \{M_i - M_n + i\epsilon\}$$

- Traverse infinite number of poles in discrete and continuous nuclear spectrum
- Natural solution is Wick rotation

K.C. Greene

Wick rotation



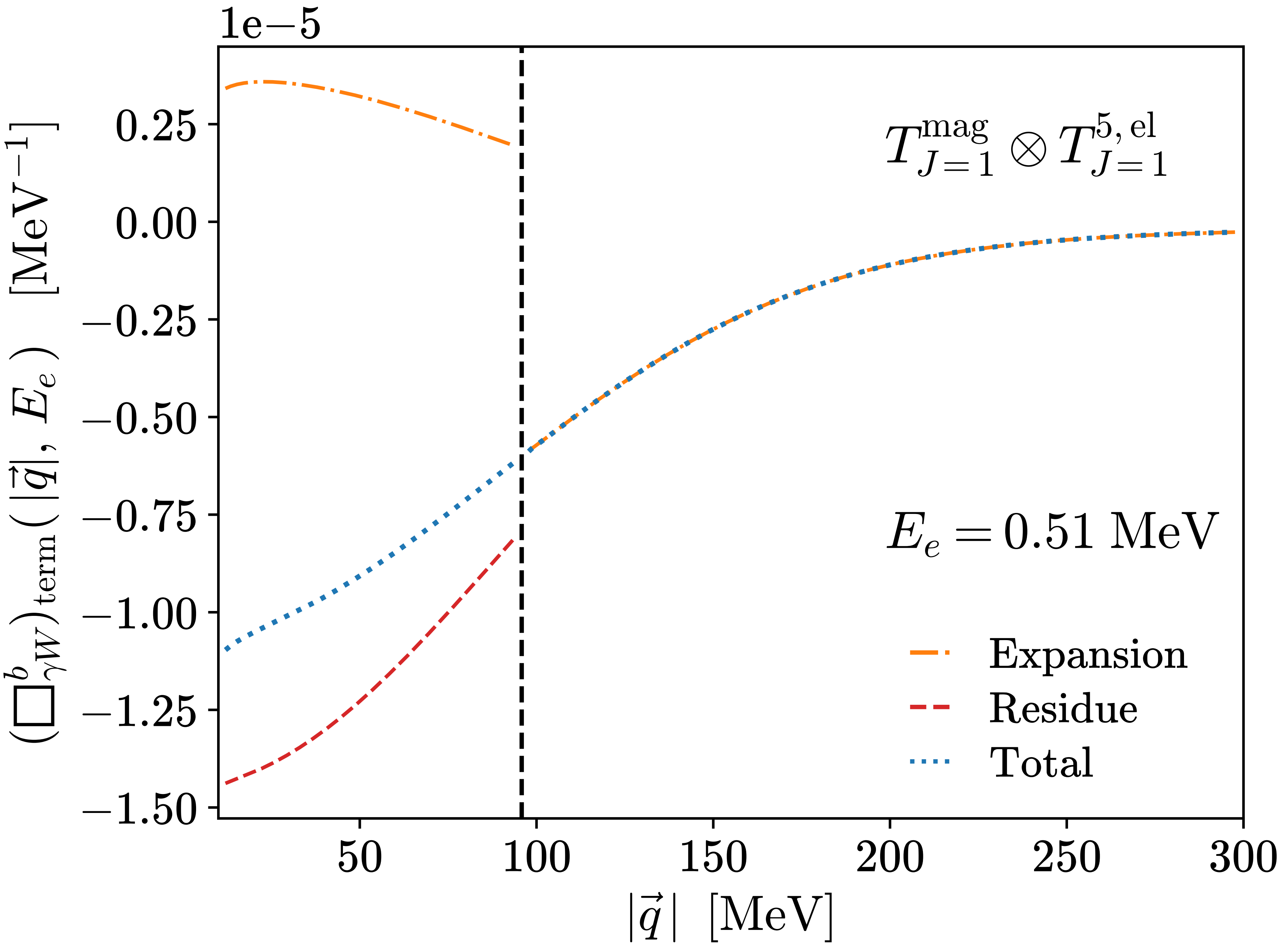
Wick rotation and electron energy expansion

$$\Box^b_{\gamma W}(E_e) = \left(\Box^b_{\gamma W}\right)_{\text{Wick}}(E_e) + \left(\Box^b_{\gamma W}\right)_{\text{Res},e}(E_e) + \left(\Box^b_{\gamma W}\right)_{\text{Res},T_3}(E_e)$$

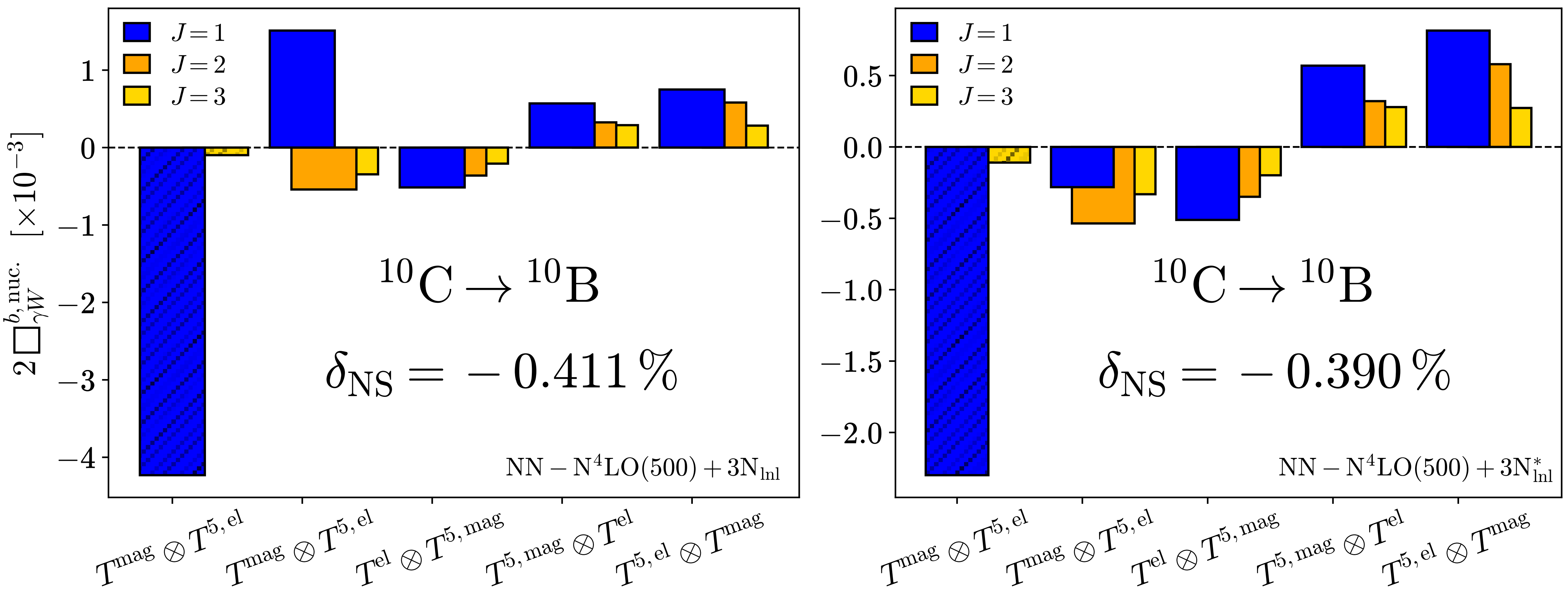
Wick rotated box diagram combined with electron propagator residue contribution regular at $E_e = 0$

T_3 residue contribution **singular**

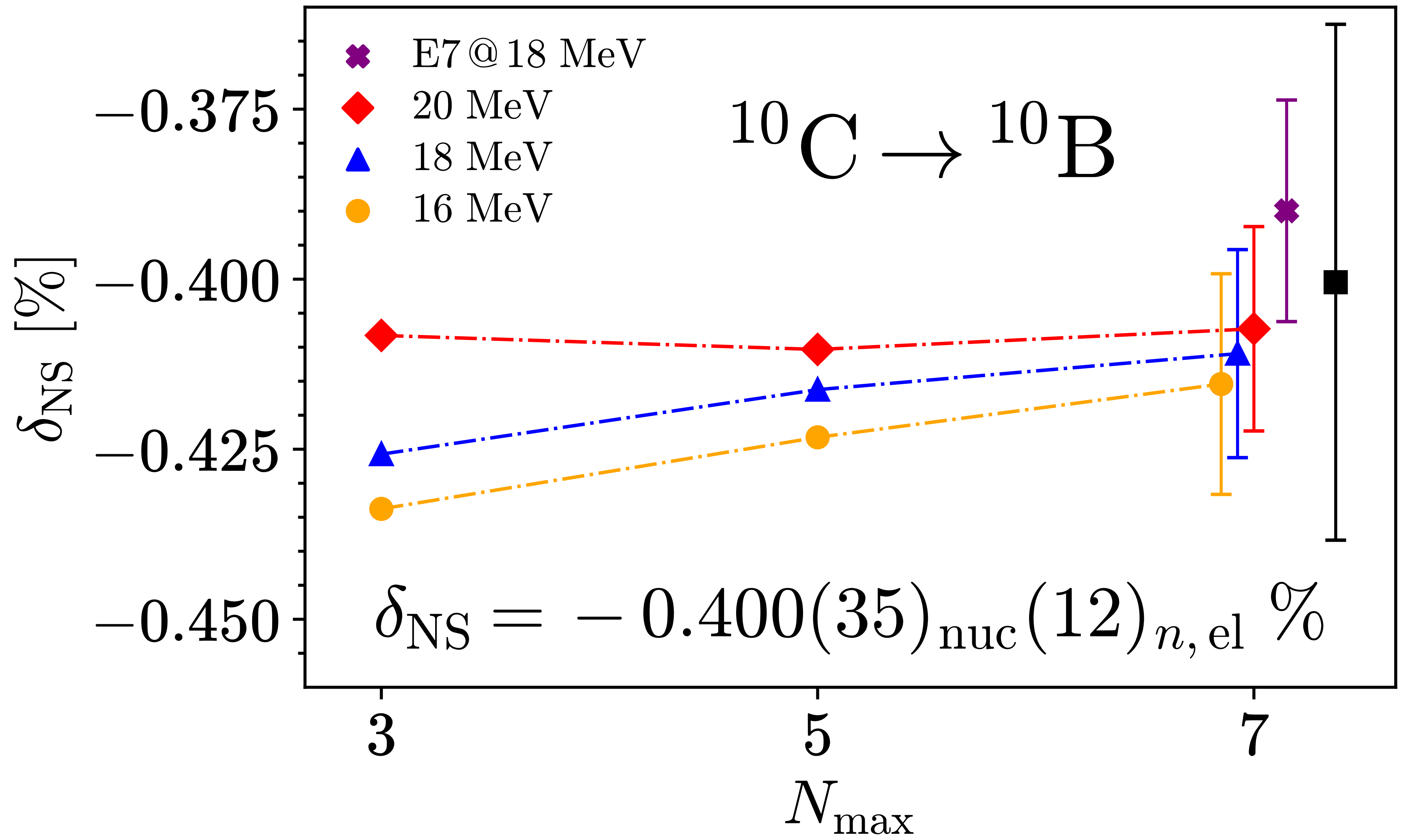
$$\Box^b_{\gamma W}(E_e) = \boxminus_0 + E_e \boxminus_1 + \left(\Box^b_{\gamma W}\right)_{\text{Res},T_3}(E_e) + \mathcal{O}(E_e^2)$$



1e−5
$T^{\mathrm{mag}}_{J=1} \otimes T^{5,\,\mathrm{el}}_{J=1}$
$E_e = 0.51$ MeV
Expansion
Residue
Total
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.25
−1.50
50
100
150
200
250
300
$|\vec{q}|$ [MeV]
$(\Box^b_{\gamma W})_{\mathrm{term}}(|\vec{q}|,\, E_e)$ [MeV^{-1}]



$J=1$
$J=2$
$J=3$
$2\square_{\gamma W}^{b,\,\mathrm{nuc.}}\ [\times 10^{-3}]$
$^{10}\mathrm{C} \to {}^{10}\mathrm{B}$
$\delta_{\mathrm{NS}} = -0.411\,\%$
$\mathrm{NN}-\mathrm{N^4LO(500)}+3\mathrm{N_{lnl}}$
$\delta_{\mathrm{NS}} = -0.390\,\%$
$\mathrm{NN}-\mathrm{N^4LO(500)}+3\mathrm{N^*_{lnl}}$
$T^{\mathrm{mag}} \otimes T^{5,\,\mathrm{el}}$
$T^{\mathrm{el}} \otimes T^{5,\,\mathrm{mag}}$
$T^{5,\,\mathrm{mag}} \otimes T^{\mathrm{el}}$
$T^{5,\,\mathrm{el}} \otimes T^{\mathrm{mag}}$


E7@18 MeV
20 MeV
18 MeV
16 MeV
$^{10}C \to {}^{10}B$
$\delta_{NS} = -0.400(35)_{nuc}(12)_{n,el}$ %
−0.375
−0.400
−0.425
−0.450
δ_{NS} [%]
3
5
7
N_{max}

- **Goal:** consistent nuclear theory corrections to Fermi transitions
- Larger basis NCSM calculations of δ_{NS} complete
- First consistent NCSM calculation, seems that residue is dominant feature
- NCSMC calculations for δ_{C} ongoing with Mack Atkinson

Outlook

- Tackle large number of many-body calculations with realistic N_{max}
  - seperate inhomogeneous Schrödinger equation at each $|\vec{q}|$
  - $N_{|\vec{q}|} \times N_{terms} \times J_{max} = 50 \times 4 \times 3 = 600$ many body calculations
- Improve limited uncertainty quantification
- Heavier transitions, e.g., $^{14}\mathrm{O} \rightarrow {}^{14}\mathrm{N}$

TRIUMF

Thank you
Merci

www.triumf.ca

Follow us **@TRIUMFLab**

Discovery, accelerated

References

1. C.Y. Seng. arXiv preprint: 2112.10942v2 (2022)
2. P.A. Zyla et al. (Particle Data Group). Prog. in Theo. and Exp. Phys. **2020**, 083C01. (2020)
3. C.Y. Seng, M. Gorchtein, H.H. Patel, & M.J. Ramsey-Musolf. PRL **121**(24), 241804. (2018)
4. M. Gorchtein. PRL **123**(4), 042503. (2019)
5. J.C. Hardy & I.S. Towner. PRC **102**, 045501 (2020)
6. D.R. Entem, R. Machleidt & Y. Nosyk. PRC **96**, 024004 (2017)
7. V. Somà, P. Navrátil, F. Raimondi, C. Barbieri & T. Duguet. PRC **101**, 014318 (2020)
8. C.Y. Seng & M. Gorchtein. PRC **107**, 035503 (2023)
9. R. Haydock. JPA **7**, 2120 (1974)
10. E. Dagotto. RMP **66**, 763 (1994)

Lanczos strength method

- Reformulate resolvent operator as inhomogeneous Schrödinger equation

$$\left(H - E\mathbb{1}\right)\left|\Phi\right\rangle = \hat{O}\left|\Psi\right\rangle$$

$$\begin{aligned}
H\mathbf{v}_1 &= \alpha_1\mathbf{v}_1 + \beta_1\mathbf{v}_2 \\
H\mathbf{v}_2 &= \beta_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \beta_2\mathbf{v}_3 \\
H\mathbf{v}_3 &= \beta_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \beta_3\mathbf{v}_4 \\
H\mathbf{v}_4 &= \beta_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \beta_4\mathbf{v}_5
\end{aligned}$$

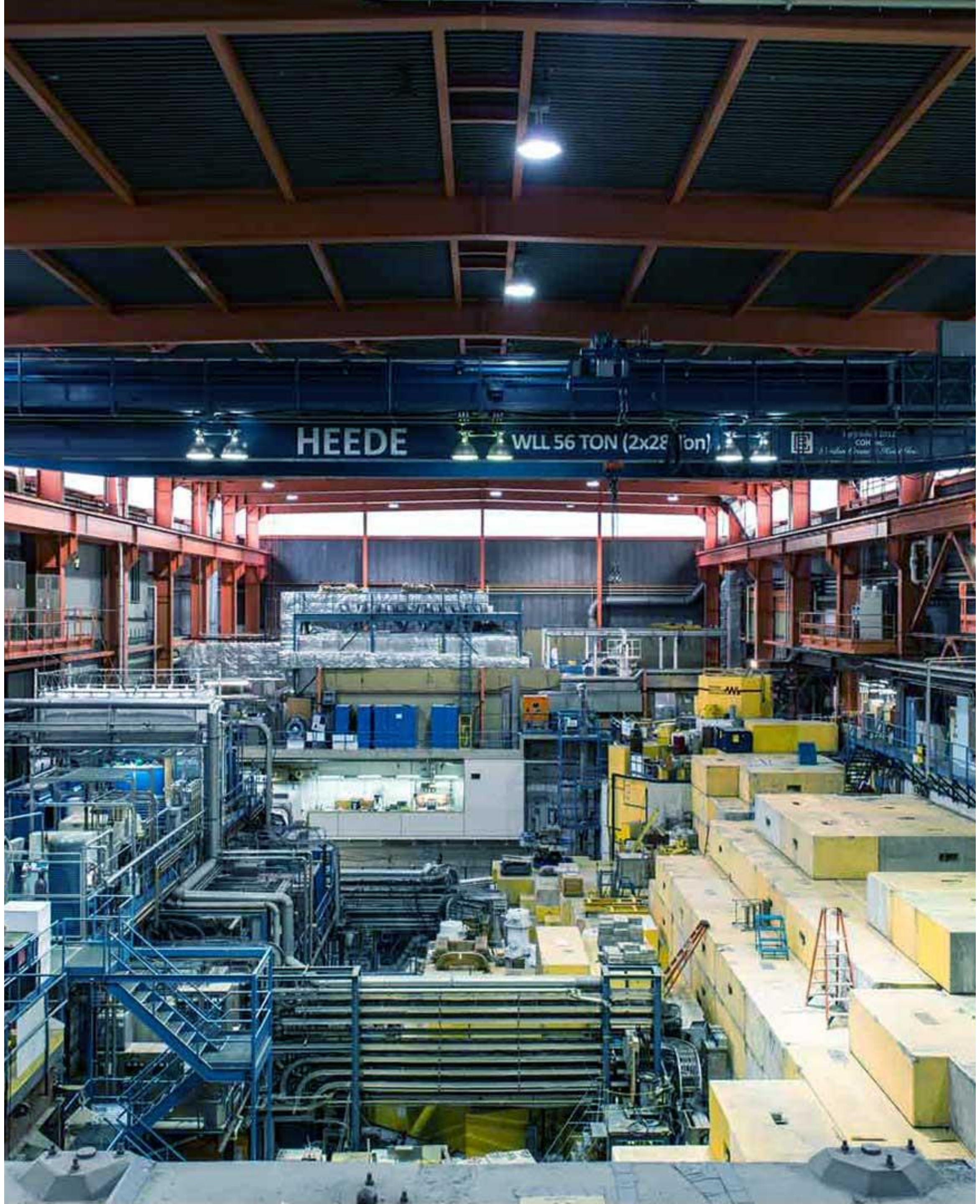
Access dynamical properties!

$$\langle J_n|\mathbf{O_1}|J_i\rangle = (\mathbf{P}^{(i)})^{\dagger}_{0n}\,||\mathbf{O_1}\,|J_i\rangle||$$

$$\langle J_i|\mathbf{O_2}|J_n\rangle = ||\mathbf{O_2}\,|J_i\rangle|| \sum \left(\mathbf{P}^{(i)}\right)_{mn} \left\langle \phi_0^{(f)} \middle| \phi_0^{(i)} \right\rangle$$

[9] Haydock (1974)
[10] Dagotto (1994)

Backup slides for multipole expansion and δ_{NS}



Discovery, accelerated

Δ_R^V and δ_{NS}

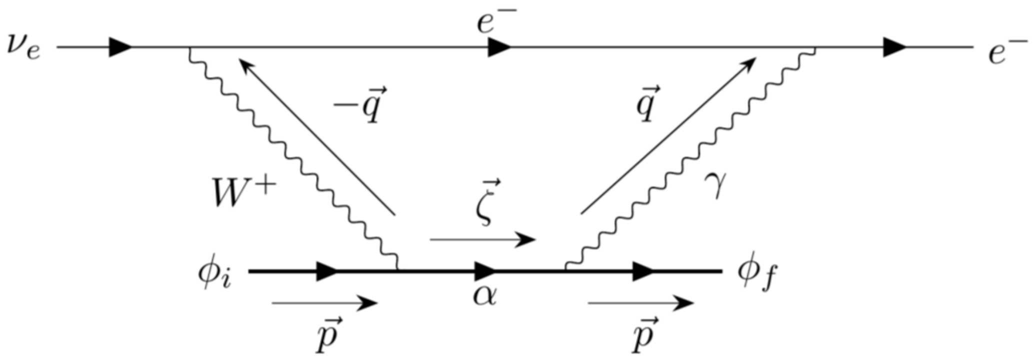
Leptonic current

NME of charged weak current

- Tree level beta decay amplitude $M_{tree} = -\frac{G_F}{\sqrt{2}} L_\lambda F^\lambda(p', p)$

- Hadronic correction in forward scattering limit

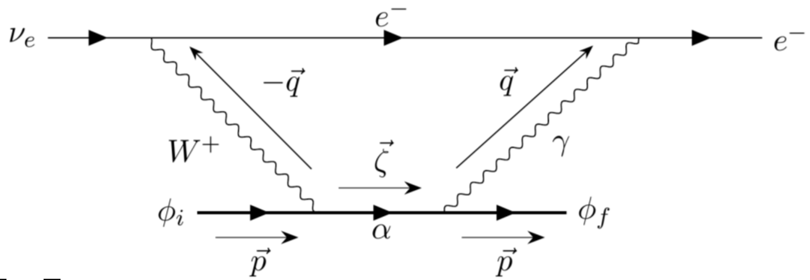
$$\delta_{NS} = 2[\Box_{\gamma W}^{\text{b,nuc}} - \Box_{\gamma W}^{\text{b,free n}}]$$



$$T^{\mu\nu}(p,q) = \frac{1}{2}\int d^4x\ e^{iq\cdot x}\langle\phi_f(p)|T[J^\mu_{\text{em}}(x)J^\nu_W(0)^\dagger]|\phi_i(p)\rangle$$

[6] Seng et al. (2023)

Nonrelativistic Compton amplitude



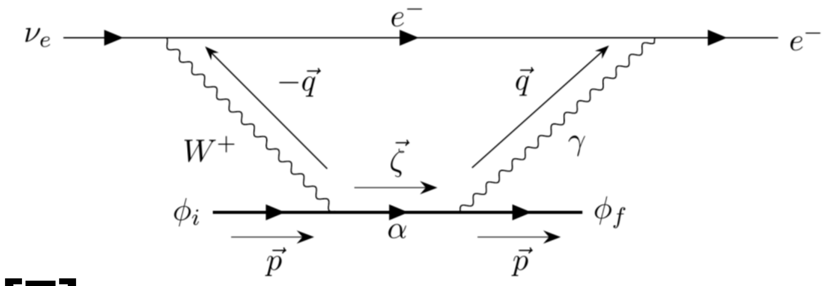
- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators

$$J^\mu(t,\vec{x}) = e^{-iHt} J^\mu(0,\vec{x})\, e^{iHt} \longrightarrow G(E) = \sum_n \frac{|n\rangle\langle n|}{E - E_n}$$

$$T^{\mu\nu}(p,q) = -\frac{i}{2}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \langle \phi_f(p) \big| J^\mu_{em}(0,\vec{x})\, \underline{G(M_f + \nu + i\epsilon)}\, J^{\dagger\,\nu}_W(0,\vec{0}) \big| \phi_i(p) \rangle - \frac{i}{2}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \langle \phi_f(p) \big| J^{\dagger\,\nu}_W(0,\vec{0})\, \underline{G(M_i - \nu + i\epsilon)}\, J^\mu_{em}(0,\vec{x}) \big| \phi_i(p) \rangle$$

[7] Haxton et al. (2007)

Nonrelativistic Compton amplitude



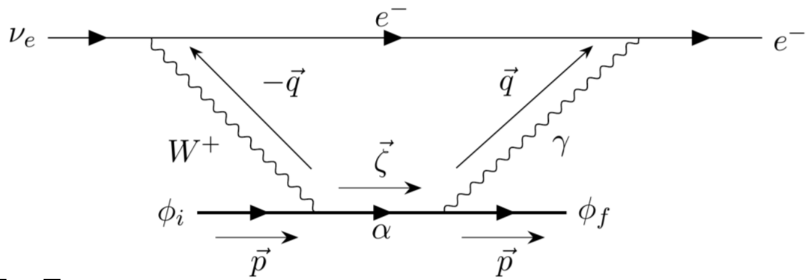
- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators
- Fourier transform currents into momentum space

$$J(\vec{r}) = \int \frac{d^3 r}{(2\pi)^3} \, e^{i\vec{q}\cdot\vec{r}} J(\vec{q})$$

$+$ Translation invariance

$$T^{\mu\nu}(p,q) = i\sqrt{M_i M_f}\,\langle \Phi_f | J^{\mu}_{em}(\vec{q})\, G(M_f + \nu + i\epsilon)\, J^{\dagger\,\nu}_{W}(-\vec{q}) | \Phi_i \rangle + i\sqrt{M_i M_f}\,\langle \Phi_f | J^{\dagger\,\nu}_{W}(-\vec{q})\, G(M_i - \nu + i\epsilon)\, J^{\mu}_{em}(\vec{q}) | \Phi_i \rangle$$

[7] Haxton et al. (2007)



Nonrelativistic Compton amplitude

- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators
- Fourier transform currents into momentum space
- General multipole expansion of currents

$$M_{JM}(q) := \int d^3r \; \mathcal{M}_{JM}(q,\vec{r})\, \rho(\vec{r})$$

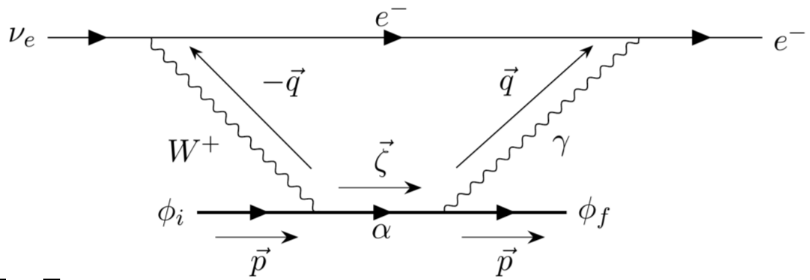
$$T^{\text{el}}_{JM}(q) := \int d^3r \; \frac{1}{q}\left(\vec{\nabla} \times \vec{\mathcal{M}}^M_{JJ}(q,\vec{r})\right) \cdot \vec{J}(\vec{r})$$

$$L_{JM}(q) := \int d^3r \; \frac{i}{q}\left(\vec{\nabla}\mathcal{M}_{JM}(q,\vec{r})\right) \cdot \vec{J}(\vec{r})$$

$$T^{\text{mag}}_{JM}(q) := \int d^3r \; \vec{\mathcal{M}}^M_{JJ}(q,\vec{r}) \cdot \vec{J}(\vec{r})$$

[7] Haxton et al. (2007)

Nonrelativistic Compton amplitude

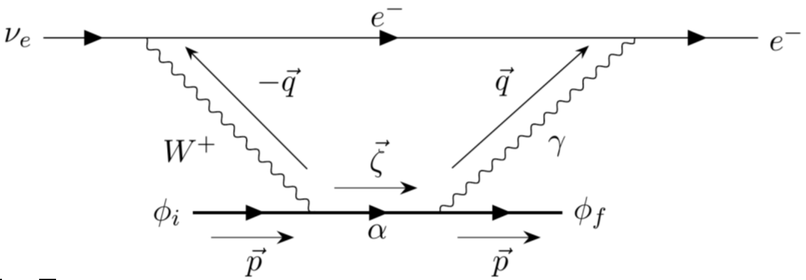


- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators
- Fourier transform currents into momentum space
- General multipole expansion of currents

$$T_3(\nu, |\vec{q}\,|) = 4\pi i \frac{\nu}{|\vec{q}\,|}\sqrt{M_i M_f}\sum_{J=1}^{\infty}(2J+1)\,\langle \Psi_f \big| \Big\{ T_{J0}^{\mathrm{mag}}\, G(\nu + M_f + i\epsilon)\, T_{J0}^{5,\mathrm{el}} + T_{J0}^{\mathrm{el}}\, G(\nu + M_f + i\epsilon)\, T_{J0}^{5,\mathrm{mag}}$$
$$+ T_{J0}^{5,\mathrm{mag}}\, G(-\nu + M_i + i\epsilon)\, T_{J0}^{\mathrm{el}} + T_{J0}^{5,\mathrm{el}}\, G(-\nu + M_i + i\epsilon) T_{J0}^{\mathrm{mag}} \Big\} (|\vec{q}\,|) \big| \Psi_i \rangle$$

[7] Haxton et al. (2007)

Nonrelativistic Compton amplitude



- **Goal**: Non-relativistic currents in momentum space **[7]**
- Rewrite currents with A-body propagators
- Fourier transform currents into momentum space
- General multipole expansion of currents

Lanczos continued fraction method to compute nuclear Green's functions **[13-14]**

$$T_3(\nu, |\vec{q}\,|) = 4\pi i \frac{\nu}{|\vec{q}\,|} \sqrt{M_i M_f} \sum_{J=1}^{\infty} (2J+1) \langle \Psi_f | \Big\{ T_{J0}^{\mathrm{mag}} \boxed{G(\nu + M_f + i\epsilon)} T_{J0}^{5,\mathrm{el}} + T_{J0}^{\mathrm{el}} \boxed{G(\nu + M_f + i\epsilon)} T_{J0}^{5,\mathrm{mag}}$$
$$+ T_{J0}^{5,\mathrm{mag}} \boxed{G(-\nu + M_i + i\epsilon)} T_{J0}^{\mathrm{el}} + T_{J0}^{5,\mathrm{el}} \boxed{G(-\nu + M_i + i\epsilon)} T_{J0}^{\mathrm{mag}} \Big\} (|\vec{q}\,|) | \Psi_i \rangle$$

[7] Haxton et al. (2007)
[13] Hao et al. (2020)
[14] Froese et al. (2021)

Electron energy expansion

$$\Box^b_{\gamma W}(E_e) = \boxminus_0 + E_e\, \boxminus_1 + \cdots + \left(\Box^b_{\gamma W}\right)_{\mathrm{Res},T_3}(E_e)$$

$$\boxminus_0 = \frac{e^2}{M}\int \frac{d^3q}{(2\pi)^3}\int \frac{d\nu_E}{2\pi}\,\frac{M_W^2}{M_W^2 - q^2}\,\frac{|\vec{q}\,|^2}{\nu_E(q^2 + i\epsilon_1)^2}\,\frac{T_3\left(i\nu_E, |\vec{q}\,|\right)}{f_+(0)}$$

$$\boxminus_1 = \frac{8}{3}\frac{e^2}{M}\int \frac{d^3q}{(2\pi)^3}\int \frac{d\nu_E}{2\pi}\,\frac{M_W^2}{M_W^2 - q^2}\,\frac{|\vec{q}\,|^2}{(q^2 + i\epsilon_1)^3}\,\frac{iT_3\left(i\nu_E, |\vec{q}\,|\right)}{f_+(0)}$$

Multipole expansion of amplitude

$$J^\mu(q) = \Big(\rho(\vec{q}), \vec{J}(\vec{q})\Big) \longrightarrow \vec{J}(\vec{q}) = \sum_\lambda J(\vec{q},\lambda)\,\vec{\epsilon}^{\,*}_\lambda$$

$$e^{-i\vec{q}\cdot\vec{r}} = 4\pi \sum_{J=0}^{\infty} \sum_{M_J} (-i)^J j_J(qr)\, Y_{JM_J}(\hat{q}) Y^*_{JM_J}(\hat{q})$$

$$\mathcal{M}_{JM}(q,\vec{r}) = j_J(qr)\,Y_{JM}(\hat{r}) \qquad \vec{\mathcal{M}}^M_{JL}(q,\vec{r}) = j_L(qr)\,\vec{Y}^M_{JL1}(\hat{r})$$

[7] Walecka (2004)

Multipole expansion of amplitude

$$\rho(\vec{q}) = \sqrt{4\pi} \sum_{J=0}^{\infty} (-i)^J \sqrt{2J+1}\, M_{J0}(q)$$

$$J(\vec{q}, \lambda = 0) = \sqrt{4\pi} \sum_{J=0}^{\infty} (-i)^J \sqrt{2J+1}\, L_{J0}(q)$$

$$J(\vec{q}, \lambda = \pm 1) = -\sqrt{2\pi} \sum_{J=1}^{\infty} (-i)^J \sqrt{2J+1} \left(\lambda T_{J\lambda}^{\mathrm{mag}}(q) - T_{J\lambda}^{\mathrm{el}}(q) \right)$$

Nuclear matrix elements of multipole operators

$$\langle N(p_f s_f m_{T_f}) | V^{\mu}_{TM_T}(0) | N(p_i s_i m_{T_i}) \rangle = \bar{u}_{s_f}(p_f) \left[F_1^{(T)} \gamma^\mu + \frac{i F_2^{(T)}}{2m_N} \sigma^{\mu\nu} (p_f - p_i)_\nu \right] u_{s_i}(p_i) \langle m_{T_f} | \Gamma_{TM_T} | m_{T_i} \rangle$$

$$\langle N(p_f s_f m_{T_f}) | A^{\mu}_{TM_T}(0) | N(p_i s_i m_{T_i}) \rangle = \bar{u}_{s_f}(p_f) \left[G_A^{(T)} \gamma^\mu \gamma_5 - \frac{G_P^{(T)}}{2m_N} \gamma_5 (p_f - p_i)^\mu \right] u_{s_i}(p_i) \langle m_{T_f} | \Gamma_{TM_T} | m_{T_i} \rangle$$

$$\mathcal{M}_{JM}(q, \vec{r}) = j_J(qr)\, Y_{JM}(\hat{r})$$

$$\Delta_{JM}(q, \vec{r}) := \vec{\mathcal{M}}^M_{JJ}(q, \vec{r}) \cdot \frac{1}{q} \vec{\nabla}$$

$$\Sigma'_{JM}(q, \vec{r}) := -i \left(\frac{1}{q} \vec{\nabla} \times \vec{\mathcal{M}}^M_{JJ}(q, \vec{r}) \right) \cdot \vec{\sigma}$$

$$\Delta'_{JM}(q, \vec{r}) := -i \left(\frac{1}{q} \vec{\nabla} \times \vec{\mathcal{M}}^M_{JJ}(q, \vec{r}) \right) \cdot \frac{1}{q} \vec{\nabla}$$

$$\Sigma''_{JM}(q, \vec{r}) := \left(\frac{1}{q} \vec{\nabla} \mathcal{M}_{JM}(q, \vec{r}) \right) \cdot \vec{\sigma}$$

$$\Sigma_{JM}(q, \vec{r}) := \vec{\mathcal{M}}^M_{JJ}(q, \vec{r}) \cdot \vec{\sigma}$$

$$\Omega_{JM}(q, \vec{r}) := \left(\mathcal{M}_{JM}(q, \vec{r})\ \vec{\sigma} \right) \cdot \frac{1}{q} \vec{\nabla}$$

Symmetry tests of T_3 amplitude

- Time reversal symmetry with exact isospin gives NME constraint
- Previously **assumed** nuclear T_3 matched nucleonic system

Nuclei

$$T_3^{(0)}(-\nu, Q^2) = -T_3^{(0)}(\nu, Q^2)$$
$$T_3^{(1)}(-\nu, Q^2) = \cdots$$

Nucleons

$$T_3^{(0)}(-\nu, Q^2) = -T_3^{(0)}(\nu, Q^2)$$
$$T_3^{(1)}(-\nu, Q^2) = T_3^{(1)}(\nu, Q^2)$$

Pions

$$T_3^{(0)}(-\nu, Q^2) = -T_3^{(0)}(\nu, Q^2)$$
$$T_3^{(1)}(\nu, Q^2) = 0$$

$$\langle {}^{10}\mathrm{B}|T_{J0}^{\mathrm{mag},(0)}(q)\, G(M+i\epsilon)\, T_{J0}^{5,\mathrm{el}}(q)|{}^{10}\mathrm{C}\rangle = \langle {}^{10}\mathrm{C}|T_{J0}^{\mathrm{mag},(0)}(q)\, G(M+i\epsilon)\, \tilde{T}_{J0}^{5,\mathrm{el}}(q)|{}^{10}\mathrm{B}\rangle$$

$$\langle {}^{10}\mathrm{B}|T_{J0}^{\mathrm{el},(0)}(q)\, G(M+i\epsilon)\, T_{J0}^{5,\mathrm{mag}}(q)|{}^{10}\mathrm{C}\rangle = \langle {}^{10}\mathrm{C}|T_{J0}^{\mathrm{el},(0)}(q)\, G(M+i\epsilon)\, \tilde{T}_{J0}^{5,\mathrm{mag}}(q)|{}^{10}\mathrm{B}\rangle$$

[6] Seng et al. (2023)