First-order thermodynamics of scalar-tensor gravity: recent progress

Valerio Faraoni[1]

[1] Dept. of Physics & Astronomy, Bishop's University

work with J. Côté, A. Giusti, S. Giardino, S. Zentarra, L. Heisenberg, A. Mentrelli, S. Jose, M. Miranda, P.-A. Graham, A. Leblanc, R. Vanderwee, J. Houle, T. Françonnet (BU, ETH Zürich, U. Heidelberg, U. Bologna, U. Sherbrooke, U. Naples)

CAP Congress, Fredericton June 2023

Outline

1. Effective fluid description of scalar-tensor/Horndeski gravity
2. Fluid is *dissipative* $\rightarrow$ 1st order thermodynamics, "temperature of gravity", approach to GR equilibrium, "hot" singularities.
3. (Other states of thermal equilibrium.)
4. ("Standard" scalar fields in GR $+$ Einstein frame ST gravity: trade temperature with chemical potential.)

Motivation

There are many motivations to modify Einstein's GR:

- Quantum corrections introduce deviations from GR as extra degrees of freedom, higher order field equations, ... (ex: Starobinski inflation, low-energy limit of bosonic string is $\omega = -1$ Brans-Dicke gravity, ...).
- Explaining the present acceleration of the cosmic expansion without the *ad hoc* dark energy. The ΛCDM model fits the data but is incomplete and completely unsatisfactory from the theoretical point of view.
- GR is not well-tested on many scales or in all regimes. Even Newtonian gravity is doubted (MOND).

Scalar-tensor gravity is the prototypical alternative to GR; $f(R)$ gravity, a subclass, is extremely popular to explain the current acceleration of the universe without an *ad hoc* dark energy.

Introduces one extra scalar (massive) d.o.f. ϕ in addition to the two massless spin 2 polarizations of GR

ST gravity has evolved into Horndeski gravity, believed to be the most general ST theory with 2nd order equations of motion (until DHOST came) → avoid Ostrogradsky instability.

$$S\left[g_{ab}, \phi\right] = \int d^4x \sqrt{-g} \left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right) + S^{(m)}$$

where $X \equiv -\frac{1}{2}\nabla_c\phi\nabla^c\phi$ and

$$
\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X)\,, \\
\mathcal{L}_3 &= -G_3(\phi, X)\,\Box\phi\,, \\
\mathcal{L}_4 &= G_4(\phi, X)R + G_{4X}(\phi, X)\left[(\Box\phi)^2 - (\nabla_a\nabla_b\phi)^2\right]\,, \\
\mathcal{L}_5 &= G_5(\phi, X)\,G_{ab}\nabla^a\nabla^b\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi\,(\nabla_a\nabla_b\phi)^3\right. \\
&\left. +2\,(\nabla_a\nabla_b\phi)^3\right]\,,
\end{aligned}
$$

the $G_i(\phi, X)$ are regular functions ($i = 2, 3, 4, 5$), $G_{i\phi} \equiv \partial G_i/\partial\phi$, *etc.*

The multi-messenger event GW1708017/GRB1708017A, a neutron star binary merger, restricts Horndeski (at least in the late universe) to the "viable" class in which gravitational waves propagate at speed c:

$$\boxed{G_5 = G_{4X} = 0}$$

(also avoids instabilities and allows an Einstein frame formulation). Contains "1st generation" scalar-tensor (*e.g.*, Brans-Dicke) and $f(R)$ gravity.

Field equations of viable Horndeski written as effective Einstein equations:

$$G_{ab} = T_{ab}[\phi] + \frac{T_{ab}^{(m)}}{G_4} \equiv T_{ab}^{(2)} + T_{ab}^{(3)} + T_{ab}^{(4)} + \frac{T_{ab}^{(m)}}{G_4}$$

where

$$\begin{aligned}
T_{ab}^{(2)} &= \frac{1}{2G_4}\left(G_{2X}\nabla_a\phi\nabla_b\phi + G_2\, g_{ab}\right) , \\
T_{ab}^{(3)} &= \frac{1}{2G_4}\left(G_{3X}\nabla_c X\nabla^c\phi - 2XG_{3\phi}\right) g_{ab} , \\
&\quad -\frac{1}{2G_4}\left(2G_{3\phi} + G_{3X}\Box\phi\right)\nabla_a\phi\nabla_b\phi \\
&\quad -\frac{G_{3X}}{G_4}\nabla_{(a}X\nabla_{b)}\phi , \\
T_{ab}^{(4)} &= \frac{G_{4\phi}}{G_4}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) + \frac{G_{4\phi\phi}}{G_4}(\nabla_a\phi\nabla_b\phi + 2X) .
\end{aligned}$$

Effective fluid description of Horndeski gravity

Known for special theories/geometries/“1st gen” scalar-tensor gravity. Possible if $\nabla^c\phi$ is timelike + future-oriented.

Kinematic quantities: Define effective **4-velocity**

$$u^c \equiv \frac{\nabla^c\phi}{\sqrt{2X}}$$

Corresponding spatial **3-metric**

$$h_{ab} \equiv g_{ab} + u_a u_b \,,$$

4-acceleration

$$\dot{u}^a \equiv u^c \nabla_c u^a = -\frac{1}{2X}\left(\nabla^a X - \frac{\nabla_c X \nabla^c X}{2X}\nabla^a\phi\right)$$

4-velocity gradient

$$\nabla_b u_a = \sigma_{ab} + \frac{\Theta}{3} h_{ab} + \cancel{\omega_{ab}} + \dot{u}_a u_b \,.$$

Decompose projected velocity gradient in the usual way:

$$V_{ab} \equiv h_a{}^c\, h_b{}^d \nabla_d u_c = \sigma_{ab} + \frac{\Theta}{3}\, h_{ab}\,,$$

where **expansion** and **shear** are

$$\Theta = \nabla_c u^c = \frac{1}{\sqrt{2X}}\left(\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{2X}\right)$$

$$\sigma_{ab} = \frac{1}{\sqrt{2X}}\left[\nabla_a\nabla_b\phi - \frac{\nabla_{(a}X\nabla_{b)}\phi}{X} - \frac{\nabla_c X\nabla^c\phi}{4X^2}\,\nabla_a\phi\nabla_b\phi - \frac{h_{ab}}{3}\left(\Box\phi - \frac{\nabla_c X\nabla^c\phi}{2X}\right)\right]$$

Restrict, for now, to viable Horndeski, rewrite field equations as effective Einstein eqs.

$$G_{ab} = T_{ab}[\phi] + \frac{T_{ab}^{(m)}}{G_4} \equiv T_{ab}^{(2)} + T_{ab}^{(3)} + T_{ab}^{(4)} + \frac{T_{ab}^{(m)}}{G_4}$$

then $T_{ab}[\phi]$ has the form of a *dissipative* fluid tensor

$$\boxed{T_{ab} = \rho u_a u_b + P h_{ab} + \pi_{ab} + q_a u_b + q_b u_a}$$

with energy density

$$\begin{aligned}
\rho \quad = \quad & T_{ab} u^a u^b = \rho^{(2)} + \rho^{(3)} + \rho^{(4)} = \frac{1}{2G_4} \left(2XG_{2X} - G_2 \right) \\
& - \frac{1}{2G_4} \left(-G_{3X} \nabla_c X \nabla^c \phi + 2XG_{3\phi} + 2XG_{3X} \Box \phi \right) \\
& + \frac{G_{4\phi}}{G_4} \left(\Box \phi - \frac{\nabla_c X \nabla^c \phi}{2X} \right) ,
\end{aligned}$$

isotropic pressure

$$
\begin{aligned}
P &= \frac{h^{ab}T_{ab}}{3} = \bar{P} + P_{viscous} = P^{(2)} + P^{(3)} + P^{(4)} \\
&= \left[\frac{G_2}{2G_4}\right] + \left[\frac{1}{2G_4}\left(G_{3X}\nabla_c X \nabla^c \phi - 2XG_{3\phi}\right)\right] \\
&\quad + \left[-\frac{G_3}{3G_4}\left(2\Box\phi + \frac{\nabla_c X \nabla^c \phi}{2X}\right) + \frac{2XG_{4\phi\phi}}{G_4}\right],
\end{aligned}
$$

trace-free anisotropic stress tensor

$$
\begin{aligned}
\pi_{ab} &= T_{cd} h_a{}^c h_b^d - P h_{ab} = \pi_{ab}^{(2)} + \pi_{ab}^{(3)} + \pi_{ab}^{(4)} \\
&= 0 + 0 + h_{ab}\left[\frac{G_{4\phi}}{3G_4}\left(2\Box\phi + \frac{\nabla_c X \nabla^c \phi}{2X}\right) - \frac{2XG_{4\phi\phi}}{G_4}\right],
\end{aligned}
$$

and heat flux density

$$
\begin{aligned}
q_a &= -T_{cd}u^c h_a{}^d = q_a^{(2)} + q_a^{(3)} + q_a^{(4)} \\
&= 0 - \frac{G_{3X}\sqrt{2X}}{2G_4}\left(\nabla_a X + \frac{\nabla_c X \nabla^c \phi}{2X}\nabla_A \phi\right) \\
&\quad + \frac{G_{4\phi}}{G_4\sqrt{2X}}\left(\nabla_a X + \frac{\nabla_c X \nabla^c \phi}{2X}\nabla_a \phi\right) .
\end{aligned}
$$

h_{ab}, π^{ab}, q^a are purely spatial.

... so far just math ...

1st order thermodynamics of viable Horndeski

With the *caveat* that we have an *effective* dissipative fluid $T_{ab}[\phi]$, take the consequences seriously. What do we know about dissipative fluids? Eckart's 1st order thermodynamics (and also Newtonian 3-D non-relativistic fluids) based on 3 constitutive relations:

$$P_{viscous} = -\zeta\,\Theta\,,$$
$$\pi_{ab} = -2\eta\,\sigma_{ab}\,,$$
$$q_a = -\mathcal{K} h_{ab}\left(\nabla^b \mathcal{T} + \mathcal{T}\dot{u}^b\right)\,,$$

$\mathcal{T}$ = temperature, $\mathcal{K}$ = thermal conductivity. For the ϕ-fluid,

$$q_a = -\frac{\sqrt{2X}\,(G_{4\phi} - XG_{3X})}{G_4}\,\dot{u}_a$$

Miracle! Eckart's constitutive relation is satisfied! $q_a = -\mathcal{K}\mathcal{T}\dot{u}_a$

$$\boxed{\mathcal{KT} = \frac{\sqrt{2X}\,(G_{4\phi} - XG_{3X})}{G_4}}$$

$\phi =$ const. $\leftrightarrow$ $\mathcal{KT} = 0 \leftrightarrow$ GR "state of equilibrium".

$$\boxed{\eta = -\frac{\sqrt{X}\,G_{4\phi}}{\sqrt{2}\,G_4}\,, \qquad \zeta = -\frac{(G_{4\phi} - 3XG_{3X})}{3G_4}\sqrt{2X}}$$

shear/bulk viscosity coefficients.

"'Temperature of gravity", eqs. for approach to equilibrium never found in Jacobson's thermodynamics of spacetime.

Example

For example, for "1st gen" ST gravity with action

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(m)}$$

we have $G_{eff} \simeq 1/\phi$ and (if ∇^c is timelike and future-oriented)

$$\begin{aligned}
\mathcal{KT} &= \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}\,, \\
\eta &= -\frac{\mathcal{KT}}{2}\,, \\
\zeta &= -\frac{\mathcal{KT}}{3}
\end{aligned}$$

$\phi = \text{const.} \implies \mathcal{KT} = 0 \quad \text{(GR)}$

Approach to GR equilibrium

One can derive a "heat equation" describing the approach to the GR equilibrium, or departures from it

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} = \left(\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\left[\mathcal{K}\mathcal{T} - \frac{(2X)^{3/2}}{G_4}\left(G_{3X} + XG_{3XX}\right)\right]$$
$$-\frac{2X}{G_4^2}\left[G_4 G_{4\phi\phi} - XG_4 G_{3X\phi} - G_{4\phi}\left(G_{4\phi} - XG_{3X}\right)\right]$$

$\tau =$ proper time along fluid lines
Similar in spirit to Jacobson's thermodynamics of spacetime, but very different:

- minimal assumptions
- less fundamental (no QFT)
- explicit $\mathcal{T}$ and equation describing approach to equilibrium.

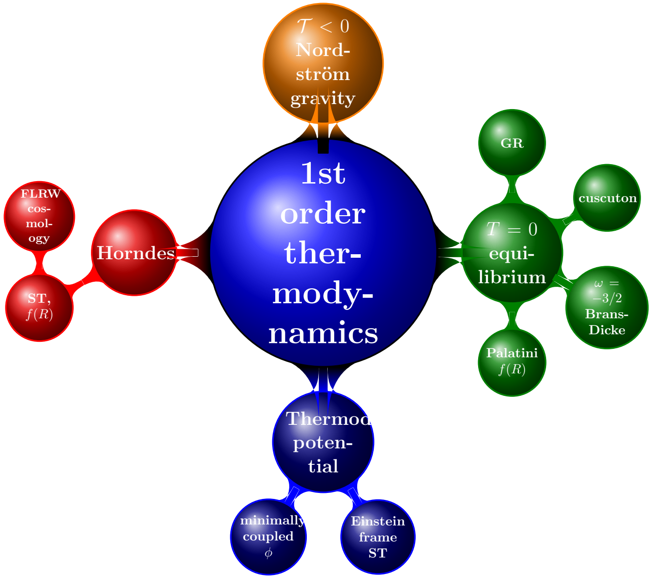
Easier to interpret in “1st gen” ST gravity and in simplified scenarios:

$$\frac{d\,(K\mathcal{T})}{d\tau} = 8\pi\,(K\mathcal{T})^2 - \Theta\,K\mathcal{T} + \frac{\Box\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}}$$

- Electrovacuum, $\omega = \text{const.}$, $V(\phi) = 0 \rightarrow \Box\phi = 0$:

$$\Theta < 0 \rightarrow \frac{d(K\mathcal{T})}{d\tau} > 8\pi(K\mathcal{T})^2$$

  or, $K\mathcal{T}$ diverges away from the GR equilibrium. Deviations of ST gravity from GR will be extreme near spacetime singularities (singularities are “hot”).
- Electrovacuum, $\Theta > 0$: then $-\Theta\,K\mathcal{T}$ can dominate $(K\mathcal{T})^2$, the solution $K\mathcal{T}$ can approach 0: diffusion to GR equilibrium (expansion cools gravity).
  But, if $K\mathcal{T}$ is large, the positive term dominates r.h.s. and drives solution away from GR:
  approach to GR equilibrium state not always expected.


$\mathcal{T} < 0$ Nordström gravity
1st order thermodynamics
GR
cuscuton
$T = 0$ equilibrium
$\omega = -3/2$ Brans-Dicke
Palatini $f(R)$
FLRW cosmology
Horndes
ST, $f(R)$
Thermod potential
minimally coupled ϕ
Einstein frame ST

Other subjects studied:

- states of thermal equilibrium $\neq$ GR: extra field is not dynamical;
- Landau frame (in which $q_a = 0$);
- tensor-multi-scalar gravity;
- stealth solutions of ST gravity, Bianchi models, other analytical solutions, ... : we attempted to falsify these ideas but so far they survive.

CONCLUSIONS

- There is an effective fluid equivalent of ST/Horndeski gravity; it is a *dissipative* fluid.
- The constitutive relations of Eckart's 1st order thermodynamics give a "temperature of gravity" $\mathcal{T}, \eta, \zeta$ and an equation describing approach to the GR equilibrium. Same spirit as emergent gravity and Jacobson's thermodynamics of spacetime, but very different.
- Theories with non-dynamical ϕ are also states of equilibrium
- Einstein frame ST: trade $\mathcal{T}$ with μ, thermal with chemical equilibrium.
- Many open problems under study.

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THANK YOU

Other states of equilibrium?

Are states of equilibrium other than GR possible?
Well, why is ST gravity an excited state w.r.t. GR? We have introduced an extra (scalar) degree of freedom ϕ in addition to the two massless spin 2 modes of GR contained in the metric g_{ab}. When this d.o.f. is dynamical and propagates we have "more dynamics" than GR and an excited states.
Some "pathological" theories of gravity are known in which ϕ is non-dynamical:

- $\omega = -3/2$ Brans-Dicke theory (in matter);
- Palatini $f(R)$gravity (in matter);
- cuscuton gravity;
- ....

They should lead to $\mathcal{KT} = 0$. Analyzed in VF, A. Giusti, S. Jose, S. Giardino PRD 106, 024029 (2022).

$\omega = -3/2$ Brans-Dicke theory

Equation of motion for ϕ in Brans-Dicke theory

$$(2\omega + 3)\,\Box\phi = 8\pi T^{(m)} + \phi\,\frac{dV}{d\phi} - 2V$$

is lost when $\omega = -3/2$ (reduces to an algebraic constraint), ϕ is non-dynamical. In vacuo the theory is known to be GR$+\Lambda$; in matter

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{\left|\nabla^c T^{(m)} \nabla_c T^{(m)}\right|}}{\phi\left|V' - \phi V''\right|}$$

becomes almost completely arbitrary.

Cuscuton gravity

A special Horndeski theory introduced to obtain dark energy. It is known that ϕ is non-dynamical

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi} \pm \mu^2 \sqrt{2X} - V(\phi) \right] + S^{(m)}$$

When you write down the field eqs. as effective Einstein eqs., you find

$$G_{ab} = 8\pi T_{ab}^{(\phi)} \,, \qquad T_{ab} = (P + \rho)\, u_a u_b + P g_{ab}$$

perfect fluid, no dissipation! So $q^a = 0$, $\mathcal{KT} = 0$

The curious case of Nordström gravity (not a ST theory)

Nordström purely scalar theory of gravity predates GR; considered interesting by Einstein, was very short-lived; sometimes used as toy model by theorists.
All solutions are conformally flat, $\tilde{g}_{ab} = \Omega^2 \eta_{ab}$. The only degree of freedom is the scalar Ω, which satisfies

$$\Box\Omega = 0$$

Compute the Einstein tensor for such metrics, obtain

$$G_{ab} = 8\pi\, \tilde{T}^{(\Omega)}_{ab} = -\frac{2\tilde{\nabla}_a\tilde{\nabla}_b\Omega}{\Omega} + \tilde{g}_{ab}\,\frac{\tilde{\nabla}^c\Omega\,\tilde{\nabla}_c\Omega}{\Omega^2}\,.$$

Assume

$$\tilde{u}_a \equiv \frac{\tilde{\nabla}_a\Omega}{\sqrt{-\tilde{g}^{cd}\,\tilde{\nabla}_c\Omega\,\tilde{\nabla}_d\Omega}}$$

timelike$+$ future-oriented, then

$$\tilde{q}^{(\Omega)}_a = -\tilde{T}^{(\Omega)}_{cd}\,\tilde{u}^c\,\tilde{h}_a{}^d = \frac{\sqrt{2\tilde{X}}}{4\pi\,\Omega}\,\dot{\tilde{u}}_a$$

and

$$\boxed{\mathcal{K}\mathcal{T} = -\frac{\sqrt{2\tilde{X}}}{4\pi\,\Omega} = -\frac{\sqrt{-\tilde{g}^{ef}\,\tilde{\nabla}_e\Omega\,\tilde{\nabla}_f\Omega}}{4\pi\,\Omega} < 0}$$

Nordström gravity has *less* d.o.f. than GR so is a de-excited state (first case in which the formalism is extended beyond ST/Horndeski gravity).

Einstein frame formulation of ST gravity

There exists another representation of ST gravity using different variables (a different "conformal frame"). Instead of

$$S_{ST} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(m)}$$

use the "Einstein frame" variables $\left(\tilde{g}_{ab}, \tilde{\phi}\right)$ defined by

$$\tilde{g}_{ab} \equiv \phi\, g_{ab}\,, \qquad d\tilde{\phi} = \sqrt{\frac{|2\omega+3|}{16\pi}}\,\frac{d\phi}{\phi}$$

the action becomes

$$S_{EF} = \int d^4x\sqrt{-g}\left[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\tilde{g}^{ab}\nabla_a\tilde{\phi}\nabla_b\tilde{\phi} - U(\tilde{\phi}) + \frac{\mathcal{L}^{(m)}}{\phi^2(\tilde{\phi})}\right]$$

with

$$U(\tilde{\phi}) = \frac{V(\phi)}{16\pi\phi^2}\Big|_{\phi=\phi(\tilde{\phi})}$$

Now the scalar field $\tilde{\phi}$ does not couple explicitly to gravity (to R) so $\tilde{T}_{ab}^{(\tilde{\phi}}$ has a *perfect* fluid structure. No dissipation, no heat flux, $\mathcal{T} = 0$ (but $\tilde{\phi}$ couples explicitly to matter).

Where did the thermal description go?

We need to understand better standard nonminimally coupled scalars in GR. The standard picture is: a minimally coupled ϕ is equivalent to a *perfect* fluid with the dictionary (Piattella 2013)

$$
\begin{aligned}
u^a &\equiv \nabla^a \phi / \sqrt{2X} \quad \text{(4-velocity)} \\
\rho &= 2X\mathcal{L}_X - \mathcal{L} \quad \text{(energy density)} \\
P &= \mathcal{L} \quad \text{(pressure)} \\
n &= \sqrt{2X}\,\mathcal{L}_X \quad \text{(particle number density)} \\
\frac{s}{n} &= \phi \quad \text{(entropy density)} \\
T &= \frac{-\mathcal{L}_\phi}{\sqrt{2X}\,\mathcal{L}_X} \quad \text{(temperature)}
\end{aligned}
$$

$$\mu = \frac{2X\mathcal{L}_X + \phi\mathcal{L}_\phi}{\sqrt{2X}\,\mathcal{L}_X} = \sqrt{2X} - \phi T \quad \text{(chemical potential)}$$

$$T_{ab}^{(\phi)} = (P + \rho)\,u_a u_b + P g_{ab}$$

where $\mathcal{L}_\phi \equiv \partial\mathcal{L}/\partial\phi$ and $\mathcal{L}_X \equiv \partial\mathcal{L}/\partial X > 0$.

Two problems:

- In general the fluid is accelerated, $\dot{u}^a \neq 0$. Then, according to Eckart's generalization of the Fourier law

$$q_a = -\mathcal{K} h_{ab} \left(\nabla^b \mathcal{T} + \mathcal{T} \dot{u}^b \right) \neq 0\,,$$

  there must be a heat flux, which contradicts the perfect fluid structure.
- Both $\mathcal{T}$ and μ can be negative

Fixed in VF, S. Giardino, A. Giusti, R, Vanderwee EPJC 83, 24 (2023).
Revised dictionary:

$$\mathcal{T} = 0$$
$$\mu = \sqrt{2X}$$

The argument: 1st law is

$$d\left(\frac{\rho}{n}\right) + P\,d\left(\frac{1}{n}\right) = T\,d\left(\frac{s}{n}\right)$$

Take s and n as independent variables →

$$T(s,n) = \frac{1}{n}\frac{\partial \rho}{\partial (s/n)}\bigg|_n = \frac{\partial \rho}{\partial s}\bigg|_n$$

perfect fluid = no dissipative effects → $T = 0$, agrees with ST thermodynamics.
Assuming $\phi = \phi(s,n)$ and $X = X(s,n)$ leads to

$$0 = \frac{\partial \rho}{\partial s}\bigg|_n = -\mathcal{L}_\phi \frac{\partial \phi}{\partial s}\bigg|_n$$

satisfied if $\mathcal{L}_\phi = 0$ or $\frac{\partial \phi}{\partial s}\Big|_n = 0$.

Since, in general, $\mathcal{L}$ contains a potential, consistency requires

$$\left.\frac{\partial \phi}{\partial s}\right|_n = 0 \rightarrow \mathcal{T} = 0$$

then $P = \mathcal{L}$ and

$$\mu = \frac{P + \rho}{n} = \sqrt{2X}\,,$$

Einstein frame ST gravity has $\mathcal{T} = 0$ but $\mu \neq 0$ and is still a non-equilibrium state. GR with $\phi =$ const. and $\mu = 0$ is still the equilibrium state.

Physical nature of dissipative fluid (Newtonian vs non-Newtonian)

Can classify Horndeski theories based on the physical nature of the effective fluid $\longrightarrow$ a new view of Horndeski gravity (M. Miranda + arXiv:2209.02727). Needed because most recent developments not dealing with gravitational waves are formal and one cannot extract much physics (not a criticism).

Newtonian fluid $\leftrightarrow$ 2 subclasses of viable Horndeski

non-Newtonian fluid (exotic) $\leftrightarrow$ all the rest

$P_{viscous}$ and π_{ab} are assumed to be related to Θ and σ_{ab} by constitutive relations. For both non-relativistic and relativistic Newtonian fluids

$$\pi_{ab} = -2\eta\,\sigma_{ab}\,, \qquad P = \bar{P} \underbrace{-\zeta\,\Theta}_{P_{viscous}}$$

so that

$$T_{ab} = \rho u_a u_b + \left(\bar{P} - \zeta\,\Theta\right) h_{ab} + q_a u_b + q_b u_a - 2\eta\,\sigma_{ab}$$

If we don't eliminate $\dot{X} \equiv u^c \nabla_c X$ there is an ambiguity in identifying $P_{viscous} \propto \Theta$. Use

$$\Box\phi = \sqrt{2X}\,\Theta + \frac{\dot{X}}{\sqrt{2X}}$$

$\rightarrow$ eliminate $\dot{X}$, obtain

$$\zeta = -\frac{\left(G_{4\phi} - 3XG_{3X}\right)}{3G_4}\sqrt{2X}$$

(bulk viscosity neglected in Giusti *et al.* 2022 PRD)

There are only 2 possibilities for $P_{viscous}, \pi_{ab}$ to depend **linearly** on $\nabla_b u_a$ (= Newtonian fluid):

1. $P_{viscous}$ does not depend on $\dot{X} \Leftrightarrow G_{4\phi} - XG_{3X} = 0$

$$\rightarrow G_3 = G_{4\phi} \ln X \leftrightarrow \mathcal{KT} = 0$$

2. $\dot{X} = \dot{X}(\phi)$ only and is **linear** in Θ:

$$\dot{X} = \underbrace{F_1(\phi, X)}_{\text{contributes to non-viscous } \bar{P}} + \underbrace{F_2(\phi, X)\Theta}_{\text{contributes to } P_{viscous}}$$

(one could also work with $\Box\phi$ and assume $\Box\phi = \tilde{F}_1 + \tilde{F}_2\Theta$)

In the second case, the effective Einstein eqs. give

$$\dot{X} = \frac{A(\phi, X) + B(\phi, X)\Theta + C(\phi, X)\Theta^2 + D(\phi, X)\,\sigma^2 + E(\phi, X)\,\dot{u}^2}{H(\phi, X) + I(\phi, X)\,\Theta}$$

where

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab}\,, \qquad \dot{u}^2 \equiv \dot{u}_a \dot{u}^a$$

($\pi_{ab} = -2\eta\,\sigma_{ab}$ is satisfied automatically).
Sub $\dot{X} = F_1 + F_2\Theta$ into this →

$$\begin{cases} F_1 H = A\,, \\ F_1 I + F_2 H = B\,, \\ (F_2 I - C)\,\Theta^2 - D\sigma^2 - E\dot{u}^2 = 0\,, \end{cases}$$

Last eq. is a 2nd order PDE for ϕ which, in general, is incompatible with the Horndeski field eq.: cannot impose it (similarly, one cannot impose $\dot{X} = 0 \Leftrightarrow \nabla_a\phi\nabla_b\phi\nabla^a\nabla^b\phi = 0$)

The only consistent way to implement the requirement of Newtonian fluid is restricting to

$$\boxed{G_{3X} = 0 \;\rightarrow\; \dot{X}\text{ linear in }\Theta \quad (C = D = E = I = 0)}$$

then $G_3 = G_3(\phi)$ and $\mathcal{L}_3 = -G_3(\phi)\Box\phi$ is absorbed into G_2 upon integration by parts,

$$-G_3(\phi)\Box\phi = 2XG_{3\phi} + \text{total divergence}$$

then (and only then) $\dot{X} = F_1 + F_2\Theta$ is not an extra eq. but coincides with the Horndeski field eq. for ϕ and

$$
\begin{cases}
F_1(\phi, X) &= \dfrac{\sqrt{2X}\left[2G_2 G_{4\phi} + XG_{4\phi}\left(6G_{4\phi\phi} - G_{2X}\right)\right]}{3G_{4\phi}^2 + G_4(G_{2X} + 2XG_{2XX})} \\
&\quad - \dfrac{\sqrt{2X}\, G_4\left(G_{2\phi} - 2XG_{2\phi X}\right)}{3G_{4\phi}^2 + G_4(G_{2X} + 2XG_{2XX})}\,, \\
& \\
F_2(\phi, X) &= -\dfrac{2X\left(3G_{4\phi}^2 + G_4 G_{2X}\right)}{3G_{4\phi}^2 + G_4(G_{2X} + 2XG_{2XX})}\,.
\end{cases}
$$

Recap:

The effective fluid of Horndeski gravity is Newtonian only if

1. $\boxed{\mathcal{L} = G_4(\phi)R + G_2(\phi, X) - G_{4\phi}\ln X} \rightarrow \mathcal{KT} = 0$

$$\rho = \frac{1}{2G_4}\left(2XG_{2X} - G_2 - 2XG_{4\phi\phi}\ln X\right) ,$$
$$P = \frac{1}{2G_4}\left[G_2 + 2XG_2 - 2XG_{4\phi\phi}\left(2 - \ln X\right)\right] ,$$
$$q^a = 0 ,$$
$$\zeta = -\frac{4\eta}{3} = \frac{2G_{4\phi}\sqrt{2X}}{3G_\phi}$$

2. $\boxed{\mathcal{L} = G_4(\phi)R + G_2(\phi, X)} \rightarrow \mathcal{KT} \neq 0$

$$\rho = \frac{1}{2G_4}\left(2XG_{2X} - G_2\right) + \sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,\Theta ,$$

$$
\begin{aligned}
\bar{P} &= -\frac{G_{4\phi}^2\left(G_2 - 2XG_{2X}\right)}{2G_4\left[3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)\right]} \\
&\quad + \frac{\left(G_2 + 4XG_{4\phi\phi}\right)\left(G_{2X} + 2XG_{2XX}\right)}{2\left[3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)\right]} \\
&\quad + \frac{G_{4\phi}\left(G_{2\phi} - 2XG_{2\phi X}\right)}{3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)}\,, \\
\eta &= -\frac{G_{4\phi}}{2G_4}\sqrt{2X}\,, \\
\zeta &= -\frac{\sqrt{2X}\,G_{4\phi}\left[3G_{4\phi}^2 + G_4\left(G_{2X} - 4XG_{2XX}\right)\right]}{3G_4\left[3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)\right]}\,, \\
q^a &= -\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,\dot{u}^a\,.
\end{aligned}
$$

Class 2) automatically excludes non-dynamical scalar fields (*e.g.*, cuscuton, $\omega = -3/2$ Brans-Dicke gravity).
Classes 1) and 2) are disconnected: they cannot be changed into one another by disformal transformations.

General Horndeski gravity

Now include $G_4(\phi, X), G_5(\phi, X)$: Newtonian fluid $\Rightarrow$ again the same classes, with $G_{4X} = G_5 = 0$ (it is sufficient to impose that all non-linear terms in ρ vanish).

Include matter

One arrives to the same equations expressing the Newtonian fluid requirement. The only difference consists of the replacement

$$\begin{aligned} A \quad \rightarrow \quad & A + \sqrt{2X}\, G_{4\phi}\, T^{(m)} \\ & -\sqrt{2X} \left(T^{(m)} + 2 T^{(m)}_{ab} u^a u^b \right) G_{3X} \,. \end{aligned}$$