Characterisation of qutrit universal gates

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Outline

- How to estimate the performance of quantum gates?
- Why is randomised benchmarking chosen?
- Universal qutrit randomised benchmarking.

Qutrits and gates

Qutrits = three level systems = span($|0\rangle, |1\rangle, |2\rangle$) single particle

- Larger Hilbert space.
- More natural: avoid the truncation of a native higher level system.
- Multiple implementations: (superconductors, ion traps, and more) [3, 6, 13, 7, 4, 14, 5, 10, 2, 9].

Gates

1. Group element $g \in G$.
2. Unitary representation U_g labelled by a group element.
3. Unitary (Schroedinger) evolution: $\rho \mapsto U_g \rho U_g^\dagger$.
4. Physical implementation labelled by a group element: $\boxed{g}$.
5. Mathematical modelling by a quantum channel: $\tilde{U} := \mathcal{E}(U_g \rho U_g^\dagger)$.

Estimating performance = characterising

Why?

- Estimate the performance to improve the gates.
- Know how many operations we can perform before the computation is purely random.

Why is characterising not easy?

- Stochastic nature of quantum operations.
- Tomography (reconstruction) is expensive (wrt scaling).

Fidelity

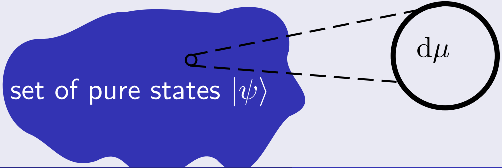
Fidelity between states

- How similar are two states ρ and σ.
- Defined by $F(\rho, \sigma) := \operatorname{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2$.

Average gate fidelity of a channel $\mathcal{E}$

$$
\begin{aligned}
\bar{F}(\mathcal{E}) &:= \int_{|\psi\rangle \in \mu} \mathrm{d}\mu F(U|\psi\rangle\langle\psi| U^\dagger, \mathcal{E}(|\psi\rangle\langle\psi|)), \\
&= \int_{|\psi\rangle \in \mu} \mathrm{d}\mu F(|\psi\rangle\langle\psi|, U^\dagger \mathcal{E}(|\psi\rangle\langle\psi|) U).
\end{aligned}
$$

$\mu :=$



Introduction to Randomised Benchmarking

Summary of randomised benchmarking

- Goal of randomised benchmarking: estimate $\bar{F}$.
- RB requires: a gate set $\mathcal{G}$, preparation of the state $|0\rangle$, a measurement $\langle 0|$.

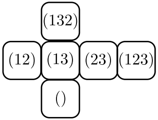
Group properties

- Composition of two gates is another gate from the gate set.
- Each gate has an inverse operation.

Group elements
$g \in G$

Physical implementation or gate
$\boxed{g} = \boxed{U_g}$

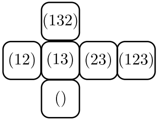
Algorithm: toy example S_3



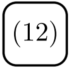
Prepare the state $|0\rangle$

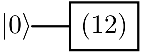
$|0\rangle$

Algorithm: toy example S_3



Roll the dice and apply the corresponding gate





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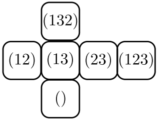
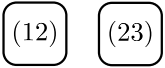
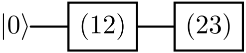
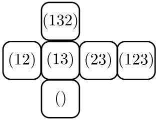


Roll a second time and apply the corresponding gate

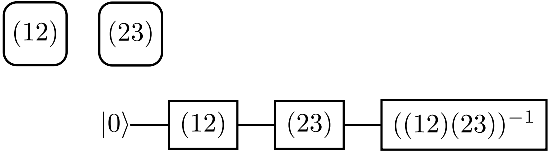




Algorithm: toy example S_3



Apply the inversion gate



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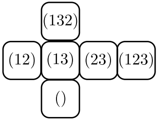
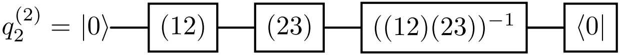
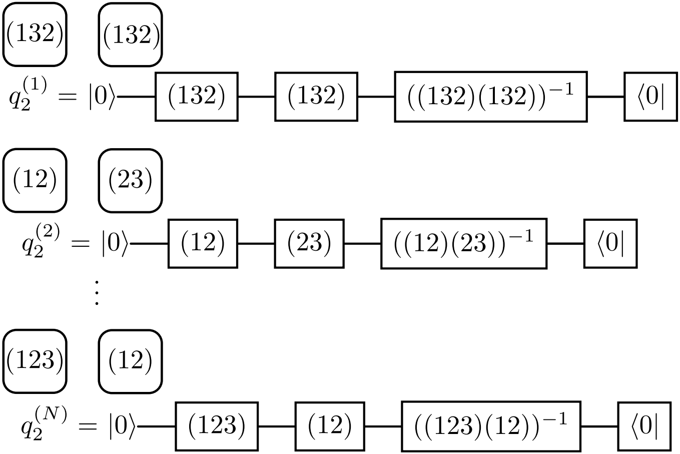


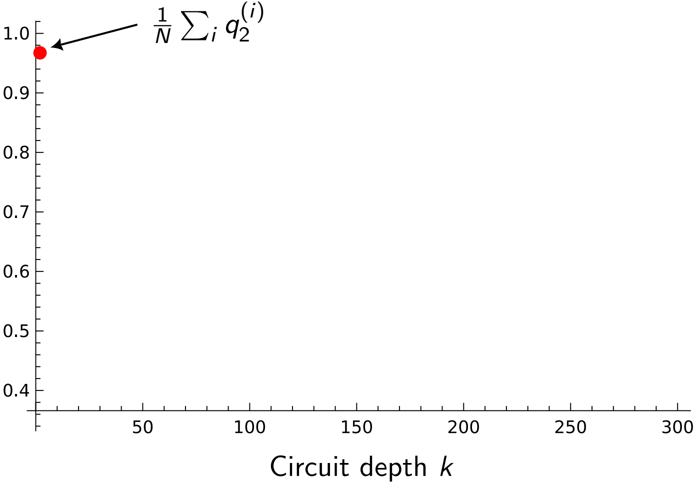
Measure with respect to the ground state $\langle 0|$



We repeat the process above N times: sample N sequences of 2 gates



Survival probability curve; Clifford



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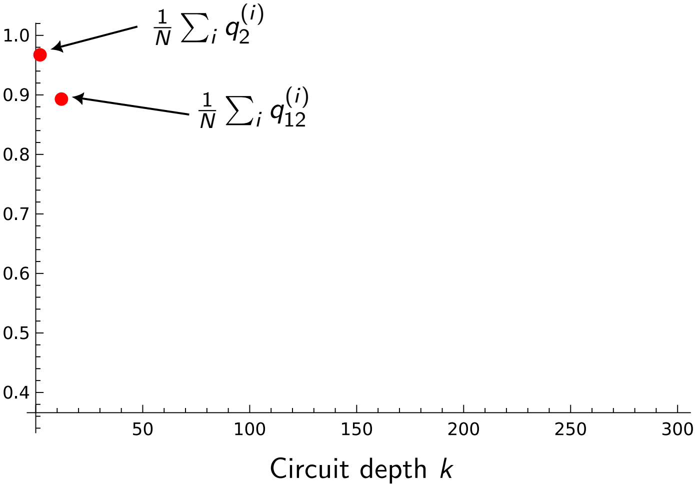

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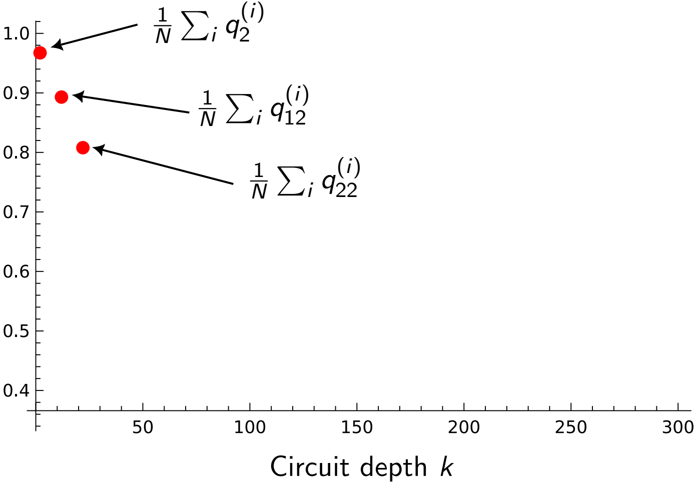

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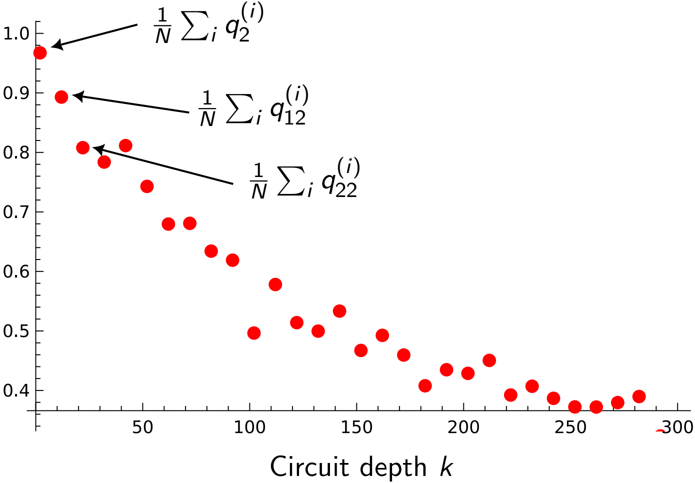

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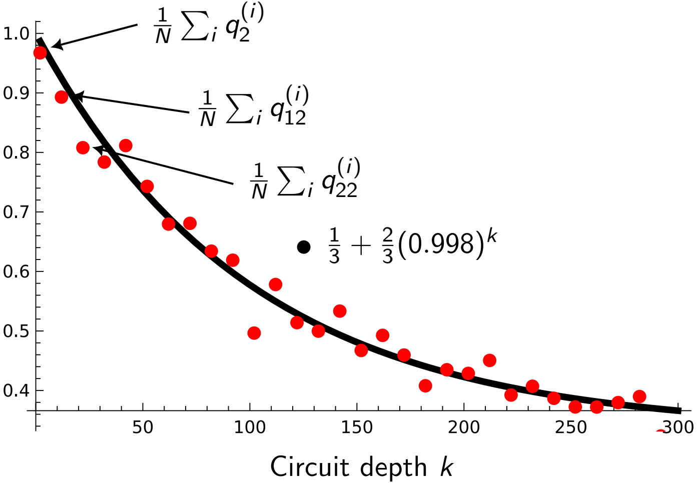
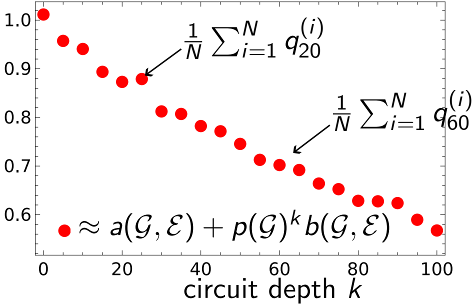


Data-analysis

- Survival probability curve.



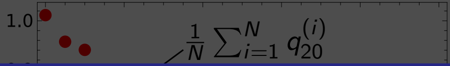
- It can be proven with some representation theory results that:

$$\bar{F}(\mathcal{G}) = \frac{3p(\mathcal{G}) - 1}{2}. \tag{1}$$

- (Recall that) $\bar{F}$ is the average fidelity over a gate set.

Data-analysis

- Survival probability curve.



Attention

Randomised benchmarking estimates $\bar{F}$ by fitting a single exponential curve to the survival probability.

- It

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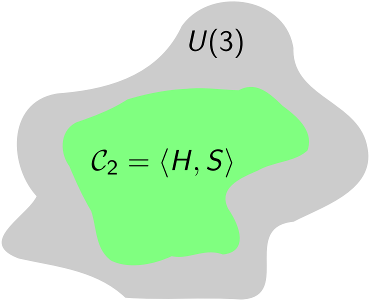
- (Recall that) $\bar{F}$ is the average fidelity over a gate set.

Clifford is not universal

Clifford and universality

- Can be efficiently simulated on a classical computer.
- Cannot efficiently approximate an arbitrary unitary.



$$H := 3^{-1/2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, S := \begin{bmatrix} 1 & & \\ & 1 & \\ & & \omega^2 \end{bmatrix}. \quad (2)$$

Universality ⇒ T gates

How do we achieve universality?

For qutrits, only two gates are necessary to approximate an arbitrary unitary operation: H (which is Clifford) and T.

T gate

- A T gate is any gate that is a generator of a universal gate set which is non-Clifford.
- Conjugating Pauli matrices by a T gate produces a Clifford gate.
- For instance,

$$T = \begin{bmatrix} 1 & & \\ & \exp(2\pi i/9) & \\ & & \exp(2\pi i/9)^8 \end{bmatrix}; \quad T^9 = \mathbb{I}. \tag{3}$$

The HDG: characterisation of T (my contribution)

HDG stands for Hyperdihedral group.

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Motivation

- Generalise dihedral benchmarking [1].
- Fewer group elements than the Clifford group.
- Smallest number of parameters in the expression for $\bar{F}$.

The HDG: characterisation of T (my contribution)

HDG stands for Hyperdihedral group.

Motivation

- Generalise dihedral benchmarking [1].
- Fewer group elements than the Clifford group.
- Smallest number of parameters in the expression for $\bar{F}$.

Construction

- Every product of X and T.
-
$$X := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & & \\ & \exp(2\pi i/3) & \\ & & \exp(2\pi i/3)^8 \end{bmatrix}. \quad (4)$$
- $X^x T^\alpha T'^\beta$, $x \in [2] := \{0, 1, 2\}$ and $\alpha, \beta \in [8]$.
- $T' := XTX^\dagger$.

Resulting scheme: HDG randomised benchmarking

The average gate fidelity (for the HDG) is $\bar{F} = \frac{9+3(1+2\lambda_0+6\lambda_+)}{36}$.

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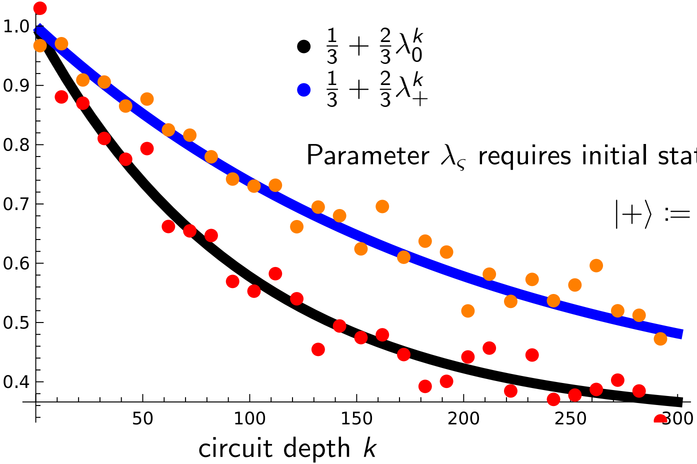

Parameter λ_ς requires initial state $|\varsigma\rangle$.

$|+\rangle := H|0\rangle$.

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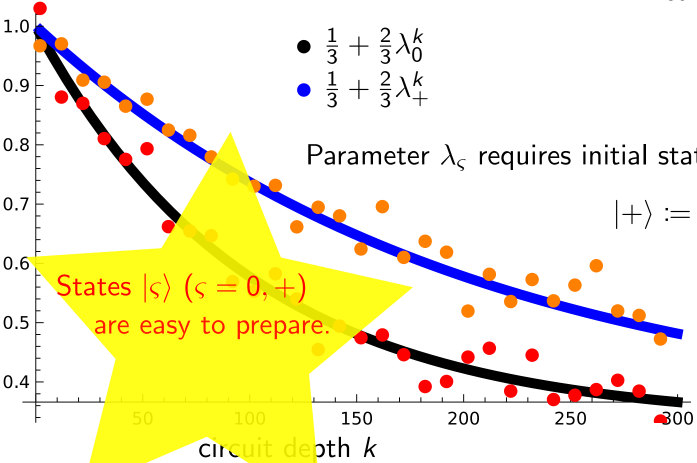

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$|+\rangle := H|0\rangle$.

States $|\varsigma\rangle$ ($\varsigma = 0, +$) are easy to prepare.

Conclusions

- Estimating the performance of quantum gates using the average gate fidelity.
- Randomised benchmarking is the Academy and industry standard for the performance quantification of quantum gates.
- I presented my generalisation, mainly done by the introduction of a group HDG, of the randomised benchmarking scheme to characterise universal qutrits gates.

Alberta NSERC CRSNG

Collaborators

- Dr. Hubert de Guise (Lakehead University).
- Dr. Barry C. Sanders (University of Calgary).

References I

[1] Arnaud Carignan-Dugas, Joel J. Wallman, and Joseph Emerson. "Characterizing universal gate sets via dihedral benchmarking". In: *Phys. Rev. A* 92 (6 Dec. 2015), p. 060302. DOI: 10.1103/PhysRevA.92.060302. URL: https://link.aps.org/doi/10.1103/PhysRevA.92.060302.

[2] Irene Fernández de Fuentes et al. "Coherent quantum control of a single ^{123}Sb atom in silicon.". In: *Bull. Am. Math. Soc.* (2022).

[3] Poolad Imany et al. "High-dimensional optical quantum logic in large operational spaces". In: *npj Quantum Inf.* 5.1 (Dec. 2019), p. 59. ISSN: 2056-6387. DOI: 10.1038/s41534-019-0173-8.

[4] A. B. Klimov et al. "Qutrit quantum computer with trapped ions". In: *Phys. Rev. A* 67.6 (June 2003), p. 062313. ISSN: 1050-2947, 1094-1622. DOI: 10.1103/PhysRevA.67.062313.

[5] M. Kononenko et al. "Characterization of control in a superconducting qutrit using randomized benchmarking". In: *Phys. Rev. Res.* 3.4 (Oct. 2021), p. L042007. ISSN: 2643-1564. DOI: 10.1103/PhysRevResearch.3.L042007.

[6] B. P. Lanyon et al. "Manipulating Biphotonic Qutrits". In: *Phys. Rev. Lett.* 100.6 (Feb. 2008), p. 060504. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.100.060504.

[7] F. M. Leupold et al. "Sustained State-Independent Quantum Contextual Correlations from a Single Ion". In: *Phys. Rev. Lett.* 120.18 (May 2018), p. 180401. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.120.180401.

[8] Junan Lin et al. "On the freedom in representing quantum operations". In: *New J. Phys.* 21.2 (Feb. 2019), p. 023006. DOI: 10.1088/1367-2630/ab075a. URL: https://doi.org/10.1088/1367-2630/ab075a.

References III

[9] Joseph Lindon et al. "Complete unitary qutrit control in ultracold atoms". arXiv. 2022. arXiv: 2208.00045 [quant-ph].

[10] A. Morvan et al. "Qutrit randomized benchmarking". In: *Phys. Rev. Lett.* 126.21 (May 2021), p. 210504. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.126.210504.

[11] Michael A Nielsen. "A simple formula for the average gate fidelity of a quantum dynamical operation". In: *Phys. Lett. A* 303.4 (2002), pp. 249–252. ISSN: 0375-9601. DOI: https://doi.org/10.1016/S0375-9601(02)01272-0. URL: https://www.sciencedirect.com/science/article/pii/S0375960102012720.

[12] Michael A Nielsen and Isaac L Chuang. *Quantum Computation and Quantum Information*. 10th anniversary ed. Cambridge, UK: Cambridge University Press, 2010. ISBN: 9781107002173.

References IV

[13] J. Randall et al. "Efficient preparation and detection of microwave dressed-state qubits and qutrits with trapped ions". In: *Phys. Rev. A* 91.1 (Jan. 2015), p. 012322. ISSN: 1050-2947, 1094-1622. DOI: 10.1103/PhysRevA.91.012322.

[14] Tanay Roy et al. "Realization of two-qutrit quantum algorithms on a programmable superconducting processor". arXiv. 2022. eprint: 2211.06523.

Why two parameters?

- $\Gamma(\mathrm{HDG}) = \Sigma_I + \Sigma_0 + \Sigma_0^* + \Sigma_+ + \Sigma_+^*$.
- Each irrep has associated a parameters in the average gate fidelity.
- For high-fidelity gates, the phase is negligible.

Representation theory

PL representation [8]

- Given an orthonormal basis for the set of 3×3 matrices: $\{W_i\}$.
- $\Gamma(\mathcal{E})_{i,j} := 3^{-1/2}\,\mathrm{Tr}(W_i^\dagger \mathcal{E}(W_j))$.
- Γ is a representation.

Irreps

For a unitary irrep γ, $\Gamma \cong \gamma \otimes \bar{\gamma}$, where $\bar{}$ denotes complex conjugate.

Average gate fidelity [11]

For a given channel $\mathcal{E}$,

$$\bar{F}(\mathcal{E}, \mathbb{I}) = \frac{3\,\mathrm{Tr}(\Gamma) + 9}{36}. \tag{5}$$

If the noise $\mathcal{E}$ is the same for each gate-set member $\bar{F}(\mathcal{E}, \mathbb{I}) = \bar{F}(\mathcal{G})$.

Survival probability theoretical curve

It can be shown that:

$$\sum_{\mathbf{g}\in G^k} \langle 0|\, \tilde{U}_{\mathrm{inv}(\mathbf{g})} \tilde{U}_{g_1} \cdots \tilde{U}_{g_k} \,|0\rangle = \langle\langle 0|\mathcal{T}(\mathcal{E})^k|0\rangle\rangle, \tag{6}$$

where $\tilde{U} := \rho \mapsto \mathcal{E}(U\rho U^\dagger)$ and

$$\mathcal{T}(\mathcal{E}) := \sum_{g\in G} \Gamma(U_g)^\dagger \Gamma(\mathcal{E}) \Gamma(U_g). \tag{7}$$

The matrix $\mathcal{T}$ is known as the twirl of $\mathcal{E}$.

Clifford

Normaliser

Let N and G groups such that $N \leq G$. G normalises N if $\forall g \in G, h \in N$, $ghg^{-1} \in N$.

Pauli group for qutrits

X and Z matrices and phases $\omega := \exp(2\pi i/3)$.

Characteristics

- Normaliser of the Pauli group.
- Can be efficiently simulated on a classical computer.
- Cannot efficiently approximate an arbitrary unitary.

Universal gates

Universal gates

- Gates with up to three qutrits.
- Finite set of gates approximate up to arbitrary accuracy any evolution.

Efficient

- Polylog with respect to the inverse of the accuracy ε.
- Meaning, increasing one digit of precision in accuracy increases polynomially.

Solovay-Kitaev theorem [12]

- Efficient computation of the circuit.
- Efficient circuit depth.