Rotational Effects on Fisher Information of Thermal Black Hole Parameter

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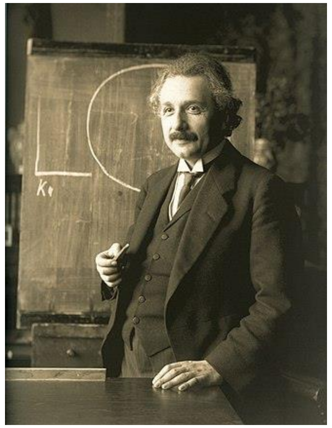
University of Waterloo | Faculty of Science

INTRODUCTION



- **RQI** as an approach to reconcile Q+G
  - How does relativity affect QI processes?
  - Using QI processes to probe relativistic systems
  - Mostly QFT in curved spacetime
- Thermal states are an important feature of the quantum vacuum in curved spacetime
  - E.g., Hawking radiation, Unruh effect
- **Fisher information** (FI) is used in Relativistic Quantum Metrology (RQM) to guide the development of experimental set ups.
  - Here we use it to 'probe' spacetime

PREVIOUS WORK

- Good understanding of FI in 4-d dS and AdS, and 3-d AdS and BTZ
  - BH mass effect
- Want to explore FI behaviour in *rotating* BH spacetime
  - Known quantum vacuum effects
  - (1) Improve ‘measurement’ procedures
  - (2) Act as a spacetime discriminant


Prepared for submission to JHEP

Fisher Information of a Black Hole Spacetime

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Abstract: Relativistic quantum metrology is the study of optimal measurement procedures within systems that have both quantum and relativistic components. Here we use Unruh-DeWitt detectors coupled to a massless scalar field as probes of thermal parameters in different spacetimes via a relativistic quantum metrology analysis. We consider both $(2+1)$-dimensional anti-de Sitter and BTZ black hole spacetimes. We compute the Fisher information to identify characteristics of the black hole spacetime and to compare it to a uniformly accelerating detector in anti-de Sitter space. We find the dependence of the Fisher information on temperature, detector energy gap, black hole mass, interaction time, and the initial state of the detector. We identify strategies that maximize the Fisher information and therefore the precision of estimation.

[1] Corresponding author.

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PHYSICAL REVIEW D 106, 045018 (2022)

Anti-Hawking phenomena around a rotating BTZ black hole

Matthew P. G. Robbins[1,2,3,*] and Robert B. Mann[1,2,3,4]

(Received 17 July 2021; accepted 25 April 2022; published 18 August 2022)

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THEORY

- Most general approach to RQI requires 3 characters:
  - Particle detector(s) – single UDW detector
  - QFT in the underlying spacetime – massless scalar field in RBTZ
  - Framework under consideration – (Thermal) Fisher information (some OQS in the background)

THEORY

- FI Definition $\mathcal{I}(\xi) = \int p(x|\xi) \left(\frac{\partial \ln p(x|\xi)}{\partial \xi} \right)^2 dx = \int \frac{1}{p(x|\xi)} \left(\frac{\partial p(x|\xi)}{\partial \xi} \right)^2 dx$

- Unruh-DeWitt (UDW) detector, interaction Hamiltonian: $H_I = \lambda \chi(\tau) \left(e^{i\Omega\tau} \sigma^+ + e^{-i\Omega\tau} \sigma^- \right) \otimes \phi[x(\tau)]$

- While the FI definition may seem a little messy, by using the Open Quantum Systems paradigm, it is solely dependent on the Response Rate [40] of the UDW detector:

$$\mathcal{F}(\Omega) = \int_{-\infty}^{\infty} d\Delta\tau \; e^{-i\Omega\Delta\tau} \, W(\Delta\tau)$$

- BTZ Wightman function is related to AdS by image sum [44]: $W_{\mathrm{BTZ}}(x, x') = \sum_{n=-\infty}^{\infty} W_{\mathrm{AdS}}(x, \Gamma^n x')$

THEORY

θ is the initial state of detector

'A' depends on Response Rate

- FI Definition $$\mathcal{I}(T) = \frac{(\partial_T a_3)^2}{1 - a_3^2}$$ where $$a_3(\tau) = -e^{-A\tau}\cos\theta - R(1 - e^{-A\tau})$$

When τ→∞, 'R' describes Fisher info

$$R = -\tanh(\frac{\Omega}{2T})$$

- While the FI definition may seem a little messy, by using the Open Quantum Systems paradigm, it is solely dependent on the Response Rate [40] of the UDW detector:

$$\mathcal{F}_{\mathrm{RBTZ}} = \frac{1}{4}\left[1 - \tanh\left(\frac{\Omega}{2T}\right)\right]\sum_{n=-\infty}^{n=\infty} e^{\frac{i\Omega n r_-}{\ell T}}\left[P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(\alpha_n^-\right) - \zeta P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(\alpha_n^+\right)\right]$$

- Where $$\alpha_n^{\pm} = (1 + 4\pi^2\ell^2T^2)\cosh(2\pi n r_+/\ell) \pm 4\pi^2\ell^2T^2$$

RESPONSE RATE – IN MORE DETAIL

- Same Response Rate, but bigger:

inner radius

$$\mathcal{F}_{\mathrm{RBTZ}} = \frac{1}{4}\left[1 - \tanh\left(\frac{\Omega}{2T}\right)\right] \sum_{n=-\infty}^{n=\infty} e^{\frac{i\Omega n r_-}{\ell T}} \left[P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(\alpha_n^-\right) - \zeta P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(\alpha_n^+\right)\right]$$

  - Where $\alpha_n^{\pm} = \left(1 + 4\pi^2\ell^2T^2\right)\cosh(2\pi n r_+/\ell) \pm 4\pi^2\ell^2T^2$

outer radius

- Note the Radii are what distinguish the black hole spacetimes
- The BTZ mass and angular momentum can be expressed in terms of these radii:

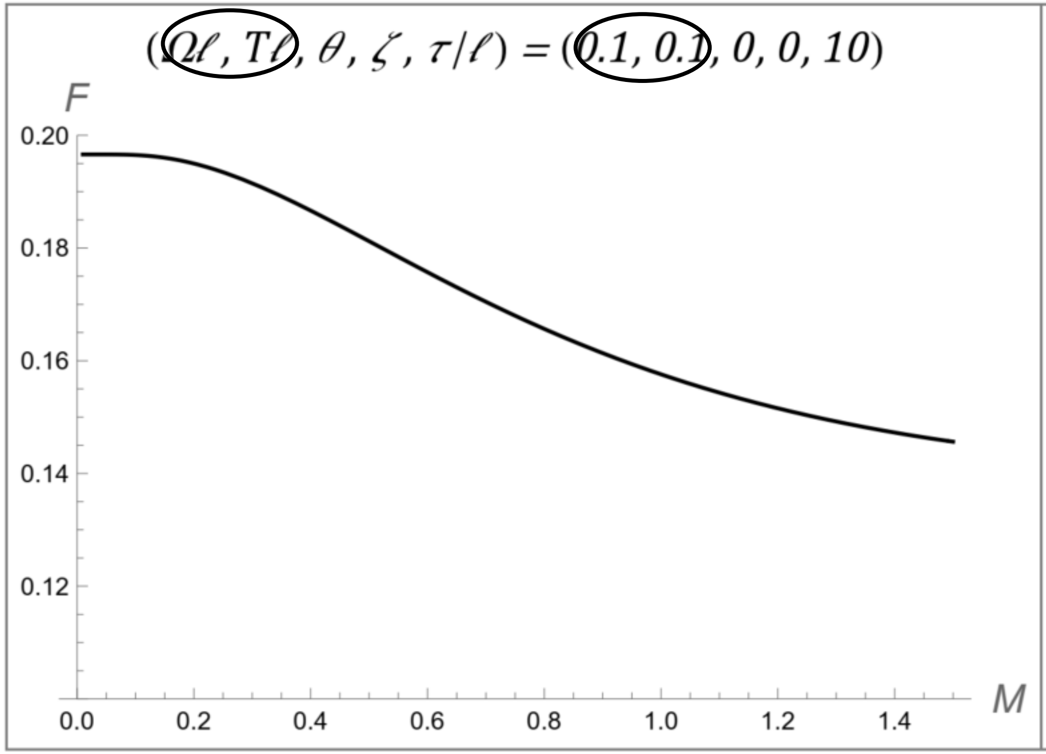
$$M = \frac{r_+^2 + r_-^2}{\ell^2}$$

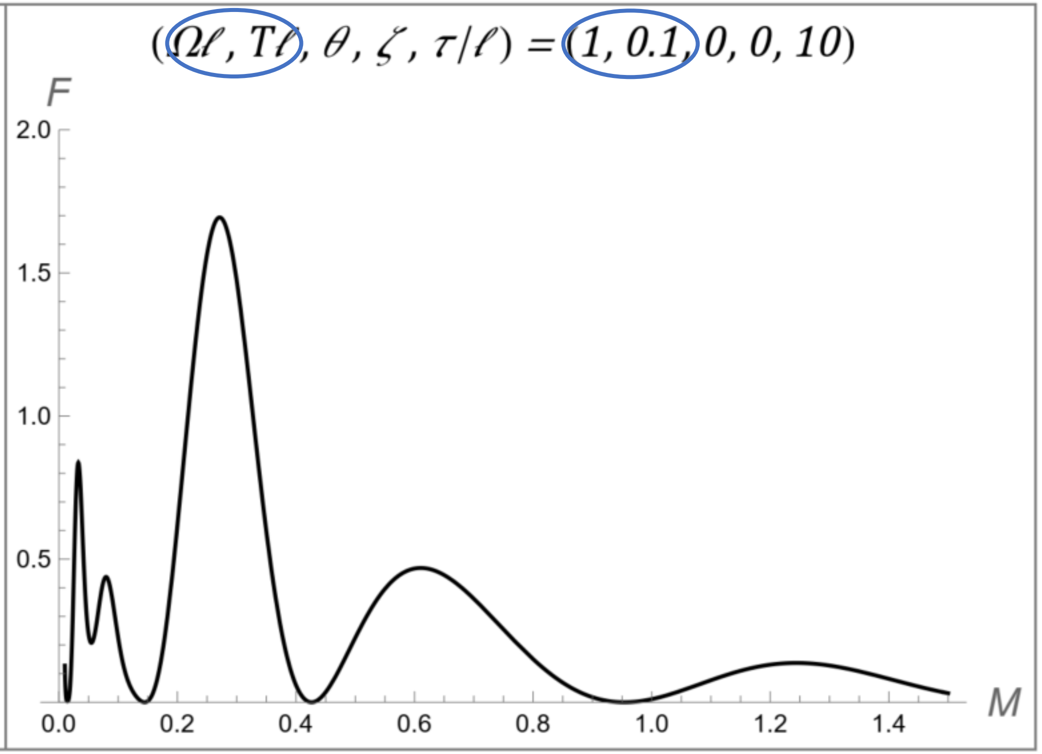
$$J = \frac{2r_+r_-}{\ell}$$

[2207.12226] Fisher Information of a Black Hole Spacetime (arxiv.org)

Static BTZ

Cold, $\Omega > T$





- When $\Omega = T$, we see a monotonic decrease in the FI for increasing mass
- Whereas for $\Omega \neq T$, we see an oscillatory behaviour
- This suggests that 'tuning' the detector to the mass and temperature can indeed improve estimation

Previous work

PHYSICAL REVIEW D **106,** 045018 (2022)

Anti-Hawking phenomena around a rotating BTZ black hole

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31 Caroline Street North, Waterloo, Ontario, Canada N2L 2Y5
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(Received 17 July 2021; accepted 25 April 2022; published 18 August 2022)

In both flat and curved spacetimes, there are weak and strong versions of the anti-Unruh/anti-Hawking effects, in which the Kubo-Martin-Schwinger field temperature is anticorrelated with the response of a detector and its inferred temperature. We investigate for the first time the effects on the weak and strong anti-Hawking effects for an Unruh-DeWitt detector orbiting a Banados-Teitelboim-Zanelli black hole in the corotating frame. We find that rotation can significantly amplify the strength of the weak anti-Hawking effect, whereas it can either amplify or reduce the strength of the strong anti-Hawking effect depending on boundary conditions. For the strong anti-Hawking effect, we find a nonmonotonic relationship between the angular momentum and detector temperature for each boundary condition. In addition, we note that the weak anti-Hawking effect is independent of a changing AdS length, while a longer AdS length increases the temperature range of the strong anti-Hawking effect.

DOI: 10.1103/PhysRevD.106.045018

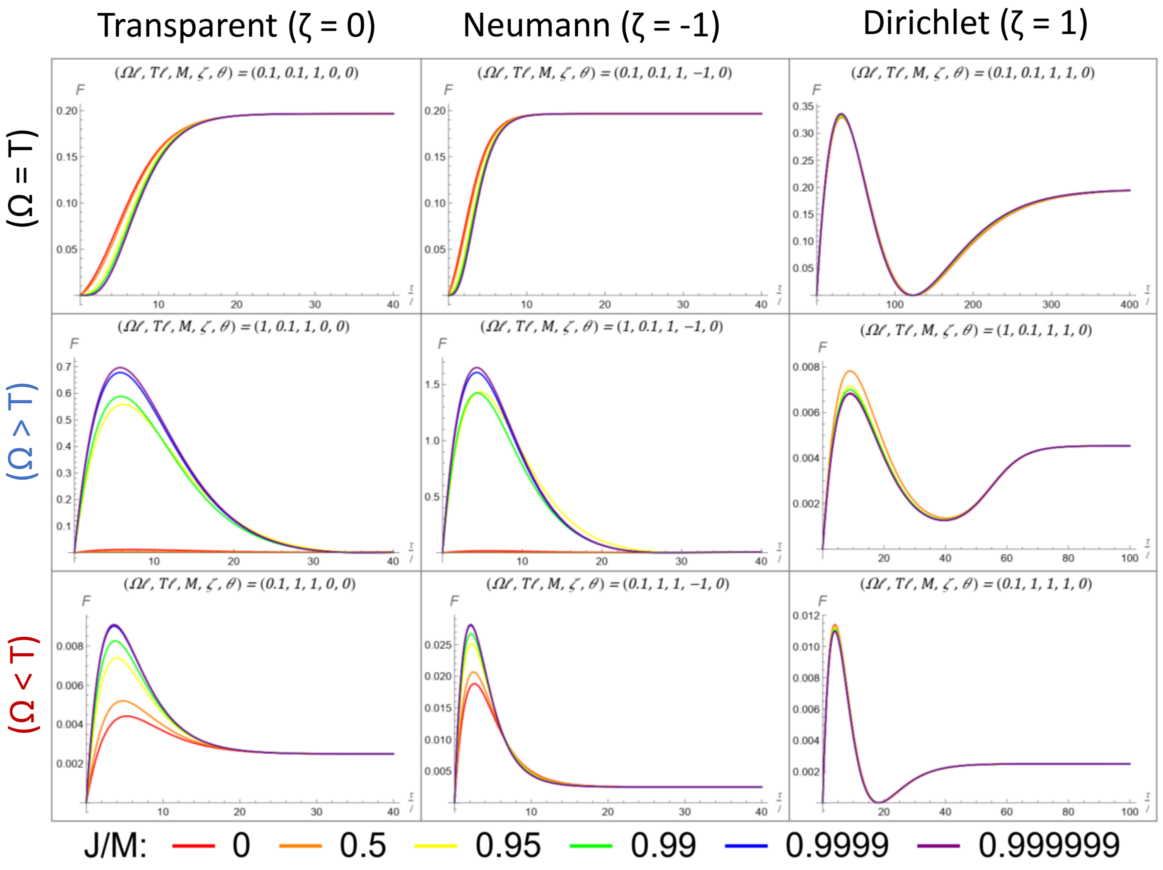
[2107.01648] Anti-Hawking Phenomena around a Rotating BTZ Black Hole (arxiv.org)

[2010.14517] Entanglement Amplification from Rotating Black Holes (arxiv.org)

Why Rotating Black Holes:

- Rotating BTZ Black Holes have exhibited
  - Amplification of Entanglement Harvesting
  - Most pronounced for small black holes
- Also demonstrate
  - Anti-Hawking effect
  - Dependent on BH mass
  - Very dependent on spacetime boundary conditions

Rotating BTZ for M=1, Varying J/M

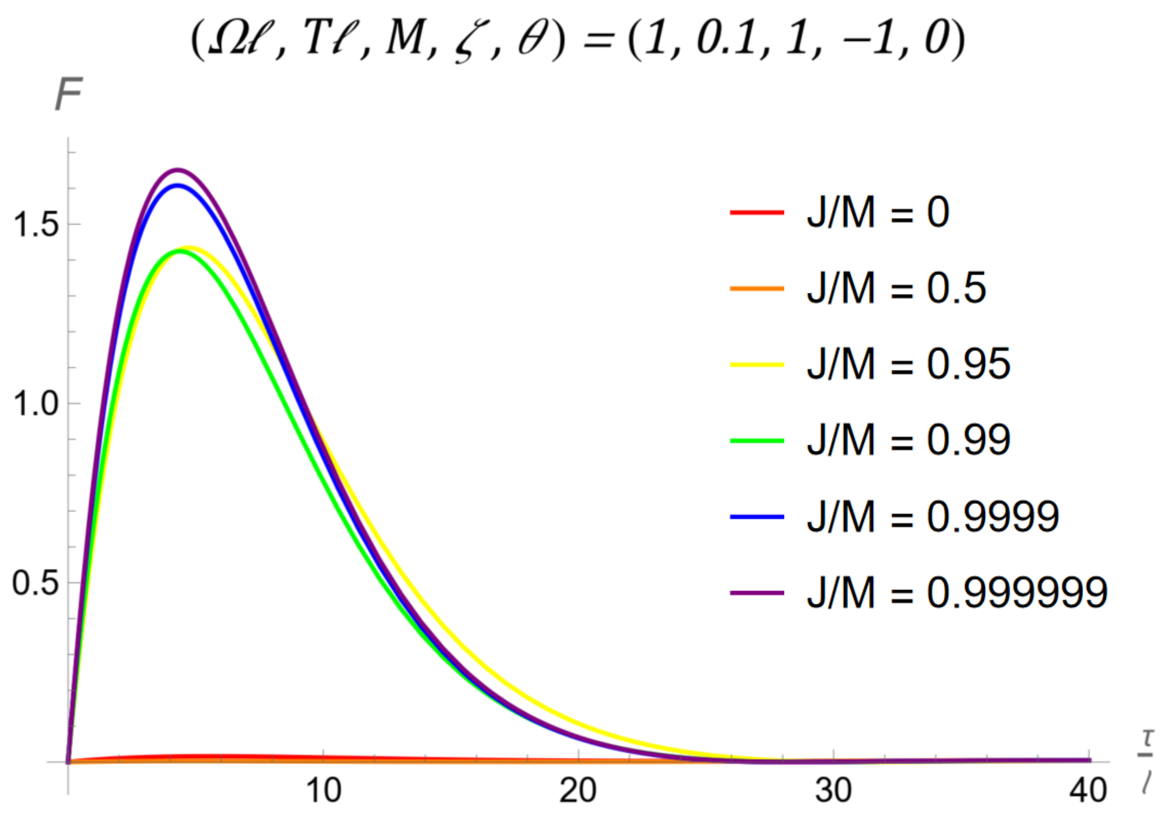


Of note:

- When **(Ω, T) = (0.1, 0.1)**, varying J leaves the FI mostly unchanged
- When **(Ω, T) = (1, 0.1)** aka Cold,
  - Increasing J leads to dramatic increase in FI for transparent (ζ = 0) and Neumann (ζ = -1) boundary conditions,
    - Most dramatic between J=0.5M and J=0.95M
  - But a slight decrease for Dirichlet (ζ = 1) boundary condition
- When **(Ω, T) = (0.1, 1)** aka Hot,
  - Increasing J leads to increase in FI for transparent (ζ = 0) and Neumann (ζ = -1) boundary conditions,
    - Though not so dramatic
  - And still a slight decrease for Dirichlet (ζ = 1) boundary condition

Rotating BTZ for M=1,

Varying J/M



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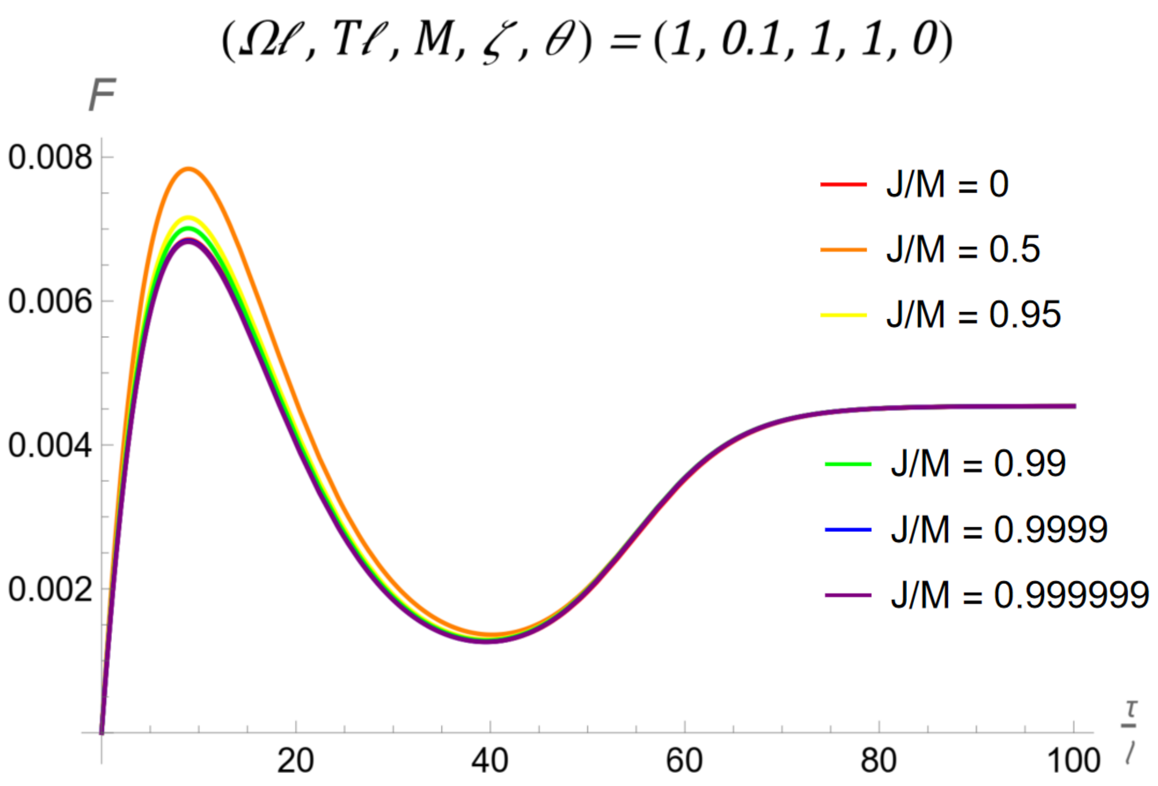
- Hot is similar to Cold

Rotating BTZ for M=1,

Varying J/M



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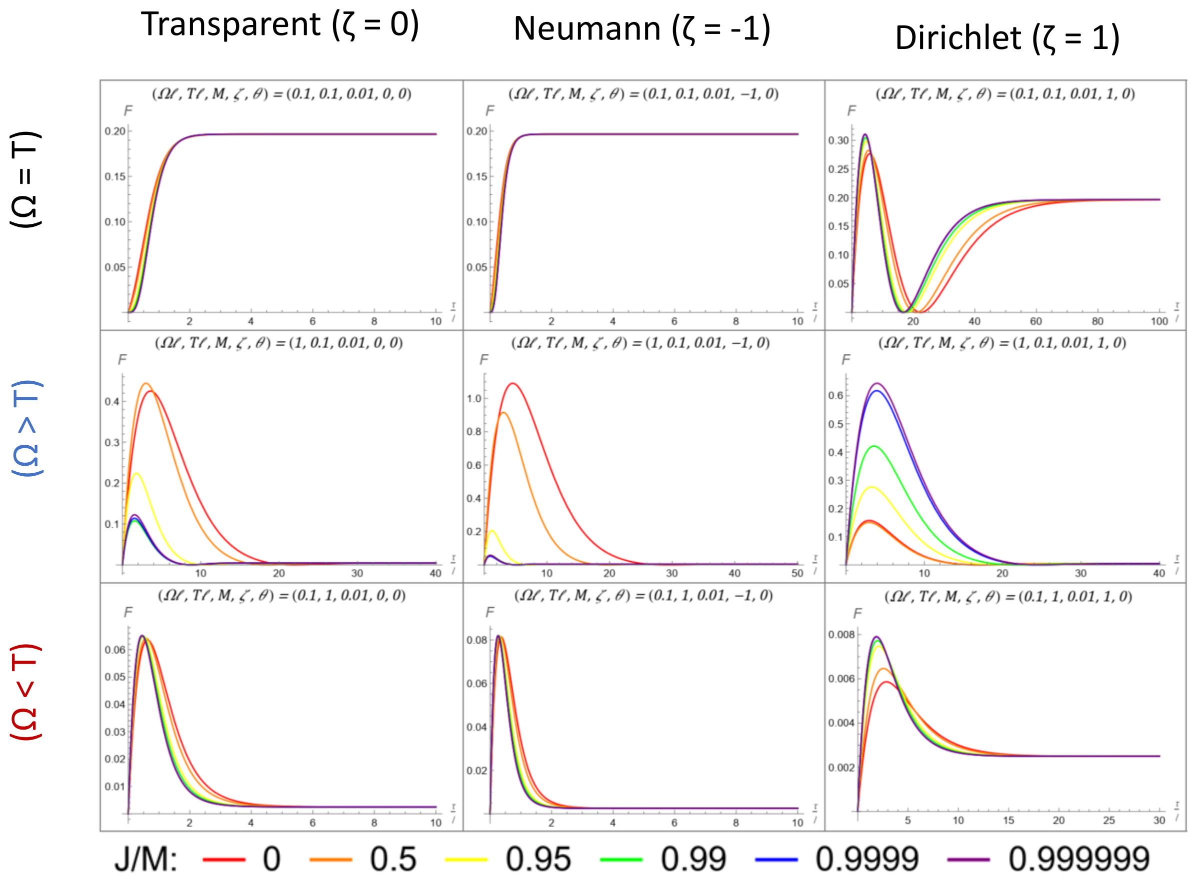
  - But a slight **decrease** for Dirichlet (ζ = 1) boundary condition
- Hot is similar to Cold

Rotating BTZ for **M=0.01**,

Varying J/M



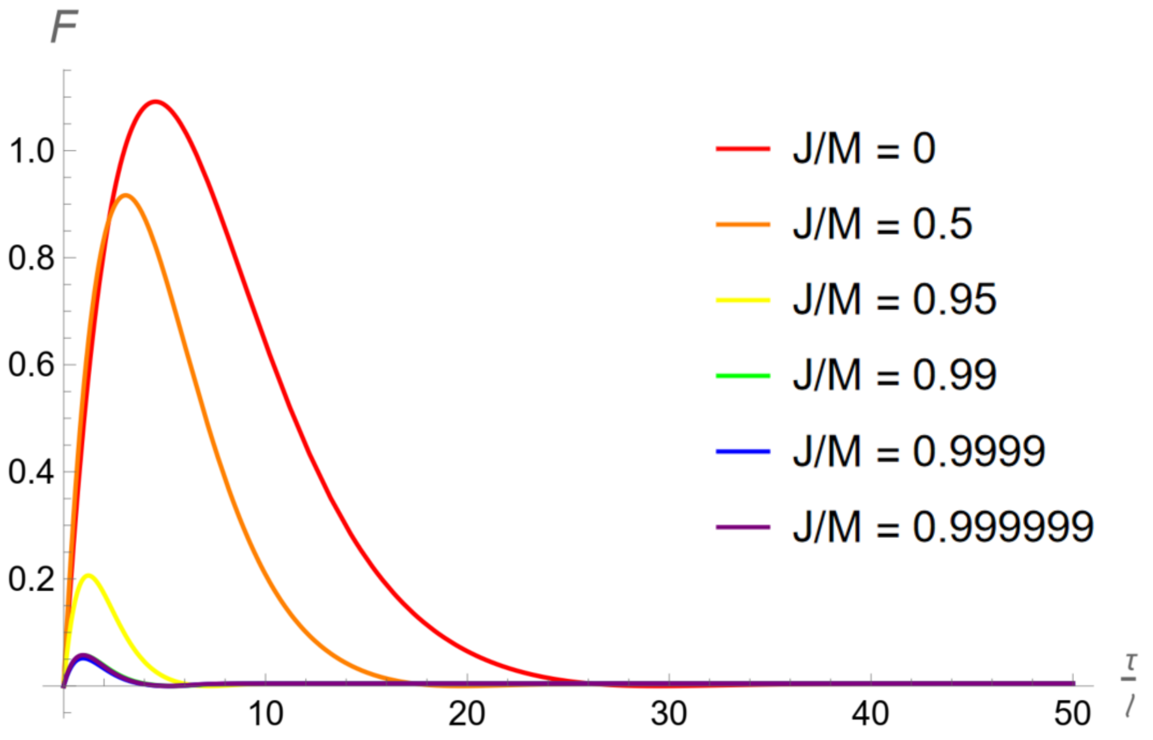
Of note:

- The distinction in the FI based on the boundary condition (ζ) persists
- However, when **(Ω, T) = (1, 0.1)** aka Cold,
  - increasing J leads to significant decrease in FI for transparent (ζ = 0) and Neumann (ζ = -1) boundary conditions,
  - but an increase for Dirichlet (ζ = 1) boundary condition
- When **(Ω, T) = (0.1, 1)** aka Hot,
  - increasing J leads to slight shift in the time of the maximal FI for all boundary conditions
  - but still a slight decrease for Dirichlet (ζ = 1) boundary condition

Rotating BTZ for **M=0.01**,

Varying J/M

$$(\Omega\ell, T\ell, M, \zeta, \theta) = (1, 0.1, 0.01, -1, 0)$$



Of note:

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- However, when **(Ω, T) = (1, 0.1)** aka Cold,
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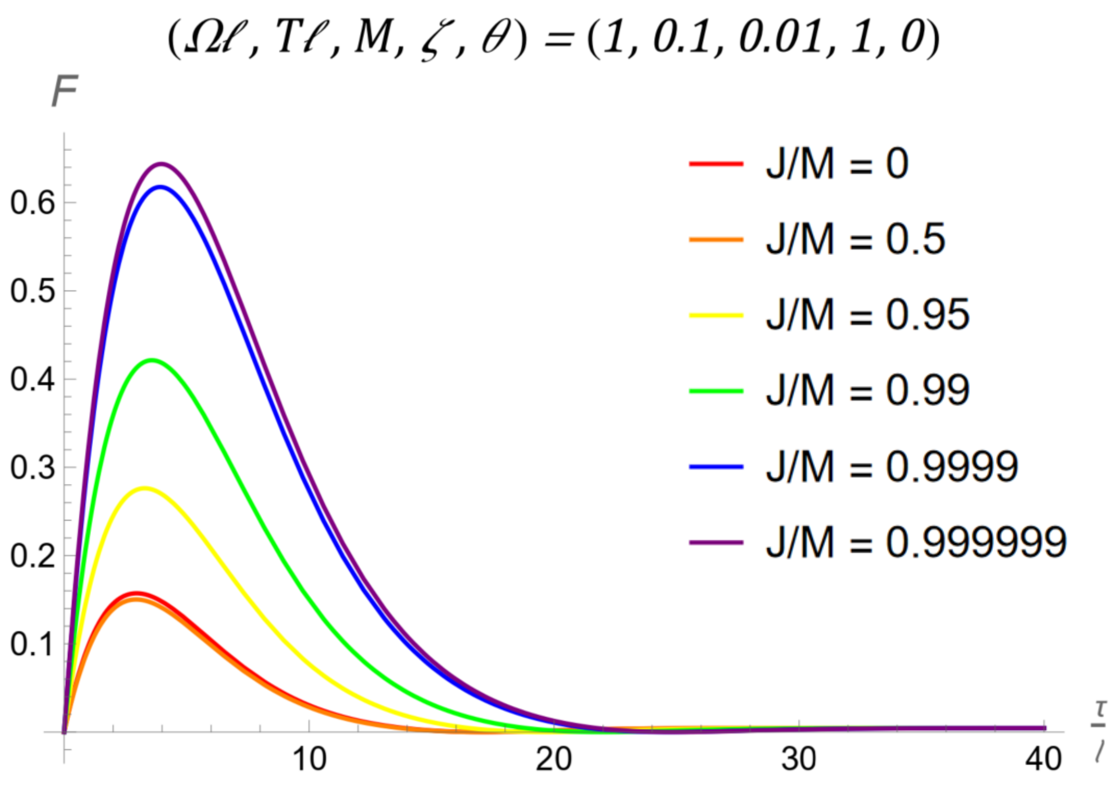

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CONCLUSION

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We find that

- Varying the angular momentum J/M does lead to change in the FI, especially for near extremal rotation
- It can both significantly amplify or de-amplify the FI depending on
  - the boundary condition ζ and
  - the BTZ mass M
  - more like anti-Hawking results than entanglement harvesting

Future work:

- Quantum Fisher information [52] analysis in 2+1 Dimensions
- More realistic Fisher information analysis: 3+1 Dimensional Spacetimes

THANKS!!

Thank you

Cited work by EP:
[2207.12226] Fisher Information of a Black Hole Spacetime (arxiv.org) ,
Analysis of the Thermal Fisher Information in 2+1 Dimensional Black Hole Spacetimes (uwaterloo.ca)

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SUPPLEMENTARY SLIDES

Rotational Effects on Fisher Information of Thermal Black Hole Parameter

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FISHER INFORMATION

- Quantification of the parameter estimation problem,
- Given (χ, Ξ), the set of possible values for the observable and underlying parameters, respectively,
- Estimator $\hat{\xi} : \chi^n \to \Xi$ is said to be unbiased if Expectation Value = True Value of Parameter
- FI Definition $$\mathcal{I}(\xi) = \int p(x|\xi) \left(\frac{\partial \ln p(x|\xi)}{\partial \xi} \right)^2 dx = \int \frac{1}{p(x|\xi)} \left(\frac{\partial p(x|\xi)}{\partial \xi} \right)^2 dx$$
- Cramér-Rao bound $$\mathrm{var}(\hat{\xi}) \geq \frac{1}{\mathcal{I}(\xi)}$$

UNRUH-DEWITT (UDW) DETECTOR

- Two-level quantum system with states $|0\rangle_D$ and $|1\rangle_D$ separated by an energy gap Ω
- Conformally couple to massless scalar field, $\phi(x)$, via interaction Hamiltonian:

$$H_I = \lambda(e^{i\Omega\tau}\sigma^+ + e^{-i\Omega\tau}\sigma^-) \otimes \phi(x(\tau))$$

- Where $\sigma^+ = |1\rangle_D\langle 0|_D, \ \sigma^- = |0\rangle_D\langle 1|_D$.
- Has response function $F(\Omega) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \, e^{-i\Omega(\tau-\tau')} W(x(\tau), x(\tau'))$,
- And response rate $\mathcal{F}(\Omega) = \int_{-\infty}^{\infty} d\Delta\tau \, e^{-i\Omega\Delta\tau} W(\Delta\tau)$, where $\Delta\tau = \tau - \tau'$.

SPACETIMES

- First consider AdS_3, then BTZ
- AdS_3 stems from induced metric on a 3-d hyperboloid $X_1^2 + X_2^2 - T_1^2 - T_2^2 = -\ell^2$
  - Where ℓ is the AdS length
- Embedded in 4-d flat geometry $dS^2 = dX_1^2 + dX_2^2 - dT_1^2 - dT_2^2$
  - With coordinates (X_1, X_2, T_1, T_2)

SPACETIME – ADS$_3$

- Transformations

$$T_1 = \ell\sqrt{\frac{r^2}{\ell^2}}\cosh\Phi\,, \quad X_1 = \ell\sqrt{\frac{r^2}{\ell^2}}\sinh\Phi\,,$$
$$T_2 = \ell\sqrt{\frac{r^2}{\ell^2}-1}\sinh\frac{t}{\ell}\,, \quad X_2 = \ell\sqrt{\frac{r^2}{\ell^2}-1}\cosh\frac{t}{\ell}$$

- Metric

$$ds^2 = -\left(\frac{r^2}{\ell^2}-1\right)dt^2 + \left(\frac{r^2}{\ell^2}-1\right)^{-1}dr^2 + r^2 d\Phi^2$$

- Wightman

$$W^{(\zeta)}_{\rm AdS}(x,x') = \frac{1}{4\pi\ell\sqrt{2}}\left(\frac{1}{\sqrt{\sigma(x,x')}} - \frac{\zeta}{\sqrt{\sigma(x,x')+2}}\right)$$

$\zeta \in \{1, 0, -1\}$

$$\sigma(x,x') = \frac{1}{2\ell^2}\left[(X_1 - X_1')^2 - (T_1 - T_1')^2 + (X_2 - X_2')^2 - (T_2 - T_2')^2\right]$$

SPACETIME – STATIONARY BTZ

- Transformations

$$T_1 = \ell\sqrt{\frac{r^2}{M\ell^2}}\cosh(\sqrt{M}\phi)\,, \quad X_1 = \ell\sqrt{\frac{r^2}{M\ell^2}}\sinh(\sqrt{M}\phi)\,,$$

$$T_2 = \ell\sqrt{\frac{r^2}{M\ell^2} - 1}\sinh\frac{\sqrt{M}t}{\ell}\,, \quad X_2 = \ell\sqrt{\frac{r^2}{M\ell^2} - 1}\cosh\frac{\sqrt{M}t}{\ell}$$

with:

identification $\phi \sim \phi + 2\pi$

- Metric

$$ds^2 = -\left(\frac{r^2}{\ell^2} - M\right)dt^2 + \left(\frac{r^2}{\ell^2} - M\right)^{-1}dr^2 + r^2 d\Phi^2$$

- Wightman

$$W_{\mathrm{BTZ}}(x, x') = \frac{1}{4\pi\sqrt{2}\ell}\sum_{n=-\infty}^{\infty}\left[\frac{1}{\sqrt{\sigma_n}} - \frac{\zeta}{\sqrt{\sigma_n + 2}}\right]$$

$$\Delta\phi := \phi - \phi'$$

$$\Delta t := t - t'$$

$$\sigma_n := \frac{rr'}{r_{\mathrm{h}}^2}\cosh\left[\frac{r_{\mathrm{h}}}{\ell}(\Delta\phi - 2\pi n)\right] - 1 - \frac{\sqrt{(r^2 - r_{\mathrm{h}}^2)(r'^2 - r_{\mathrm{h}}^2)}}{r_{\mathrm{h}}^2}\cosh\left[\frac{r_{\mathrm{h}}}{\ell^2}\Delta t\right]$$

SPACETIME – ROTATING BTZ

- Very similar to Stationary BTZ

- Metric: $ds^2 = -\left(N^{\perp}\right)^2 dt^2 + f^{-2}dr^2 + \left(d\phi + N^{\phi}dt\right)^2$
  - Where $N^{\perp} = f = \sqrt{-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}}$, $M = \frac{r_+^2 + r_-^2}{\ell^2}$, and r_- and r_+ are the inner and outer radii,

    $N^{\phi} = -\frac{J}{2r^2}$ $J = \frac{2r_+ r_-}{\ell}$ is the angular momentum.

- Wightman $W_{\mathrm{BTZ}}(x, x') = \frac{1}{4\pi\sqrt{2}\ell} \sum_{n=-\infty}^{\infty} \left[\frac{1}{\sqrt{\sigma_n}} - \frac{\zeta}{\sqrt{\sigma_n + 2}}\right]$

$$\Delta\phi := \phi - \phi'$$

$$\Delta t := t - t'$$

SPACETIME – ROTATING BTZ

- Wightman function appears to be unchanged

$$W_{\mathrm{BTZ}}(x,x') = \frac{1}{4\pi\sqrt{2}\ell}\sum_{n=-\infty}^{\infty}\left[\frac{1}{\sqrt{\sigma_n}} - \frac{\zeta}{\sqrt{\sigma_n+2}}\right]$$

- But the squared distance is now

$$\sigma_n(x,x') = -1 + \sqrt{\alpha(r)\alpha(r')}\cosh\left[\frac{r_+}{\ell}(\Delta\phi - 2\pi n) - \frac{r_-}{\ell^2}(\Delta t)\right] - \sqrt{(\alpha(r)-1)(\alpha(r')-1)}\cosh\left[\frac{r_+}{\ell^2}(\Delta t) - \frac{r_-}{\ell}(\Delta\phi - 2\pi n)\right]$$

  - Where $\alpha(r) = \frac{r^2-r_-^2}{r_+^2-r_-^2}$
  - And $\Delta\phi := \phi - \phi'$

    $\Delta t := t - t'$

DERIVATIONS – ADS RESPONSE RATE

- For AdS, we consider the constantly accelerating trajectory
  - Corresponding to a stationary detector in Rindler coordinates

$$x_D(\tau) := \{t = \frac{\tau}{\sqrt{f(R_D)}},\, r = R_D,\, \phi = \Phi_D\}$$

- The Wightman function is then given by
  - Where we substituted R_D by the KMS temperature

$$T = \frac{\sqrt{a^2\ell^2 - 1}}{2\pi\ell} = \frac{1}{2\pi\ell}\frac{1}{\sqrt{f(R_D)}}$$

$$\begin{aligned} W_{\text{AdS}}(x, x') &= \frac{1}{8\pi\ell\sqrt{f(R_D)}}\left(\frac{1}{\sqrt{-\sinh^2(\Delta\tau/(2\sqrt{f(R_D)}\ell))}}\right. \\ &\quad \left. - \frac{\zeta}{\sqrt{1/f(R_D) - \sinh^2(\Delta\tau/(2\sqrt{f(R_D)}\ell))}}\right) \\ &= \frac{T}{4}\left(\frac{1}{\sqrt{-\sinh^2(\Delta\tau\pi T)}} - \frac{\zeta}{\sqrt{4\pi^2\ell^2T^2 - \sinh^2(\Delta\tau\pi T)}}\right) \end{aligned}$$

DERIVATIONS – ADS RESPONSE RATE

- The response rate can then be expressed as

$$\mathcal{F}_{\text{AdS}} = \frac{1}{4} - \frac{i}{4\pi}\text{PV}\int_{-\infty}^{\infty} dz \frac{e^{-i\Omega z/(\pi T)}}{\sinh z} - \frac{\zeta}{2\pi\sqrt{2}}\text{Re}\int_{0}^{\infty} dz \frac{e^{-i\Omega z/(2\pi T)}}{\sqrt{1 + 8\pi^2\ell^2T^2 - \cosh z}}$$

  - Where we performed the substitution $z = \pi T \Delta\tau$

- Performing the integrals, we find that

$$\mathcal{F}_{\text{AdS}} = \frac{1}{4}\left[1 - \tanh\left(\frac{\Omega}{2T}\right)\right] \times \left\{1 - \zeta P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(1 + 8\pi^2\ell^2T^2\right)\right\}$$

DERIVATIONS – STATIONARY BTZ RESPONSE RATE

- For BTZ, we consider the stationary detector trajectory $x_D(\tau) := \{t = \tau/\gamma_D,\, r = R_D,\, \phi = \Phi_D\}$
  - Where $\gamma_D = \sqrt{\frac{R_D^2}{\ell^2} - M}$ is the redshift factor
- Following a very similar approach to that employed in Ads, we find that the response rate is given by

$$\mathcal{F}_{\text{BTZ}} = \frac{1}{4}\left[1 - \tanh\left(\frac{\Omega}{2T}\right)\right] \sum_{n=-\infty}^{n=\infty} \left[P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(\alpha_n^-\right) - \zeta P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(\alpha_n^+\right)\right]$$

  - Where $\alpha_n^{\pm} = \pm 4\pi^2\ell^2T^2 + (1 + 4\pi^2\ell^2T^2)\cosh\left[2\pi n\sqrt{M}\right]$

X

DERIVATIONS – ROTATING BTZ RESPONSE RATE

- For BTZ, we consider the co-rotating detector trajectory $x_D(\tau) := \{t = \ell\tau/\gamma_D,\, r = R_D,\, \phi = r_-\tau/(r_+\gamma_D)\}$
  - Where $\gamma_D = \sqrt{(r^2 - r_+^2)(r_+^2 - r_-^2)}/r_+$ is the redshift factor
- Once more, following a similar approach to that employed in AdS, we find that the response rate to be

$$\mathcal{F}_{\text{RBTZ}} = \frac{1}{4}\left[1 - \tanh\left(\frac{\Omega}{2T}\right)\right] \sum_{n=-\infty}^{n=\infty} e^{\frac{i\Omega n r_-}{\ell T}} \left[P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(\cosh\alpha_n^-\right) - \zeta P_{-\frac{1}{2}+\frac{i\Omega}{2\pi T}}\left(\cosh\alpha_n^+\right)\right]$$

  - Where $\alpha_n^{\pm} = \left(1 + 4\pi^2\ell^2T^2\right)\cosh(2\pi n r_-/\ell) \pm 4\pi^2\ell^2T^2$

DERIVATIONS – SPACETIMES

- For **AdS**, we consider the constantly **accelerating** trajectory $x_D(\tau) := \{t = \frac{\tau}{\sqrt{f(R_D)}}, r = R_D, \phi = \Phi_D\}$
  - Corresponding to a stationary detector in Rindler coordinates

- For **Stationary BTZ**, the detector trajectory is **stationary**: $x_D(\tau) := \{t = \tau/\gamma_D, r = R_D, \phi = \Phi_D\}$

- For **Rotating BTZ**, the detector trajectory is **co-rotating**: $x_D(\tau) := \{t = \ell\tau/\gamma_D, r = R_D, \phi = r_-\tau/(r_+\gamma_D)\}$

- In these, $\sqrt{f(R_D)}$ and γD are the redshift factors in the appropriate spacetimes and are dependent on the radial position of the detector, R_D

DERIVATIONS – FISHER INFORMATION

- Overall Hamiltonian $H = H_D + H_\phi + H_I$

$$H_D = \frac{1}{2}\Omega a_D^\dagger a_D = \frac{1}{2}\Omega(|0\rangle_D \langle 0|_D - |1\rangle_D \langle 1|_D)$$

$$H_\phi = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$$

$$H_I = \lambda(e^{i\Omega\tau}\sigma^+ + e^{-i\Omega\tau}\sigma^-) \otimes \phi(x(\tau))$$

- von Neumann eq.

$$\frac{\partial \rho_{\text{tot}}}{\partial \tau} = -i[H, \rho_{\text{tot}}]$$

States:

$$\rho_{\text{tot}}(0) = \rho_D(0) \otimes |0\rangle \langle 0|$$

- Master equation of Kossakowski-Lindblad form

$$\frac{\partial \rho_D(\tau)}{\partial \tau} = -i[H_{\text{eff}}, \rho_D(\tau)] + L[\rho_D(\tau)]$$

$$\rho_D = \text{tr}_\phi \, \rho_{\text{tot}}.$$

State of the detector

Describes time-evolution of detector

FI DERIVATION

- In more detail…

$$\frac{\partial \rho_D(\tau)}{\partial \tau} = -i[H_{\text{eff}}, \rho_D(\tau)] + L[\rho_D(\tau)]$$

$$H_{\text{eff}} = \tfrac{1}{2}\tilde{\Omega}(|0\rangle_D \langle 0|_D - |1\rangle_D \langle 1|_D)$$

$$\tilde{\Omega} = \Omega + i\left[\mathcal{K}(-\Omega) - \mathcal{K}(\Omega)\right]$$

$$\mathcal{K}(\Omega) = \frac{1}{i\pi}\text{PV}\int_{-\infty}^{\infty} d\omega \frac{F(\omega)}{\omega - \Omega}$$

$$L[\rho] = \frac{1}{2}\sum_{i,j=1}^{3} C_{ij}\left(2\sigma_j \rho \sigma_i - \sigma_i \sigma_j \rho - \rho \sigma_i \sigma_j\right)$$

$$C_{ij} = \begin{pmatrix} A & -iB & 0 \\ iB & A & 0 \\ 0 & 0 & A+C \end{pmatrix}$$

Kossakowski matrix elements depend on the response rate ⇨

$$A = \frac{1}{2}[\mathcal{F}(\Omega) + \mathcal{F}(-\Omega)]$$

$$B = \frac{1}{2}[\mathcal{F}(\Omega) - \mathcal{F}(-\Omega)]$$

$$C = \mathcal{F}(0) - A$$

FI DERIVATION

- Surprisingly, $\frac{\partial \rho_D(\tau)}{\partial \tau} = -i[H_{\text{eff}}, \rho_D(\tau)] + L[\rho_D(\tau)]$ admits an analytic solution
- Given the initial state $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$, parametrized by θ
- We have $\rho(\tau) = \frac{1}{2}(I + \boldsymbol{a}(\tau)\cdot\boldsymbol{\sigma})$, where

$$\boldsymbol{a} = (a_1, a_2, a_3)$$

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$a_1(\tau) = e^{-A\tau/2}\sin\theta\cos\tilde{\Omega}\tau,$$

$$a_2(\tau) = e^{-A\tau/2}\sin\theta\sin\tilde{\Omega}\tau,$$

$$a_3(\tau) = -e^{-A\tau}\cos\theta - R(1 - e^{-A\tau})$$

$$R = B/A = -\tanh\left(\frac{\Omega}{2T}\right)$$

The third Bloch vector term is dependent on a ratio of Kossakowski matrix elements

FI DERIVATION

- Fisher information $$\mathcal{I}(\xi) = \frac{1}{p}\left(\frac{\partial p}{\partial \xi}\right)^2 + \frac{1}{1-p}\left(-\frac{\partial p}{\partial \xi}\right)^2 = \frac{1}{p(1-p)}\left(\frac{\partial p}{\partial \xi}\right)^2$$
  - For 2-d system
  - Where $$p = \mathrm{Tr}(\rho|0\rangle_D\langle 0|_D) = \frac{1}{2}(1+a_3)$$ $$1-p = \tfrac{1}{2}(1-a_3)$$
- Thus, $\mathcal{I}(\xi) = \dfrac{(\partial_\xi a_3)^2}{1-a_3^2}$, which for us will be

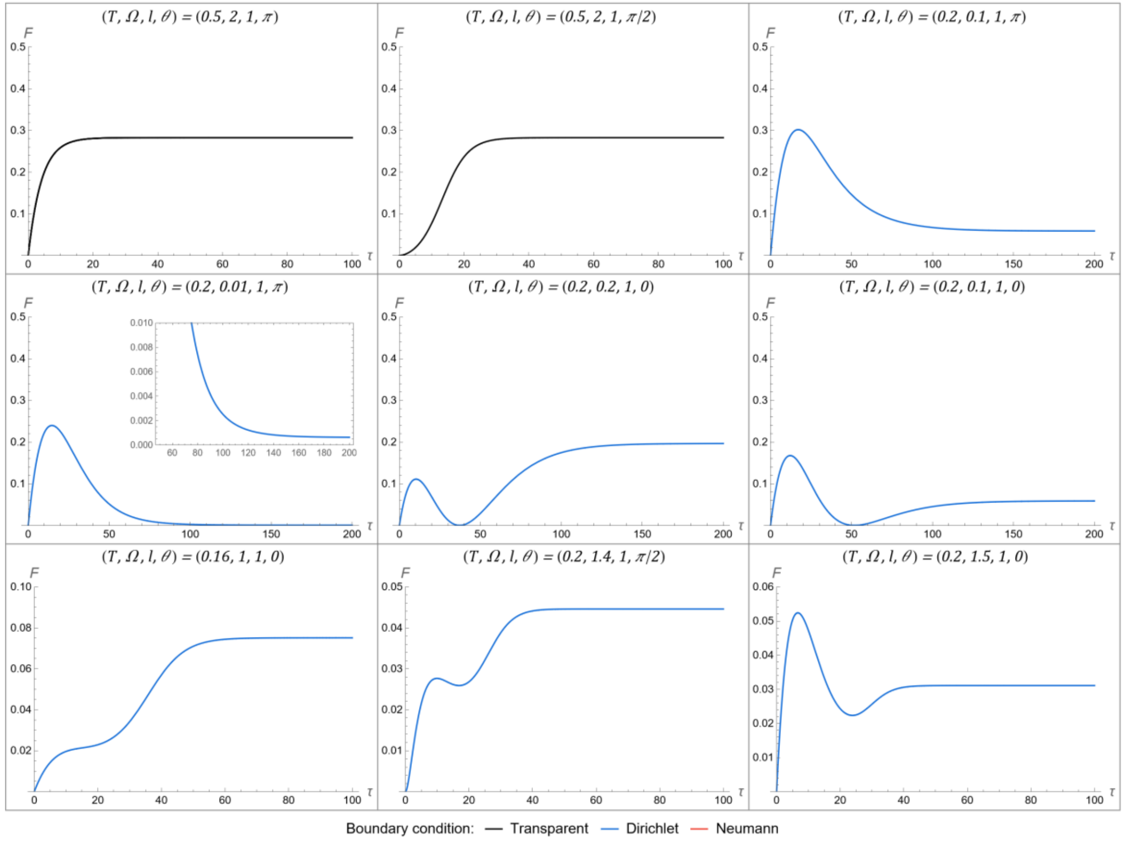
Fisher Information

$$\mathcal{I}(T) = \frac{(\partial_T a_3)^2}{1-a_3^2}$$

- Parameters at play: $(T, \Omega, \theta, \ell, \zeta, (M))$ and J

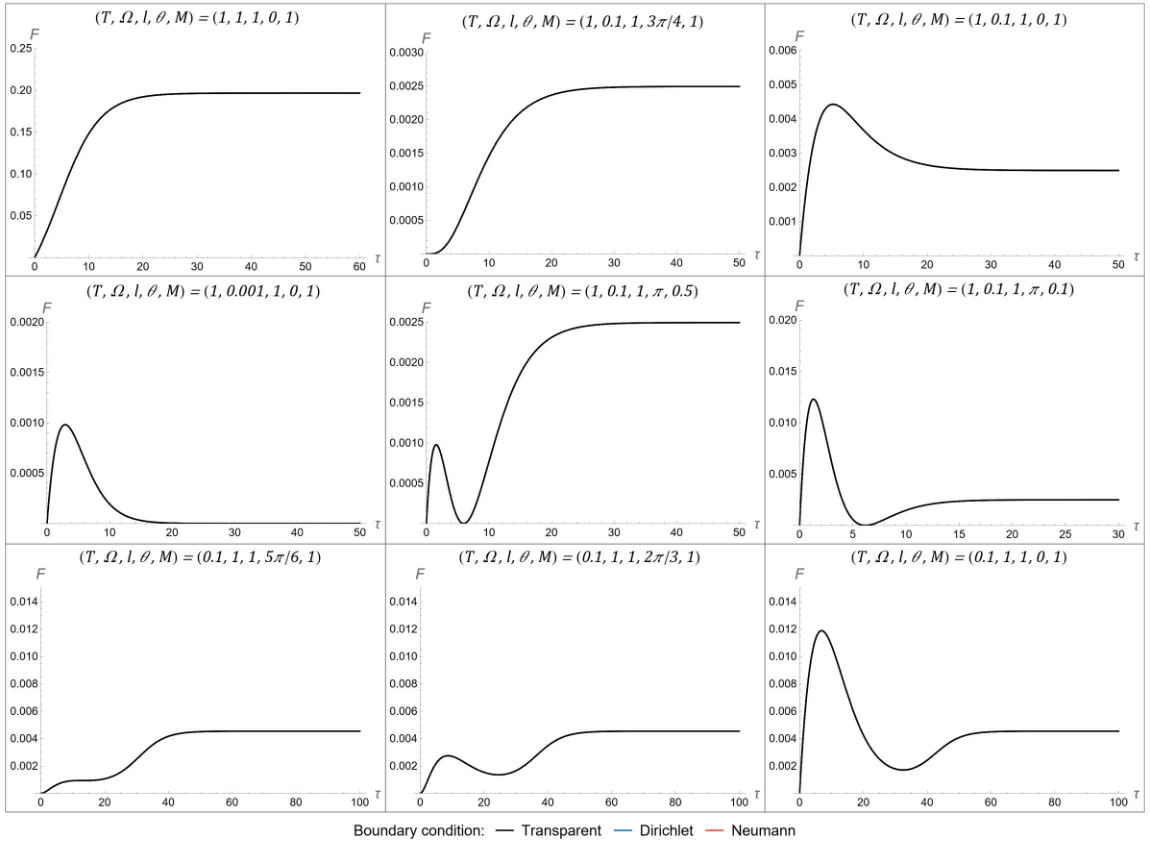
Temperature (T), Energy gap (Ω), Initial state (θ), AdS length (ℓ), Boundary condition (ζ), BTZ Mass (M), Angular momentum (J)

AdS



- There are 8* distinct qualitative behaviours
  - Behaviour 4 is the same as behaviour 3
- Curve colour indicates boundary condition
  - Black = Transparent ($\zeta=0$)
  - Blue = Dirichlet ($\zeta=1$)
  - Red = Neumann ($\zeta=-1$)

BTZ



- Same 8 qualitative behaviours are present in BTZ
- All can be achieved using only the transparent boundary condition and M=1

4-d AdS

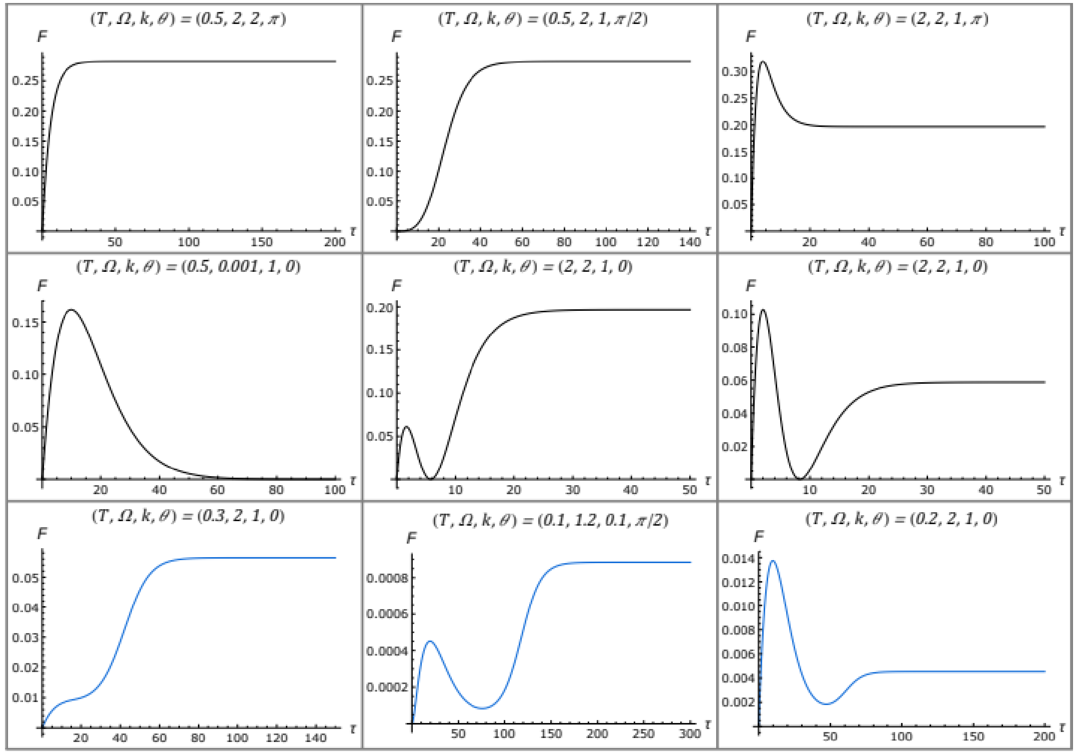
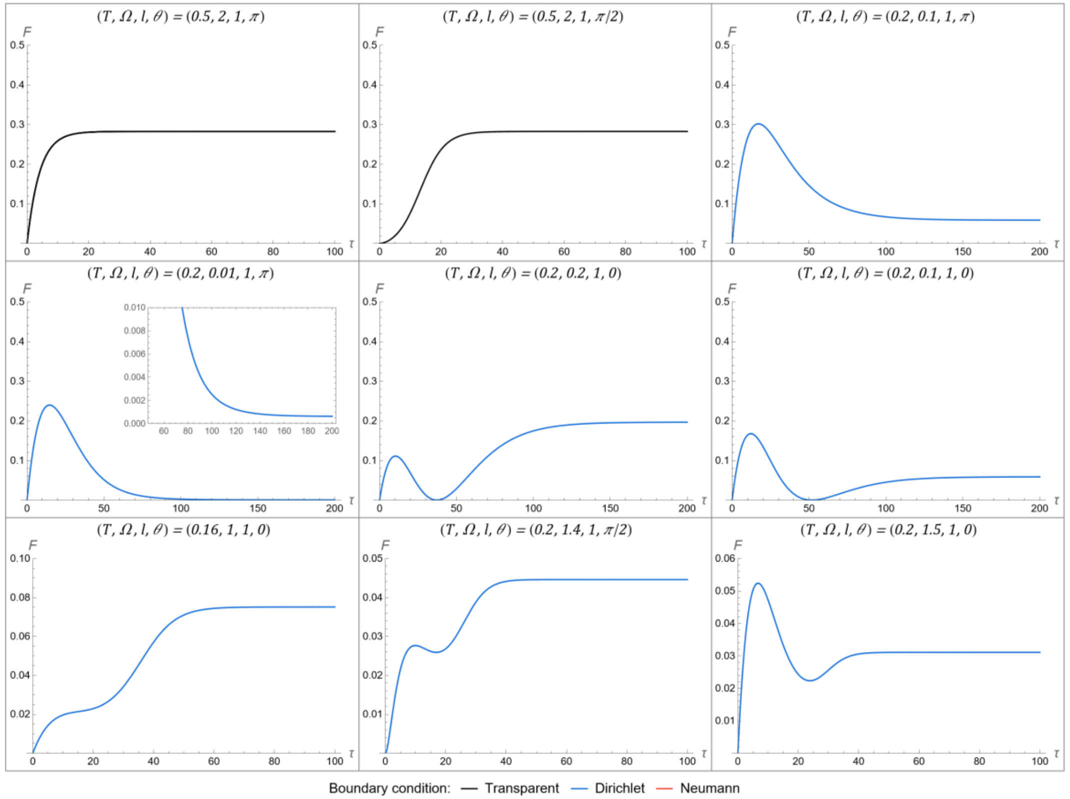


Figure 5: A gallery of different qualitative behaviors of the time evolution of Fisher information

[1] H. Du and R.B. Mann, *Fisher information as a probe of spacetime structure: relativistic quantum metrology in (A)dS*, *JHEP* **112**, 2021.

$$\mathcal{F}^{\mathrm{AdS}}_{\mathrm{asym}} = \frac{\Omega^2}{4T^2}\,\mathrm{sech}^2\left(\frac{\Omega}{2T}\right)$$

3-d AdS

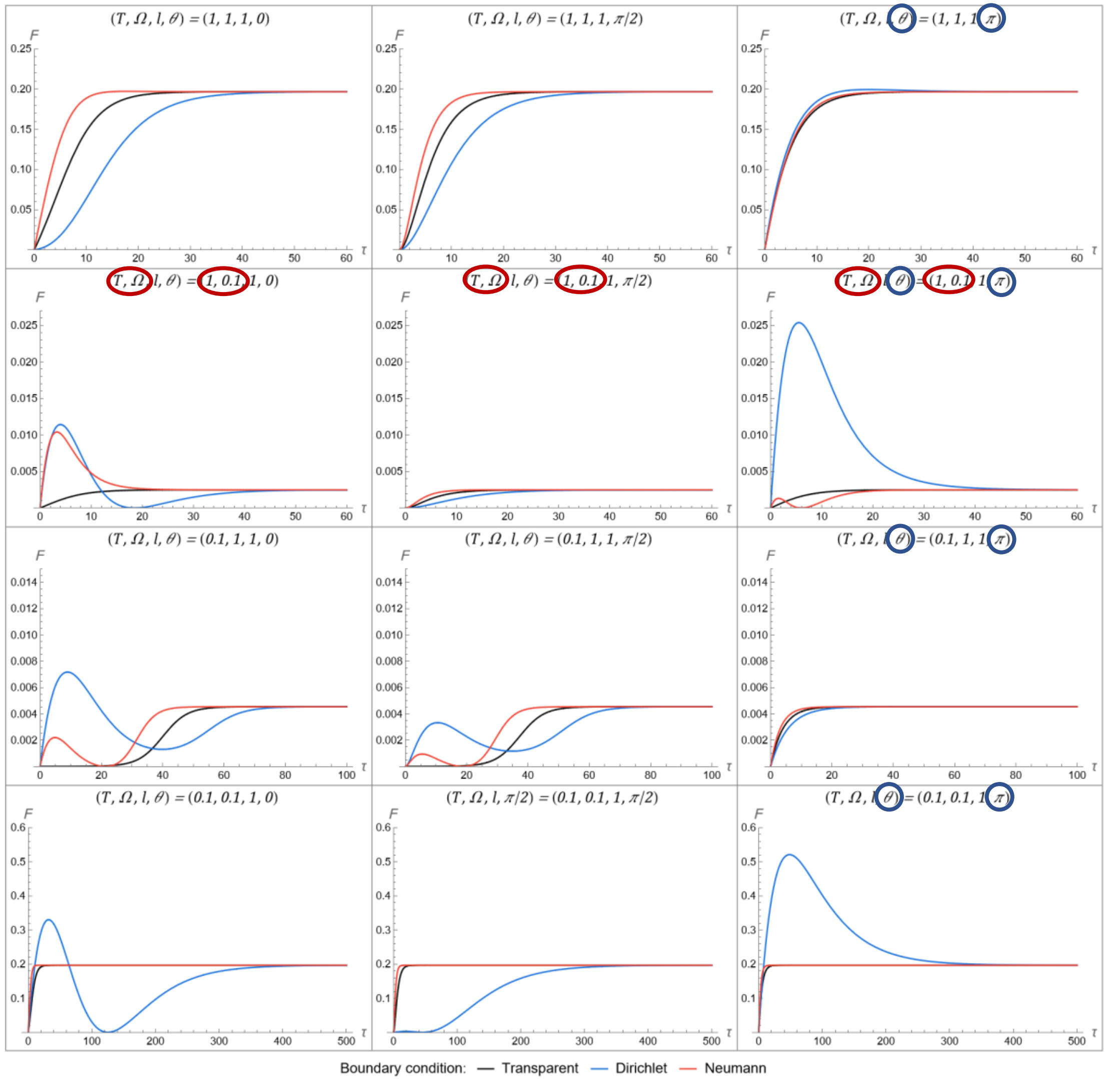


- There are 8 distinct qualitative behaviours
  - Behaviour 4 is the same as behaviour 3
- Curve colour indicates boundary condition
  - Black = Transparent (ζ=0)
  - Blue = Dirichlet (ζ=1)
  - Red = Neumann (ζ=-1)

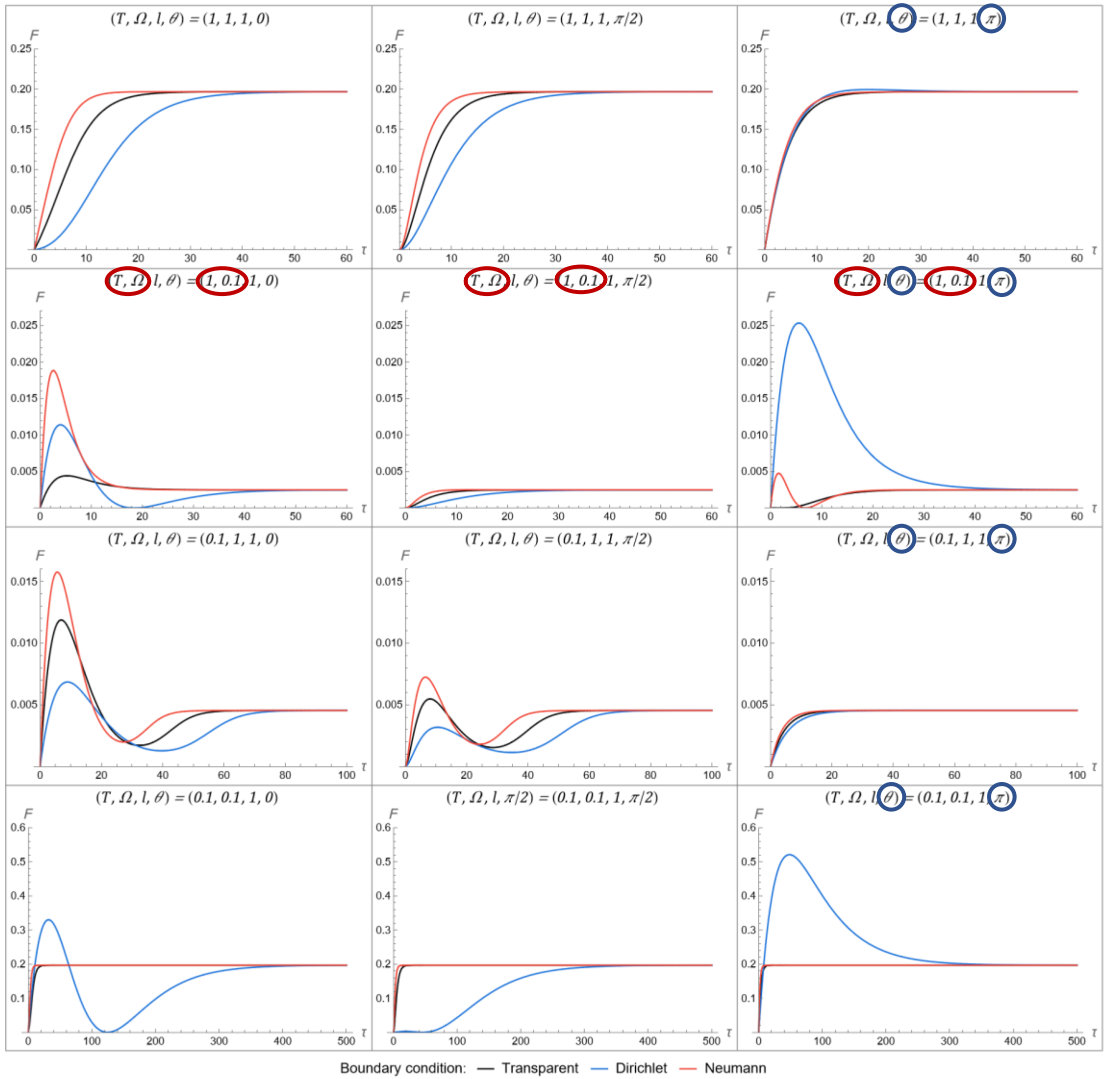
AdS

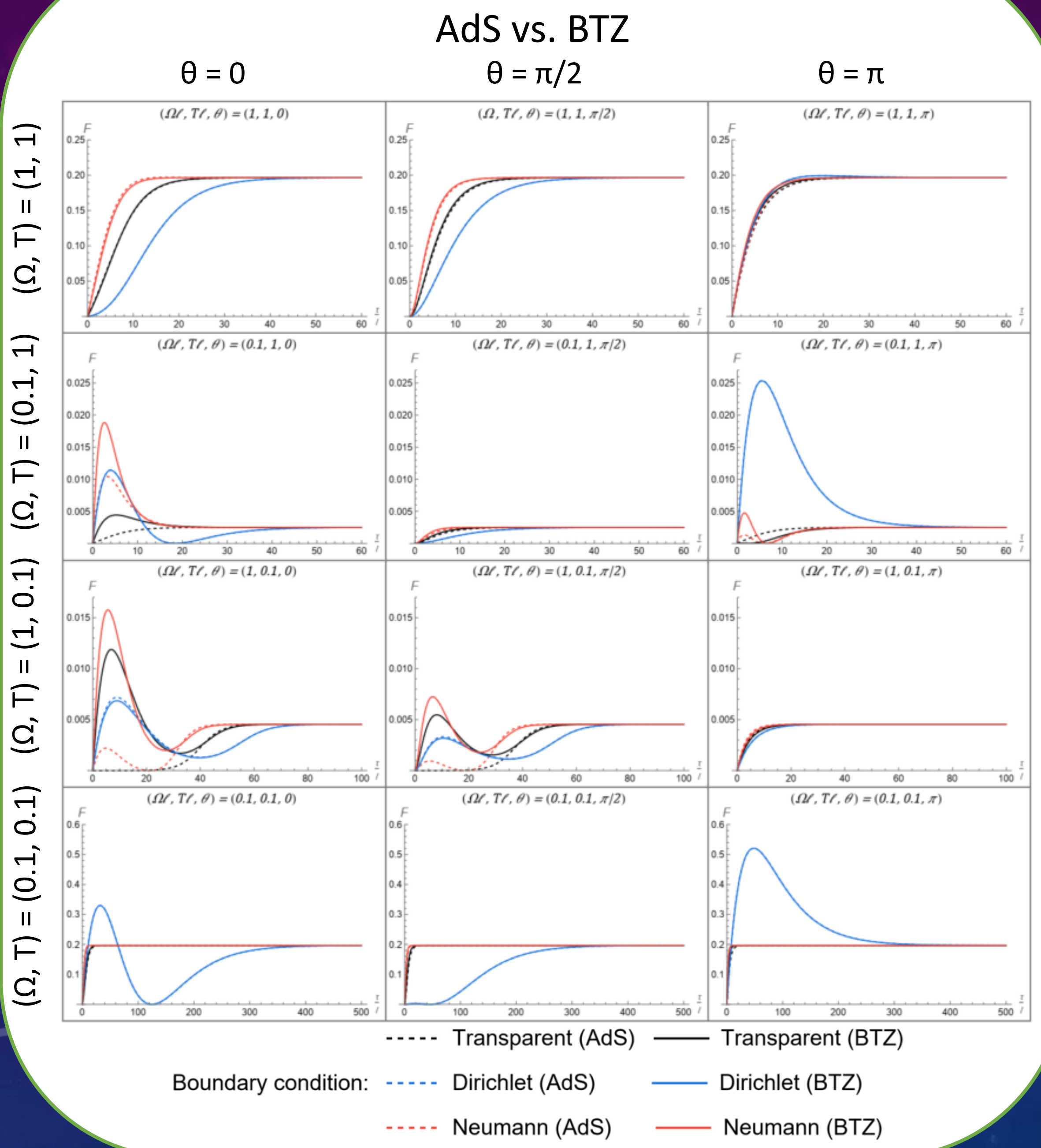
Rows have fixed Ω,T

Columns have fixed θ



BTZ

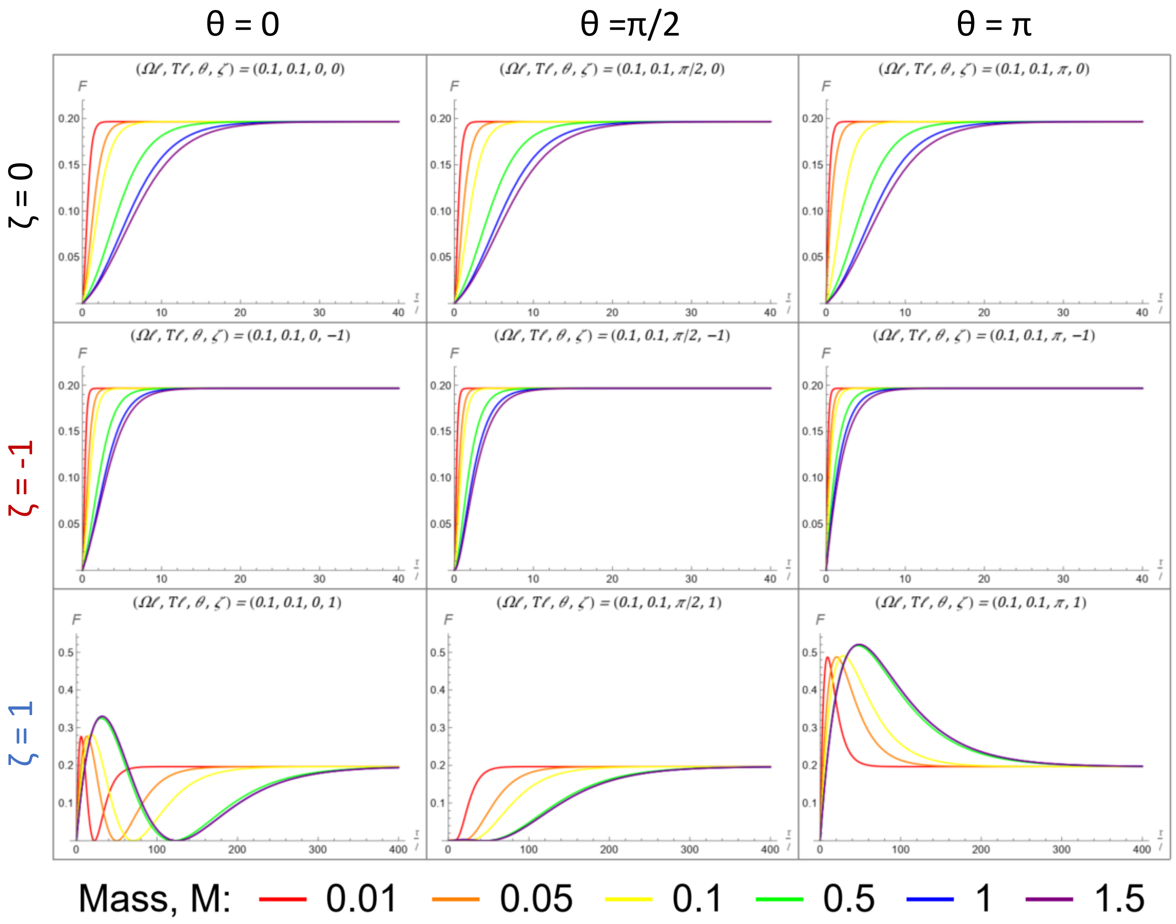




Of note:

- AdS is dashed lines; BTZ is solid lines
- When **Ω=T**, AdS and BTZ coincide
  - As shown in 1st and 4th row
- When Ω ≠ T, discrepancy between AdS and BTZ
  - 'Hot' when Ω=0.1 **<** T=1
    - Dirichlet actually coincides here
  - 'Cold' when Ω=1 **>** T=0.1
    - Qualitatively distinct behaviours present for all boundary conditions
- BTZ mass is fixed at M=1 here
  - What if we vary the mass…

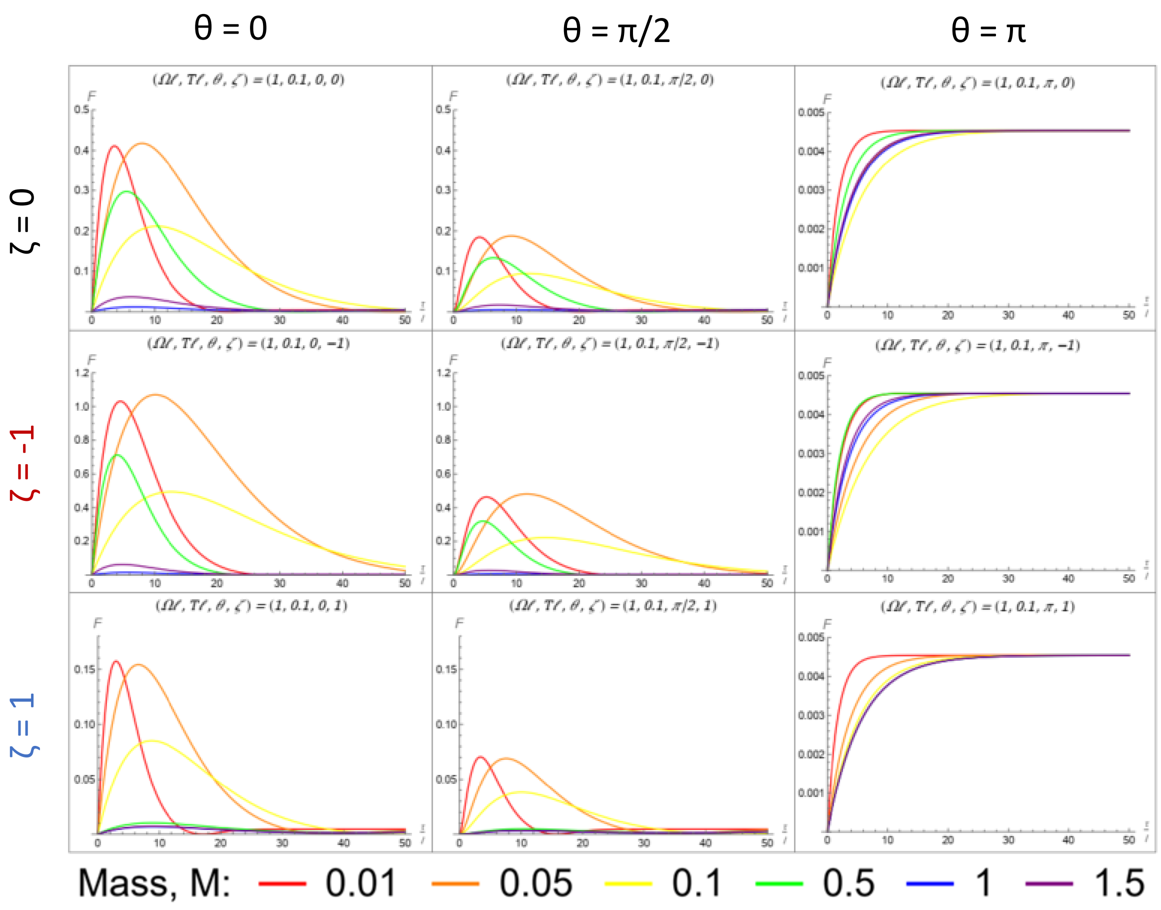
BTZ for Ω=T =0.1



Of note:

- Different masses are represented by different colours
  - Increasing mass from M=0.01 to M=1.5 corresponds to moving along the rainbow spectrum (ROYGBV)
- Here we have Ω=T -> likely more simple
  - No (significant) change in qualitative behaviour
- Increasing M seems to shift curve right
- Slight amplification at times
  - between 0.1 and 0.5 (yellow-> green)

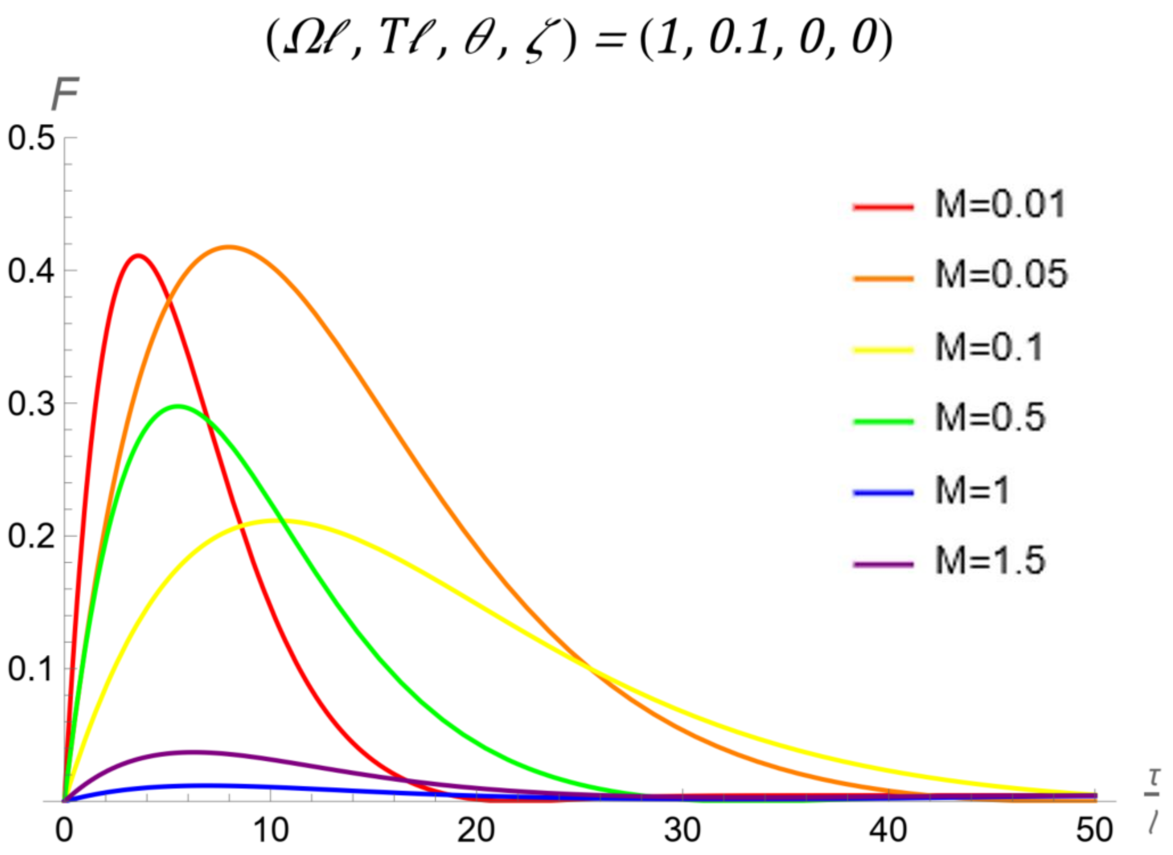
BTZ for Ω > T ‘Cold’



Of note:

- Different masses are represented by different colours
  - Increasing mass from M=0.01 to M=1.5 corresponds to moving along the rainbow spectrum (ROYGBV)
- No clear trend in Fisher information alongside the changing mass
- Most drastic behaviour appears again between 0.1 and 0.5 (yellow-> green)

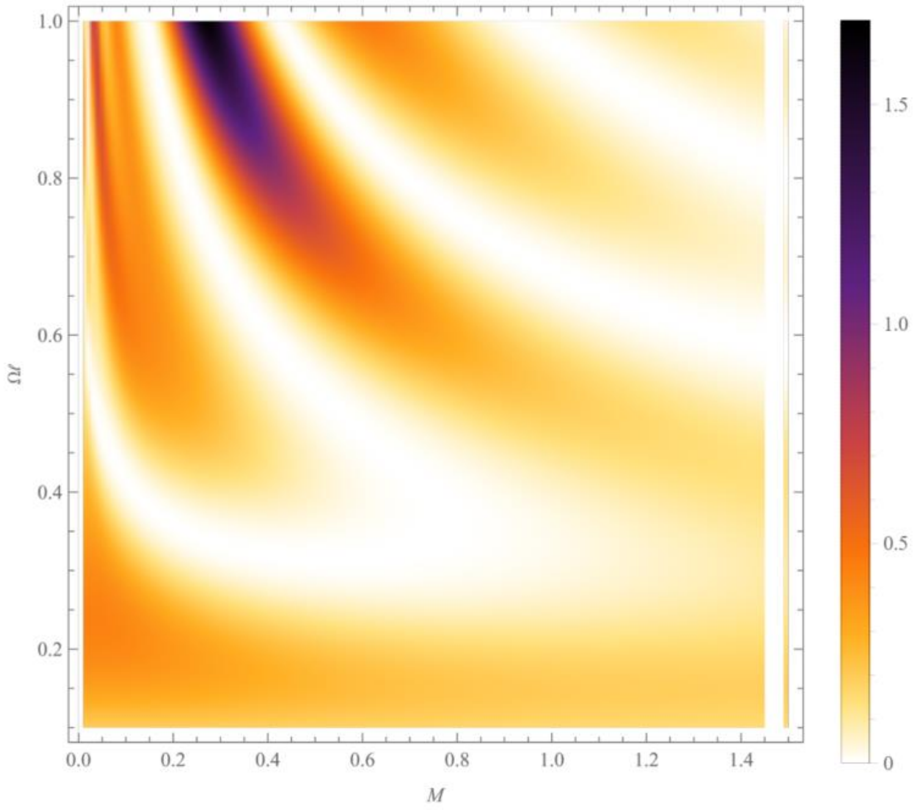
BTZ for Ω > T ‘Cold’

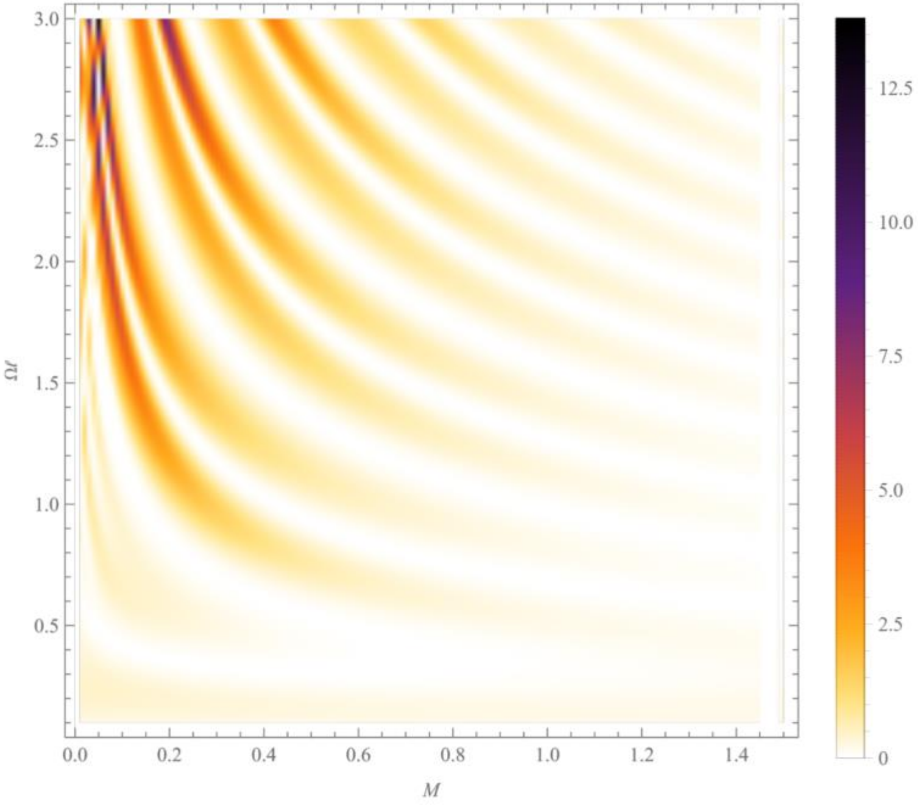


Of note:

- Different masses are represented by different colours
  - Increasing mass from M=0.01 to M=1.5 corresponds to moving along the rainbow spectrum (ROYGBV)
- No clear trend in Fisher information alongside the changing mass
- Most drastic behaviour appears again between 0.1 and 0.5 (yellow-> green)
- ZOOMED

Fixed time, varying mass - Density Plots





- Oscillatory behaviour can be seen in the ‘wave’ pattern
- For larger Ω, we see even greater amplitudes and more oscillations
- Note the change in scale

BONUS CONTENT

Rotational Effects on Fisher Information of Thermal Black Hole Parameter

Everett Patterson
CAP Congress 2023, UNB
June 20, 2023

BONUS CONTENT – WHAT IS THE DEFN OF QFI

- In essence QFI is the maximal FI obtained over all possible measurements

before. For a given measurement scheme on the quantum system within state ρ, FI relates with a measurement outcome ξ of a positive operator valued measurement (POVM) $\{\hat{E}(\xi)\}$, and takes the form of

$$\mathcal{F}_C(\beta) = \sum_{\xi} p(\xi|\beta)\left(\frac{\partial \ln p(\xi|\beta)}{\partial \beta}\right)^2 \tag{10}$$

where $p(\xi|\beta)$ is the conditional probability of obtaining ξ w.r.t. a chosen POVM and given initial state (7). From (8), we observe that the initial states of the detector characterized by θ and evolving time τ would play an important role in the metrology process, and eventually determine the ultimate bound on precision. Optimizing (10) over all the possible quantum measurements of the state (7), we define the QFI of estimation as $\mathcal{F}_Q(\beta) \equiv \mathrm{Max}_{\{\hat{E}(\xi)\}}\mathcal{F}_C(\beta)$, saturated by an *optimal* POVM and can be calculated in terms of the symmetric logarithmic derivative (SLD) operator as $\mathcal{F}_Q(\beta) = \mathrm{Tr}[\rho(\beta)L_\beta^2]$, where SLD L_β satisfies $\partial_\beta\rho = \frac{1}{2}\{\rho, L_\beta\}$. In particular, for a density matrix admitting decomposition (9), QFI can be further explicitly expressed as [50, 51]

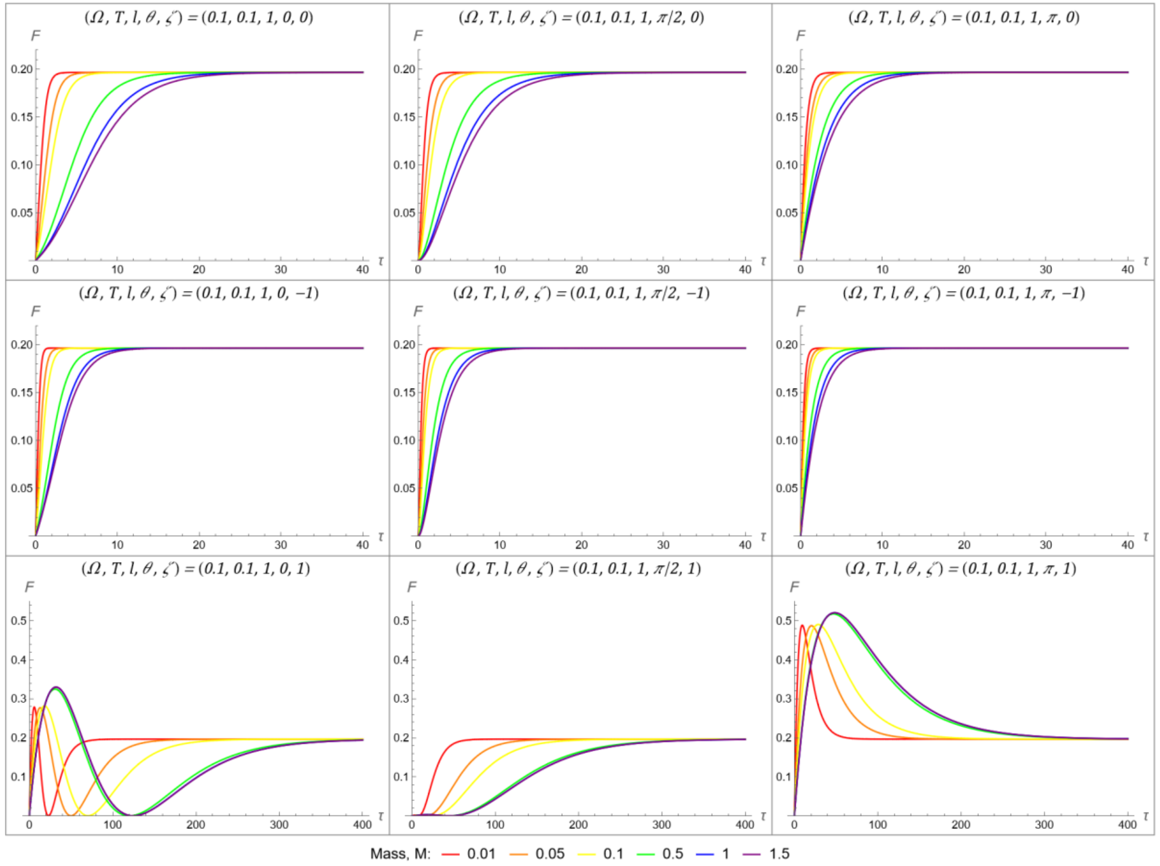
$$\mathcal{F}_Q(\beta) = \sum_{i=\pm}\frac{(\partial_\beta\lambda_i)^2}{\lambda_i} + \sum_{i\neq j=\pm}\frac{2(\lambda_i-\lambda_j)^2}{\lambda_i+\lambda_j}|\langle\psi_i|\partial_\beta\psi_j\rangle|^2 \tag{11}$$

where the summations involve sums over all $\lambda_i \neq 0$ and $\lambda_i + \lambda_j \neq 0$, respectively.

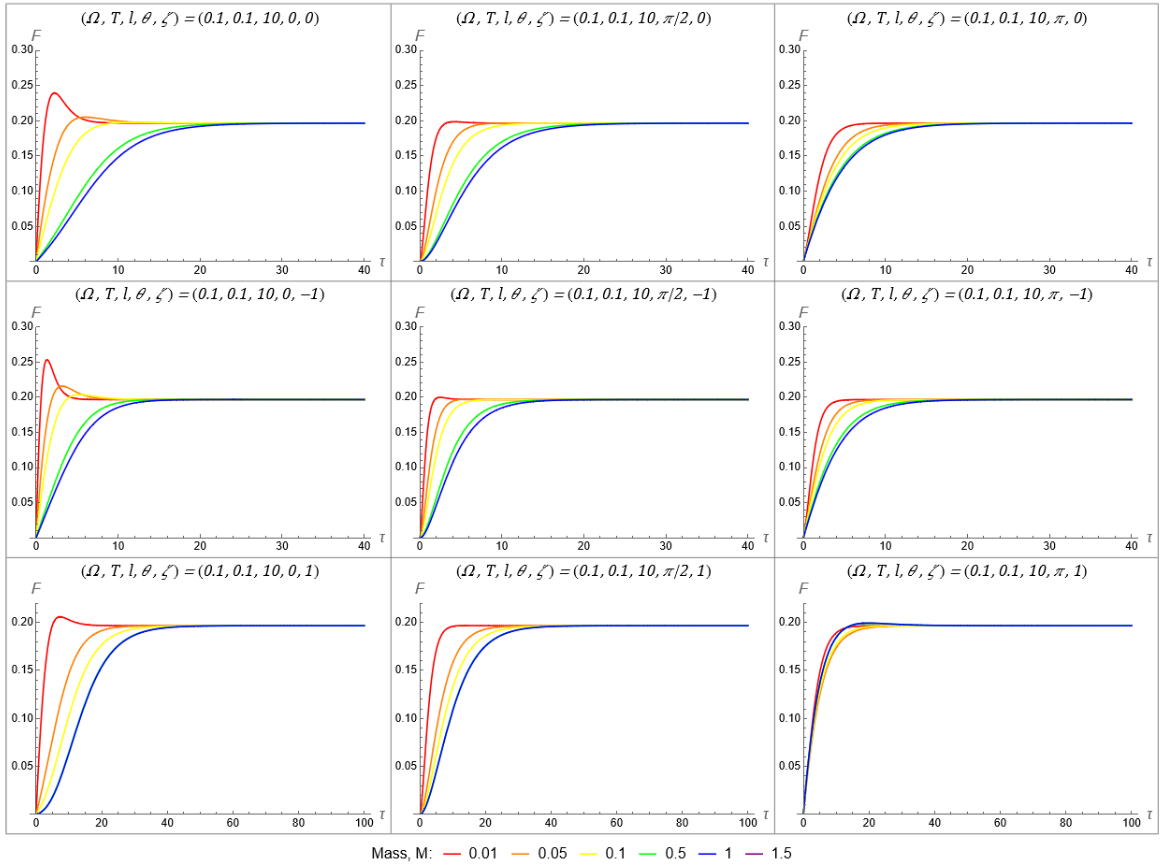
BONUS CONTENT – BOUNDARY CONDITIONS

- Dirichlet
  - The field vanishes at the boundary
- Neumann
  - The field’s normal derivative vanishes at the boundary
- Transparent
  - This is described in detail in a 1978 PRD paper by Avis, Isham, and Storey.
  - It is a little peculiar in its definition, but it enables conservation of energy, angular momentum, etc.

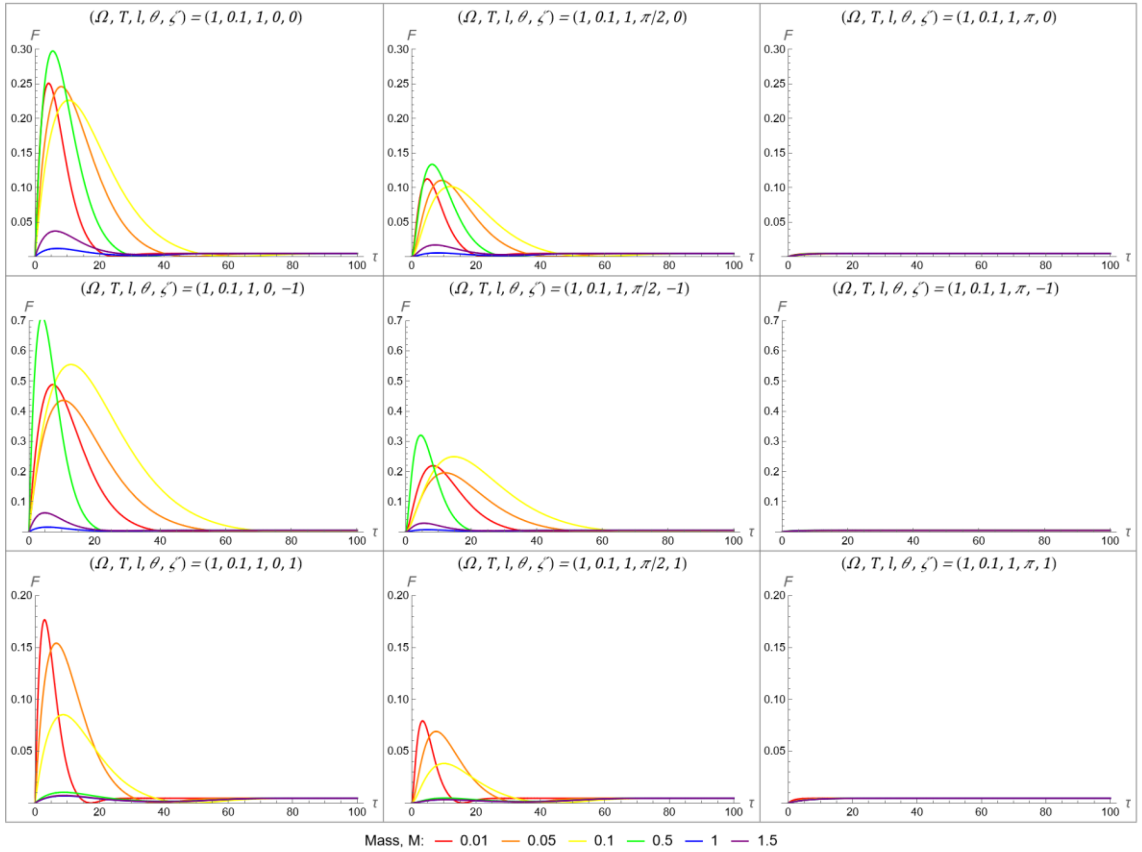
BTZ; ell=1 , Omega=T=0.1



BTZ; ell=10 , Omega=T=0.1



BTZ; ‘cold’, varying M



BTZ; ‘hot’, varying M