A Novel Approach to Account for the Fano Factor

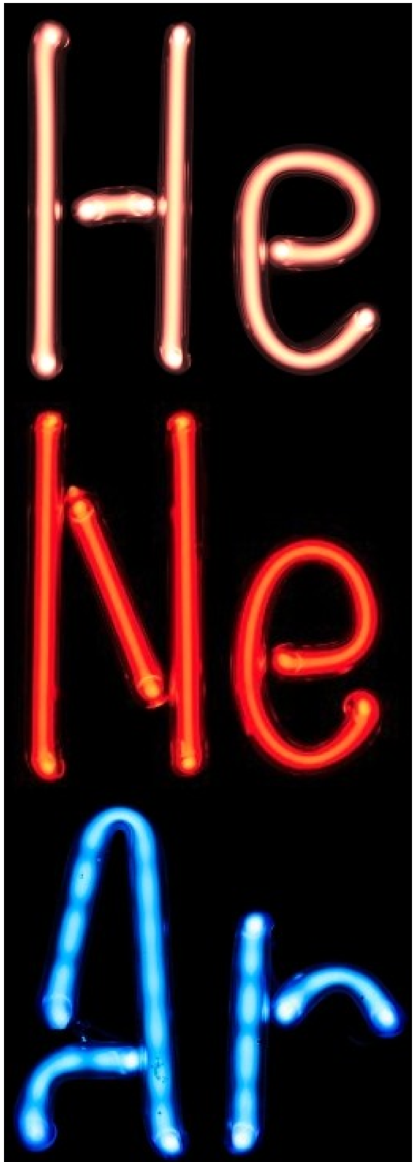
Daniel Durnford

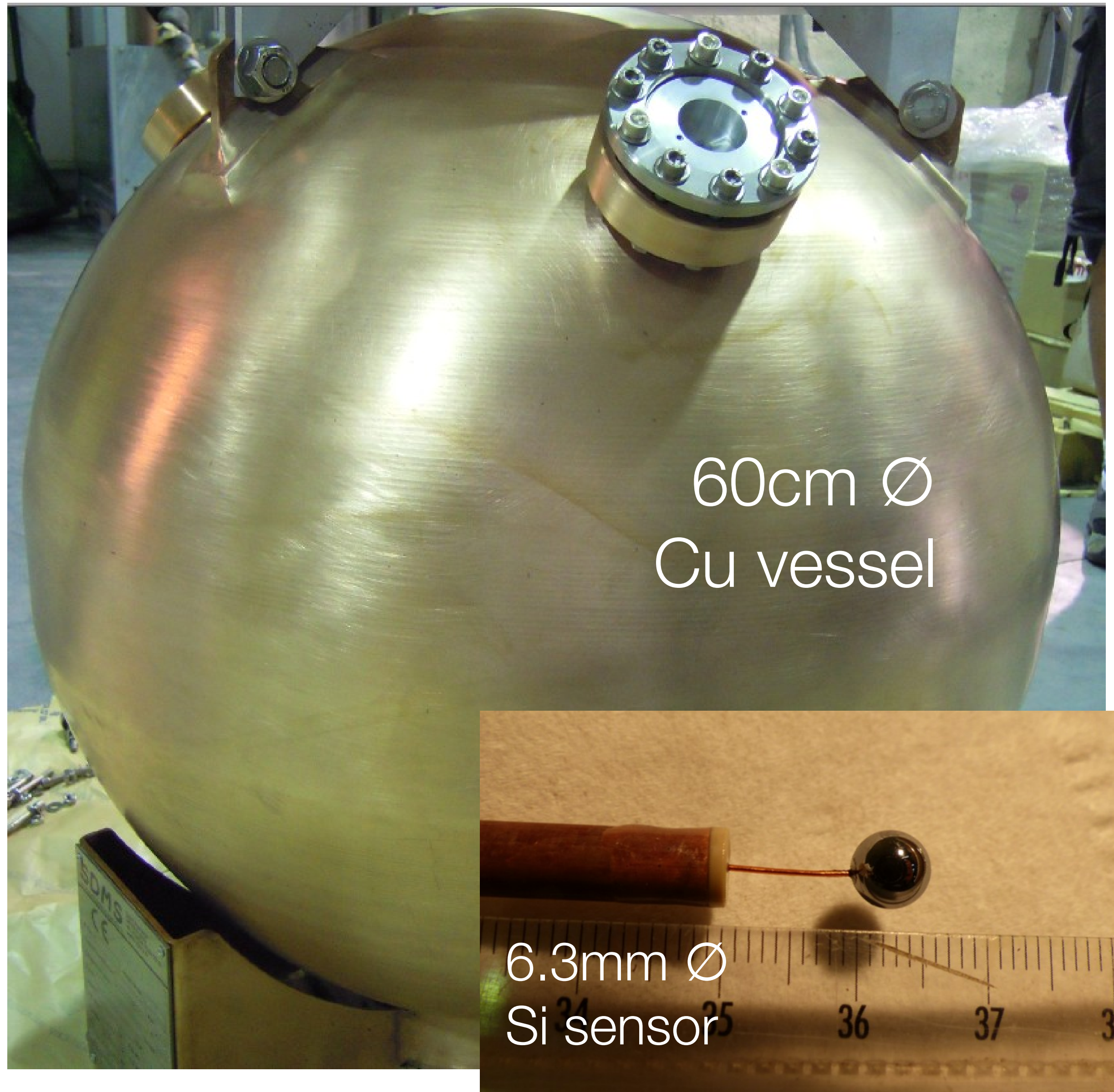
CAP Congress 2018

June 12th 2018, Halifax

SPCs to search for low mass dark matter

NEWS-G





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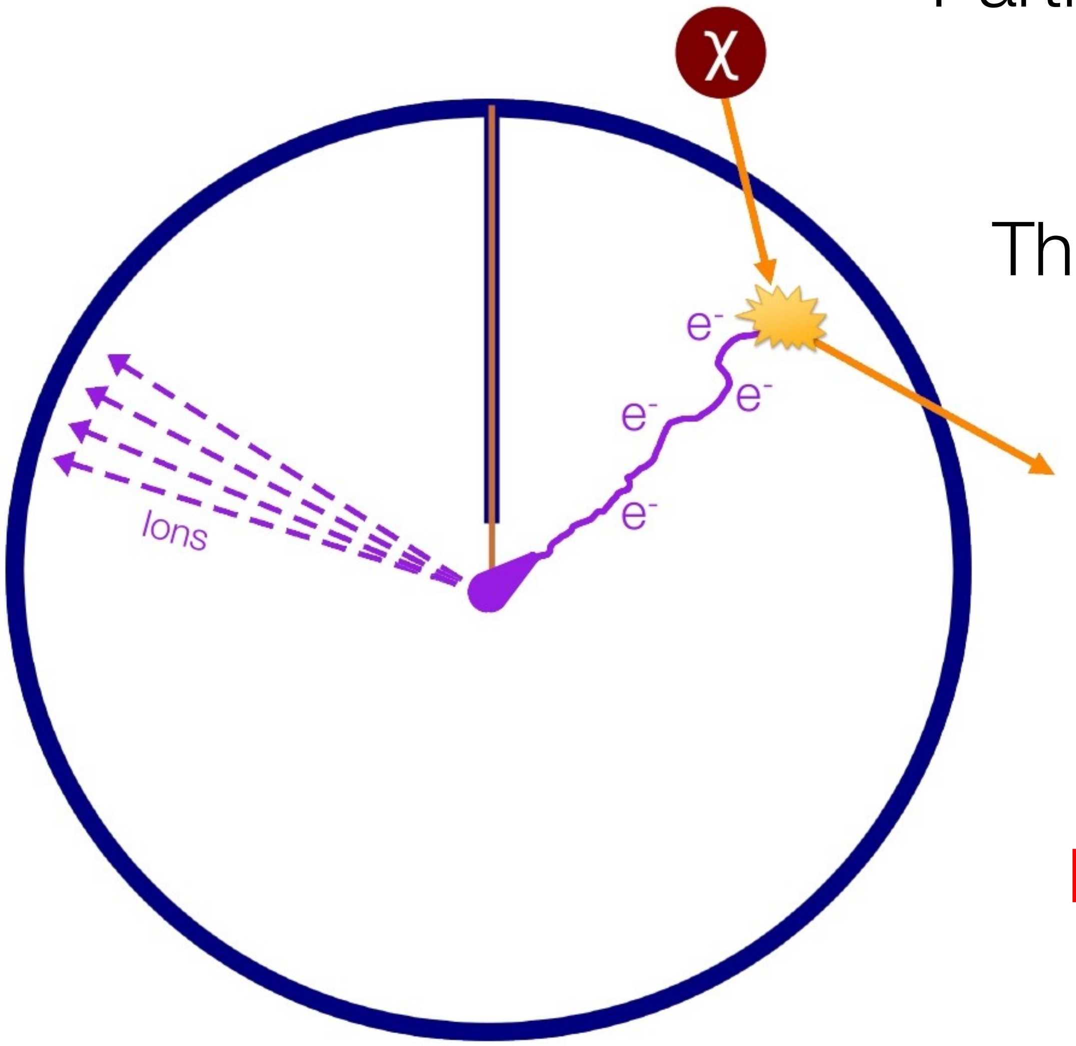


Particle interactions ionize gas molecules

These primary electrons induce a charge avalanche at the anode/sensor

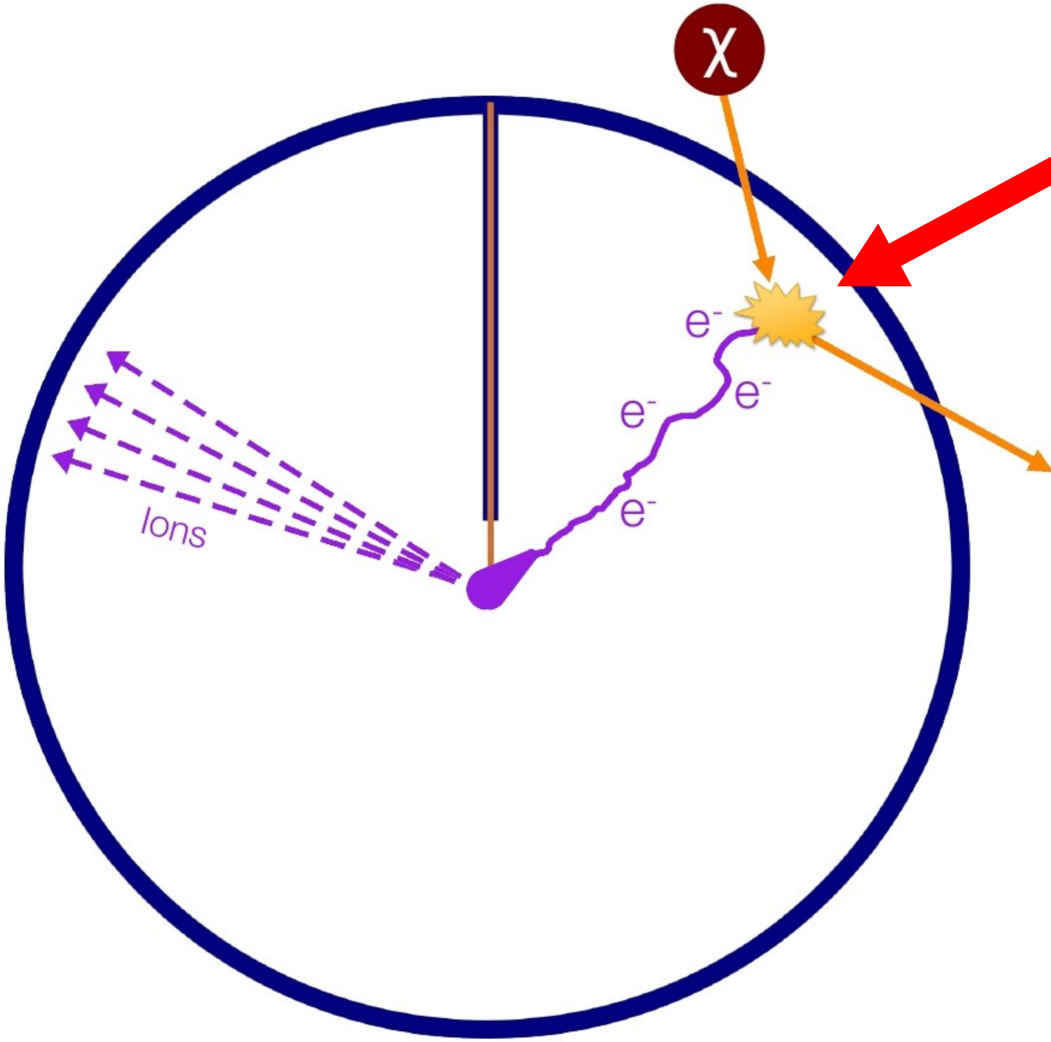
Large gain →
Low energy threshold

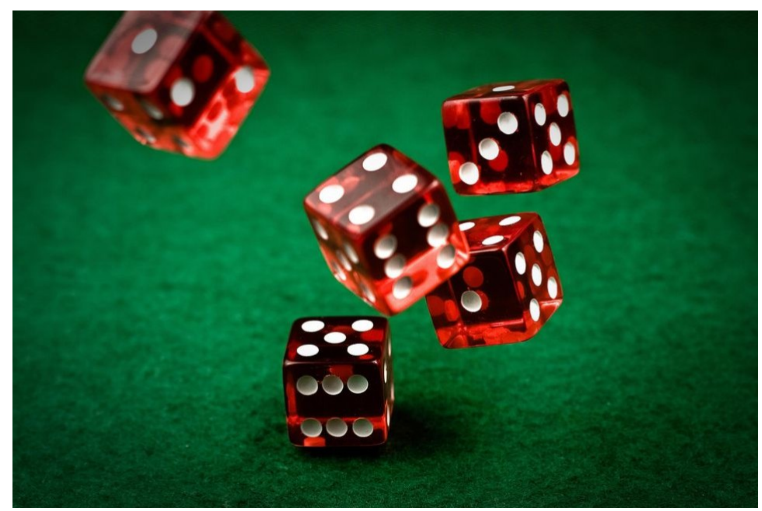
Gaseous Detectors

Primary ionization is a stochastic process

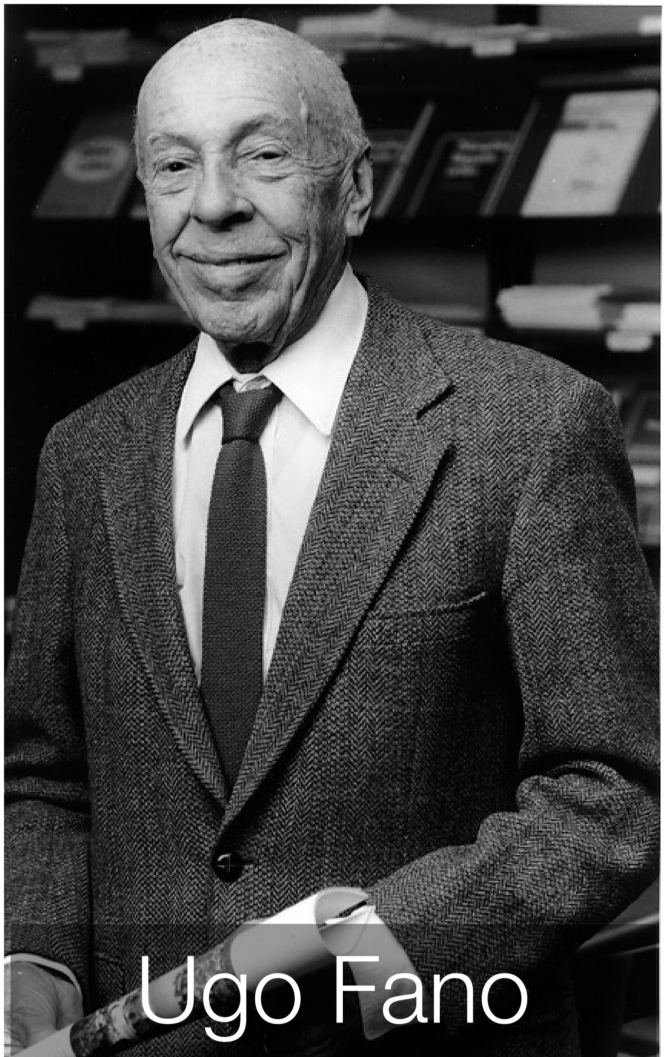
$$\langle \#PE \rangle = \frac{E}{W(E)}$$

(For neon: W_γ = 36eV/pair)





Primary ionization is a stochastic process

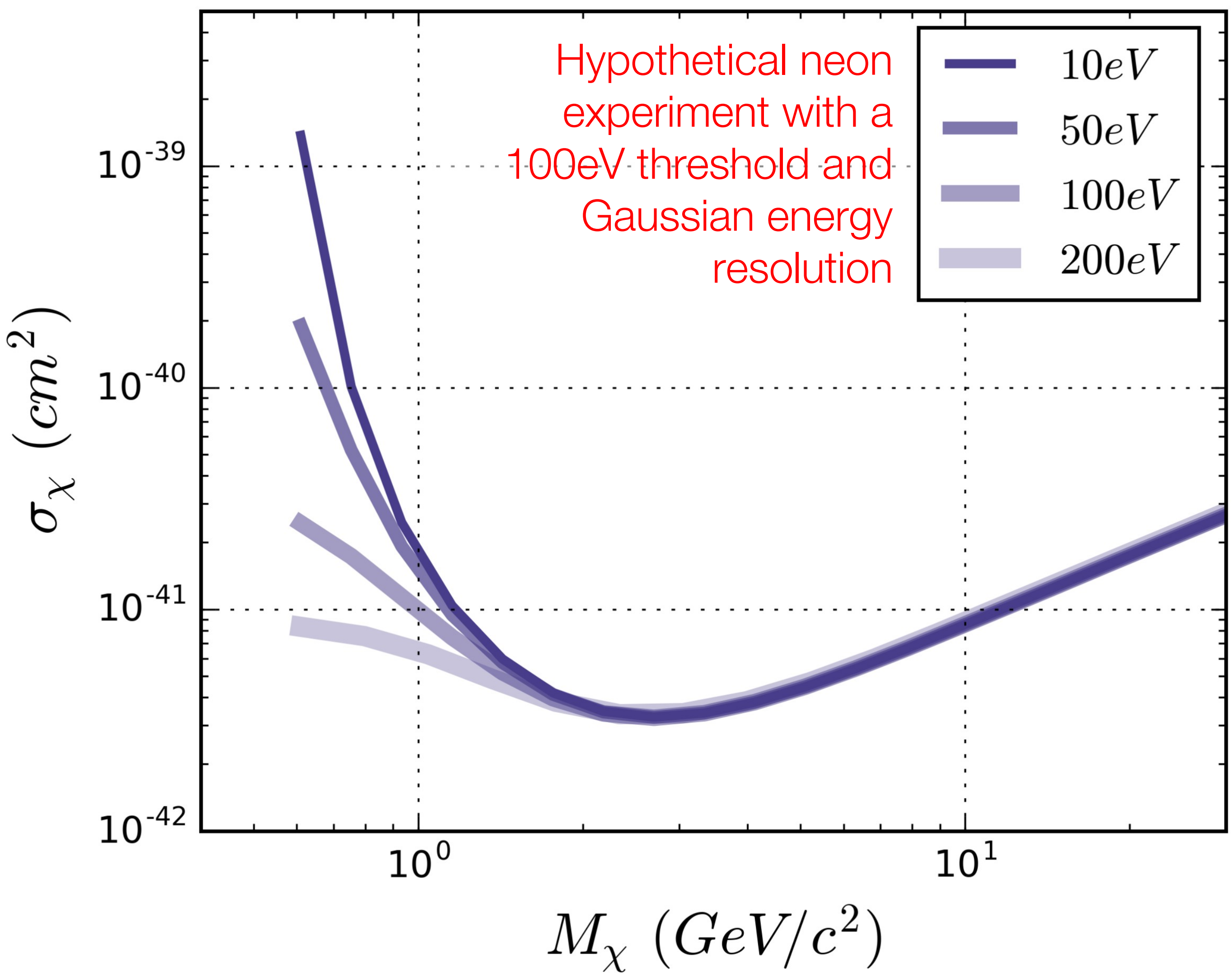


The dispersion of this process is described by the Fano Factor:

$$F = \frac{\sigma^2}{\mu}$$

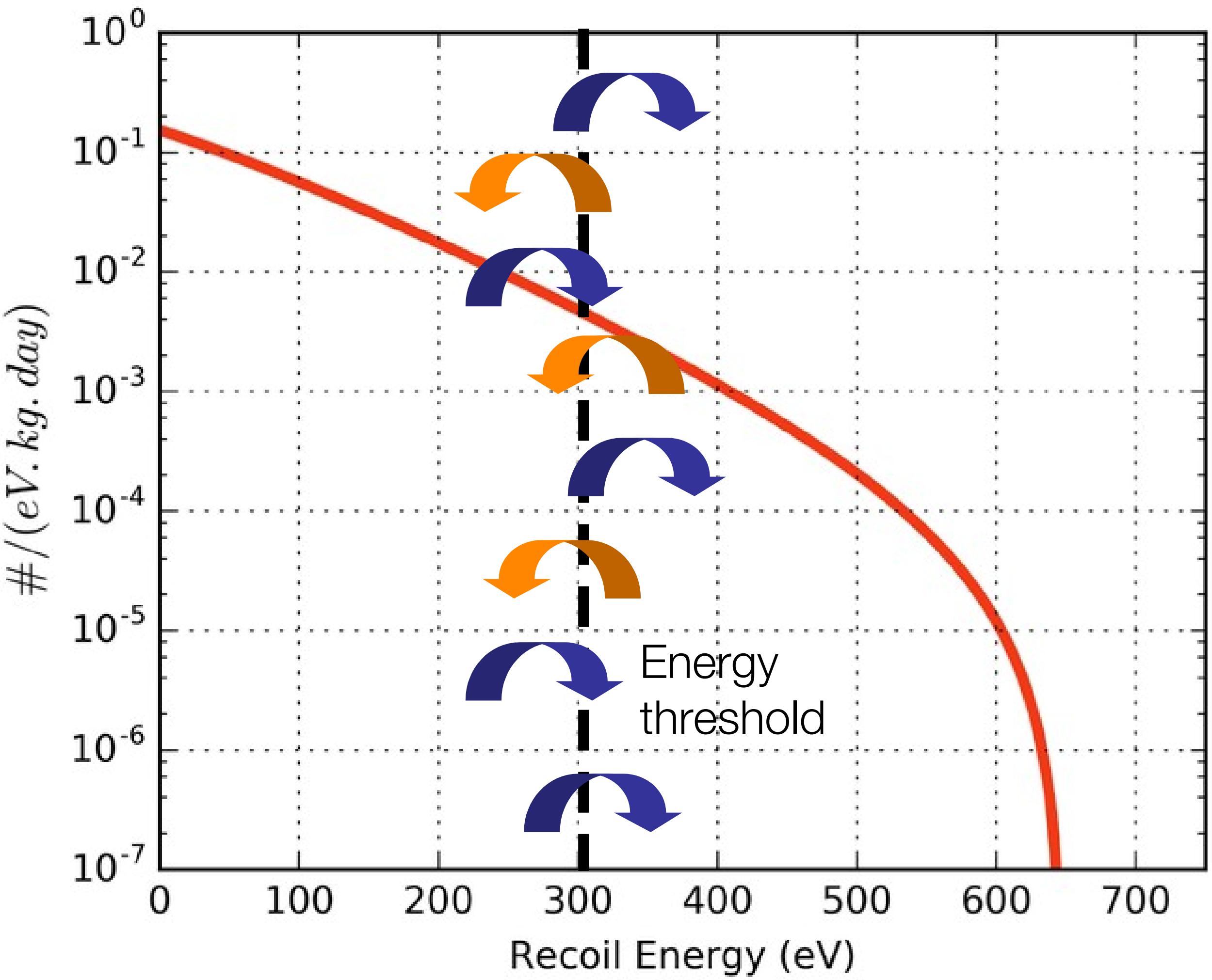
For noble gases, F ~ 0.2

Why should we care?



Energy resolution can have a significant effect on low mass dark matter sensitivity!

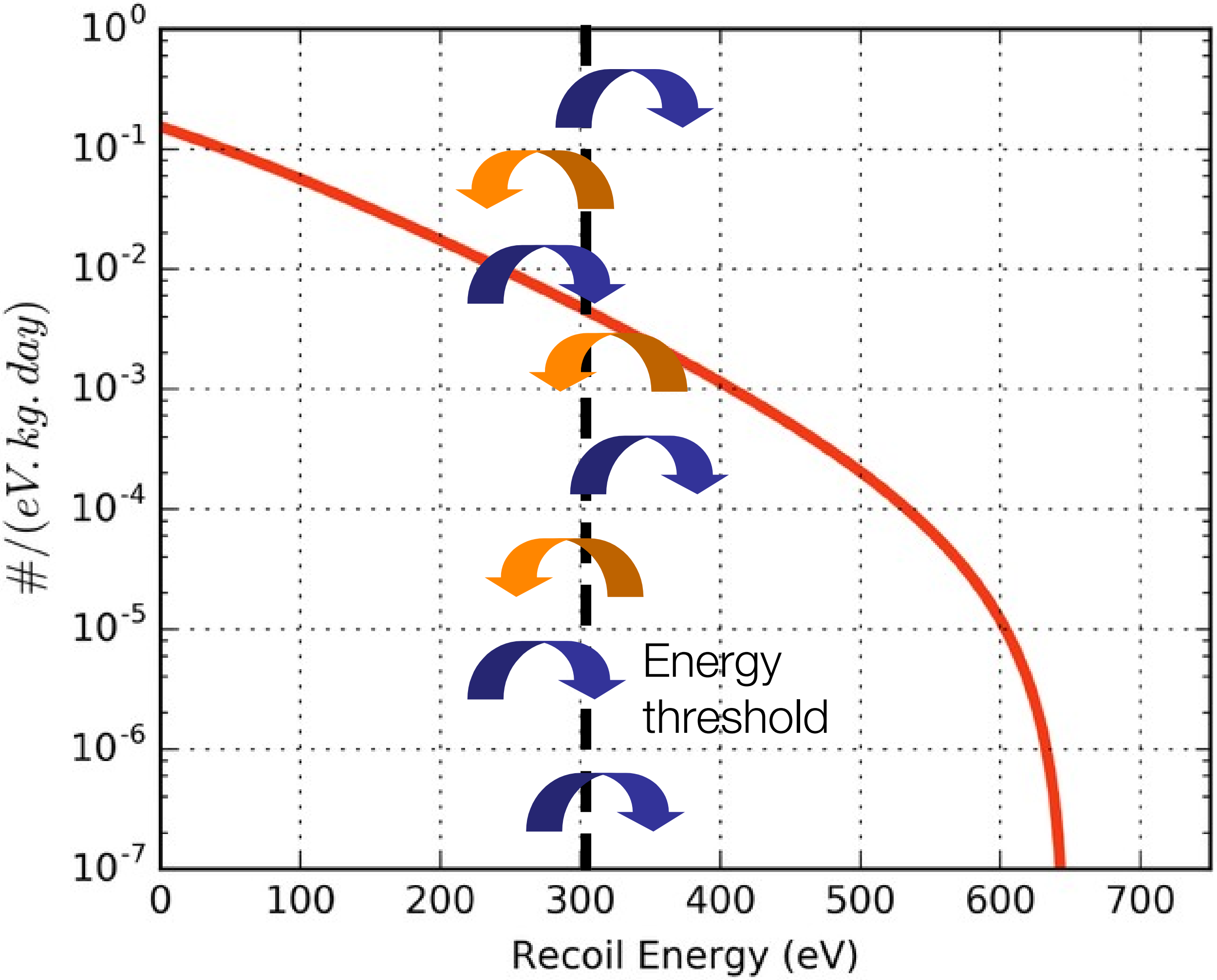
Why should we care?



Because the WIMP recoil spectrum is asymmetric, sometimes having a poor energy resolution can improve sensitivity to WIMPs!

Example: Recoil energy spectrum of a 1 GeV WIMP in Neon

Why should we care?



How do we model this at the level of primary ionization?

Example: Recoil energy spectrum of a 1 GeV WIMP in Neon

The Fano Factor

To account for the Fano Factor in simulations, we need a probability distribution $P(x|\mu,F)$ that:

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Mean and variance controlled separately

Not Poisson, Binomial

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Mean and variance controlled separately

Not Poisson, Binomial

Works for $F < 1$ (down to $F = 0.1$)

Not Generalized Poisson, Negative Binomial, Double Binomial

We found the COnway Maxwell - Poisson (COM-Poisson) distribution!

$$P(x|\lambda, \nu) = \frac{\lambda^x}{(x!)^\nu Z(\lambda, \nu)}$$

$$Z(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu} \qquad \lambda \in \{\mathbb{R} > 0\}, \quad \nu \in \{\mathbb{R} \geq 0\}$$

We found the COnway Maxwell - Poisson (COM-Poisson) distribution!

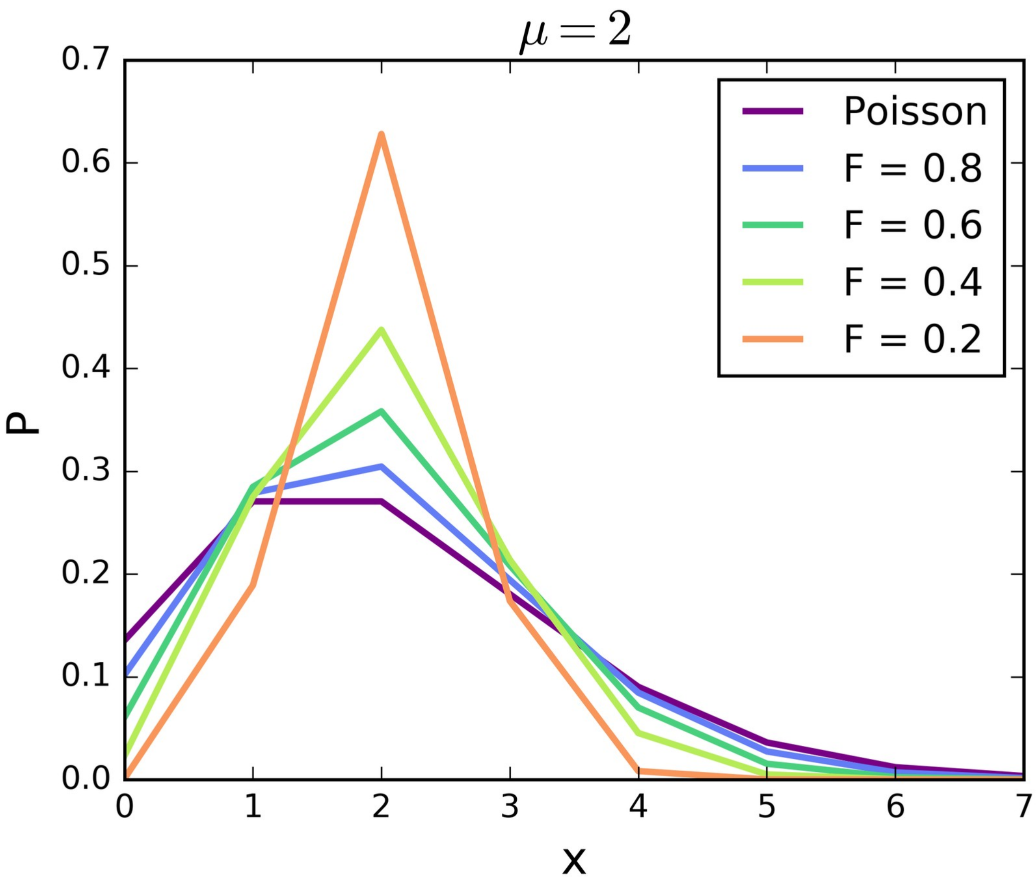
The mean and variance are:

$$\mu\left(\lambda,\nu\right)=\sum_{j=0}^{\infty}\frac{j\lambda^{j}}{\left(j!\right)^{\nu}Z\left(\lambda,\nu\right)}$$

$$\sigma^{2}\left(\lambda,\nu\right)=\sum_{j=0}^{\infty}\frac{j^{2}\lambda^{j}}{\left(j!\right)^{\nu}Z\left(\lambda,\nu\right)}-\mu\left(\lambda,\nu\right)^{2}$$

Used for linguistics, economics, marketing...

It can model under-dispersion (F < 1) when **ν** > 1



F	λ	ν
1.0	2.00	1.000
0.8	2.74	1.327
0.6	4.71	1.904
0.4	11.8	2.912
0.2	233	6.131

$$P(x|\lambda,\nu) = \frac{\lambda^x}{(x!)^\nu Z(\lambda,\nu)}$$

$$Z(\lambda,\nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu} \quad \lambda \in \{\mathbb{R} > 0\}, \quad \nu \in \{\mathbb{R} \geq 0\}$$

The problem...

We have:

$$P(x|\lambda, \nu)$$

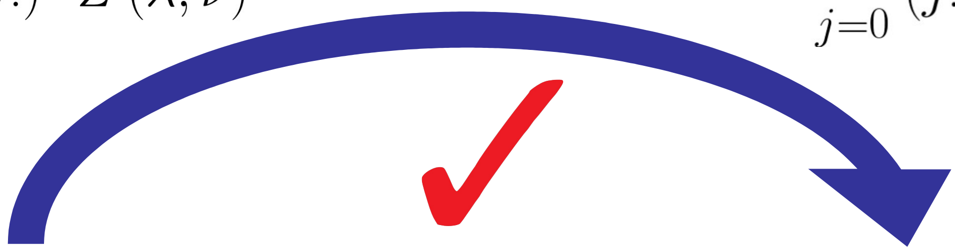
We want:

$$P(x|\mu, F)$$

The problem...

$$\mu(\lambda,\nu) = \sum_{j=0}^{\infty} \frac{j\lambda^j}{(j!)^\nu Z(\lambda,\nu)}$$

$$\sigma^2(\lambda,\nu) = \sum_{j=0}^{\infty} \frac{j^2\lambda^j}{(j!)^\nu Z(\lambda,\nu)} - \mu(\lambda,\nu)^2$$



$$P(x|\lambda,\nu) \qquad\qquad P(x|\mu,F)$$

The problem...

$$\mu(\lambda,\nu) = \sum_{j=0}^{\infty} \frac{j\lambda^j}{(j!)^\nu Z(\lambda,\nu)}$$

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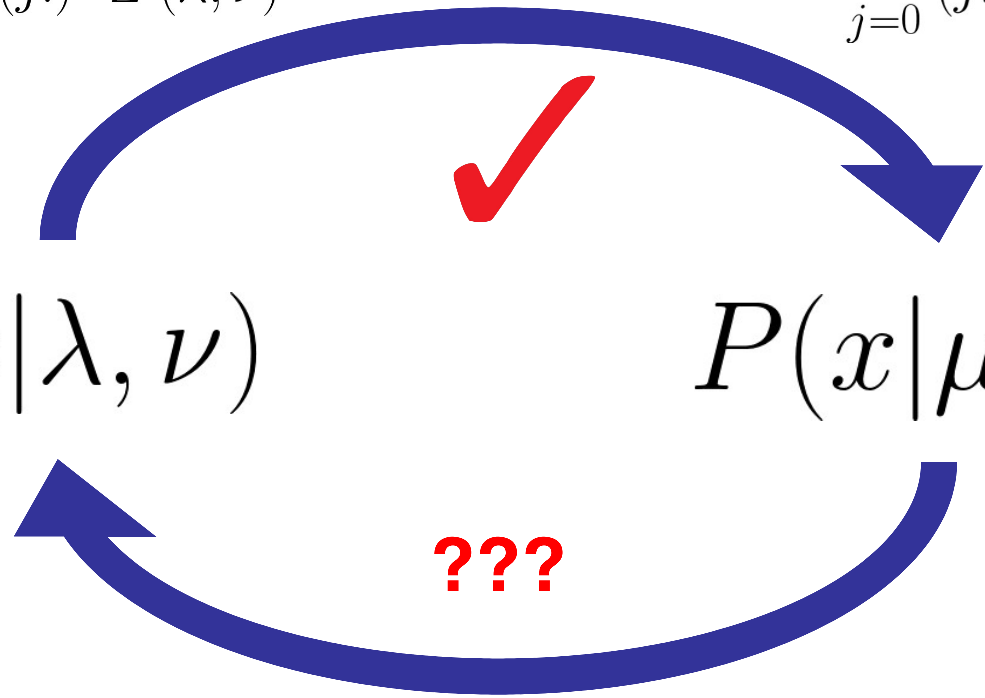
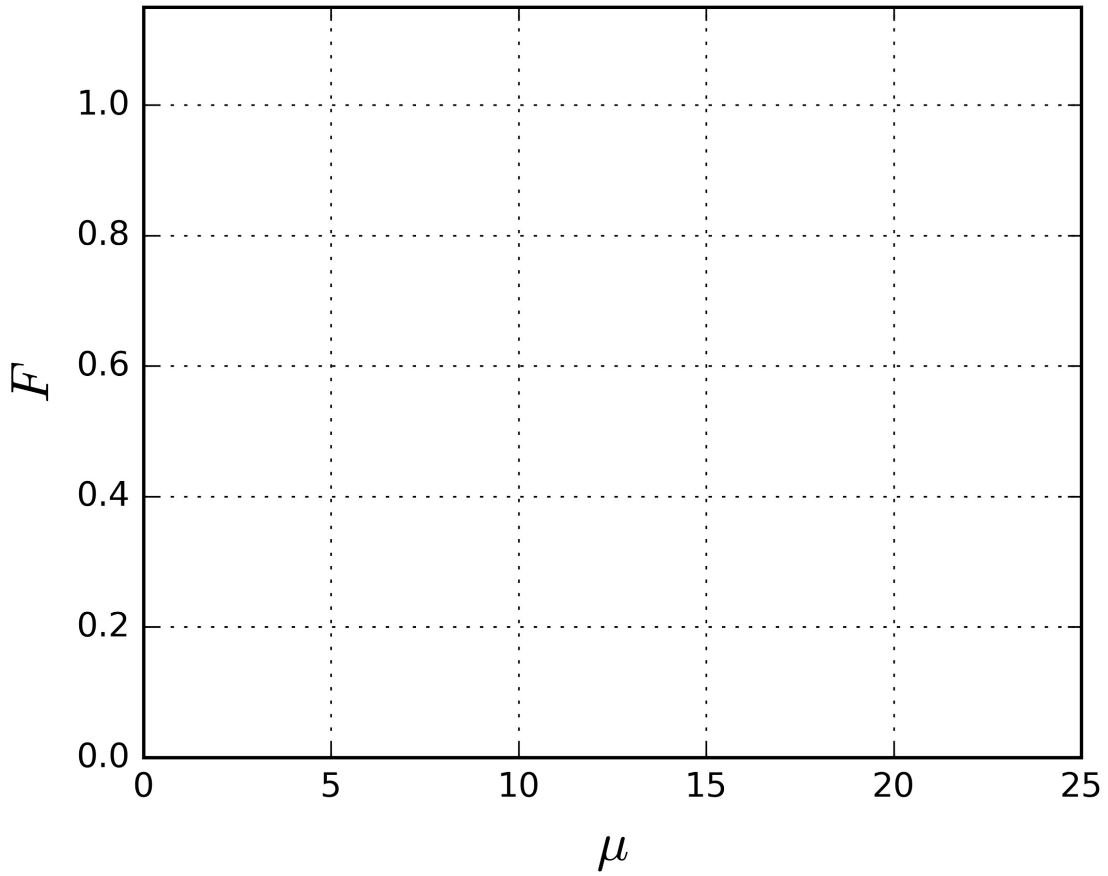


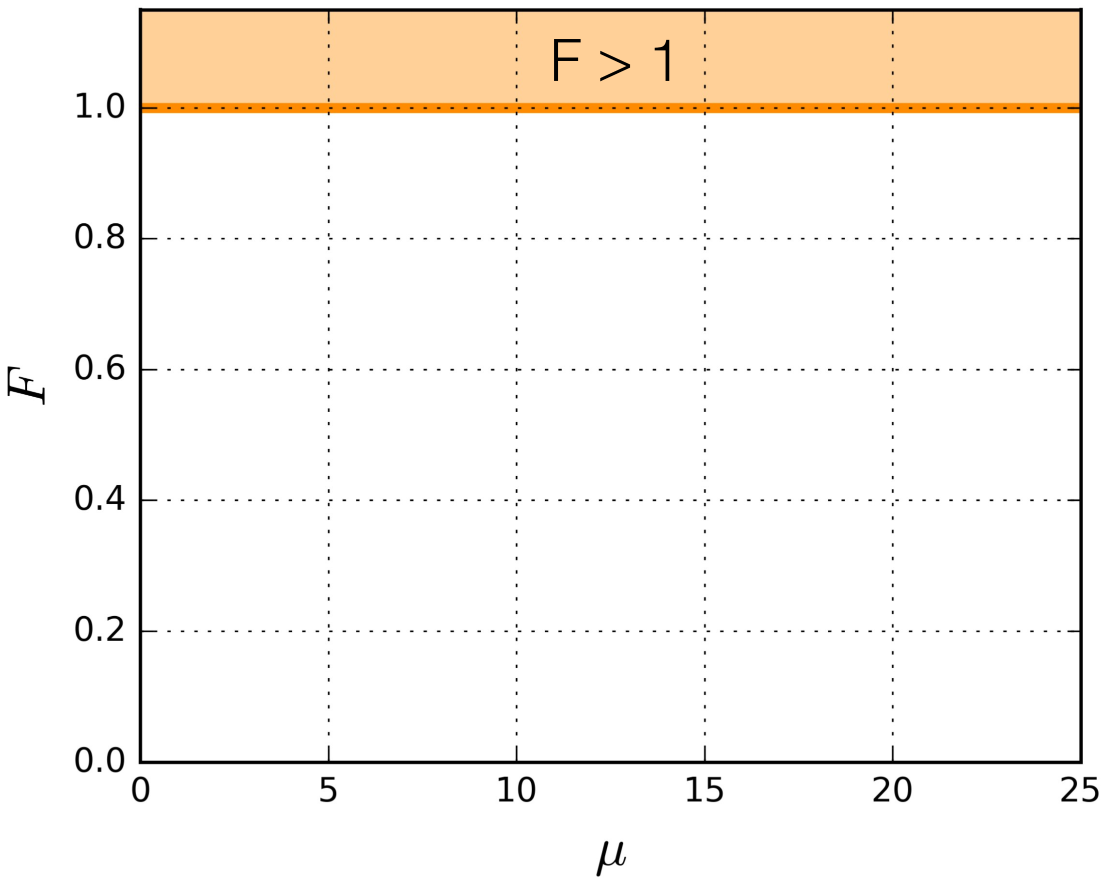
$$P(x|\lambda,\nu) \qquad P(x|\mu,F)$$

$$??? \quad \lambda(\mu,F) \quad \nu(\mu,F) \quad ???$$

Using COM-Poisson

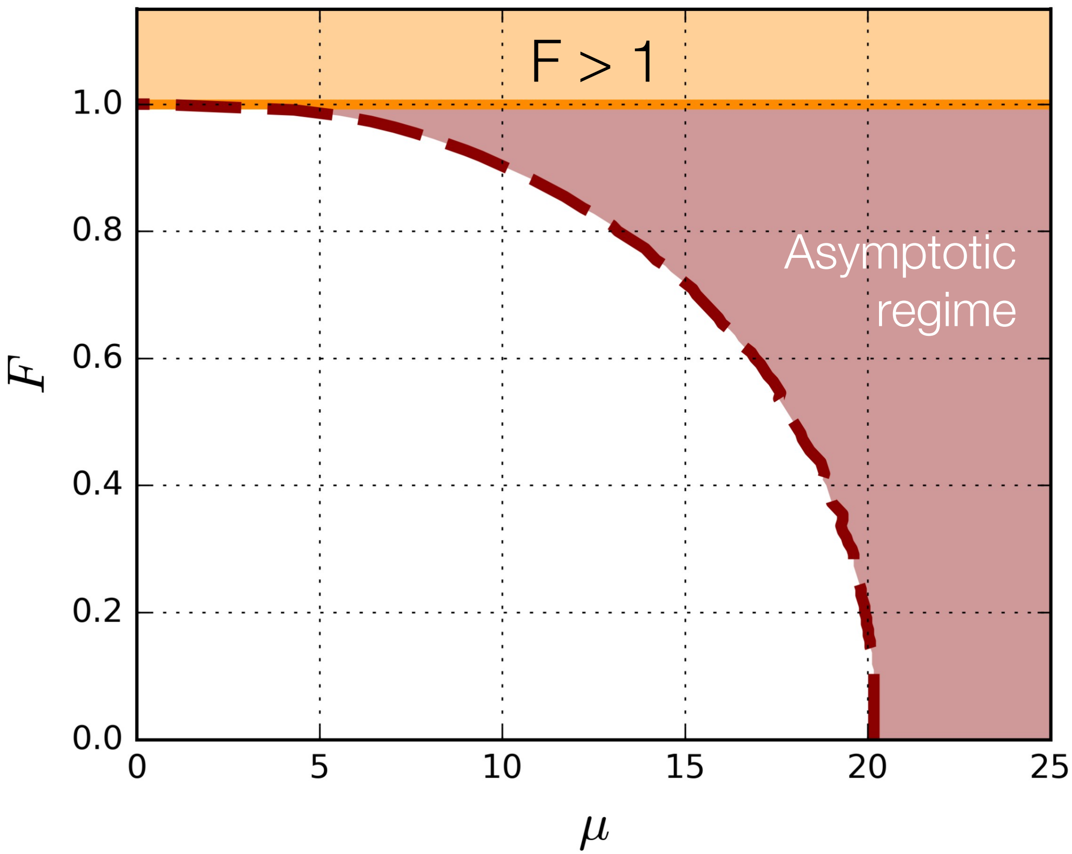


Using COM-Poisson



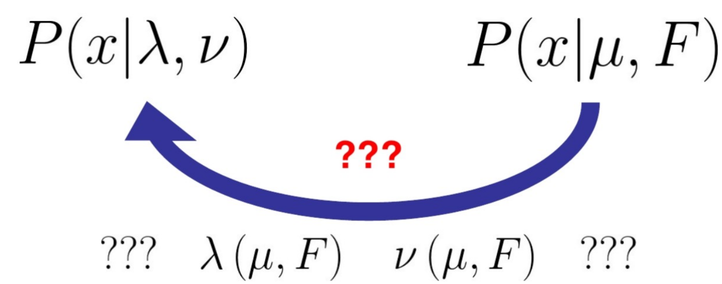
Empirically, we know F < 1 (sub-Poissonian dispersion)

Using COM-Poisson



At high μ/F, there are asymptotic expressions we can use!

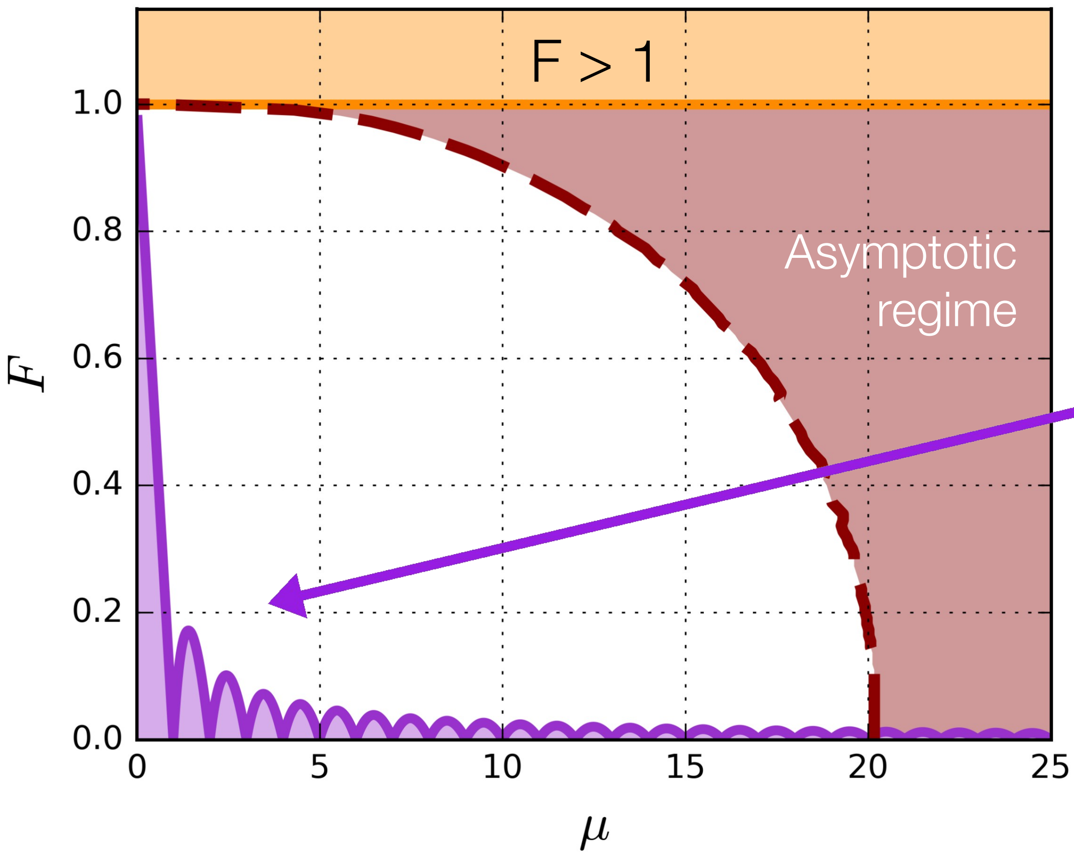
Solves this problem:



$$\lambda(\mu, F) \approx (\nu\mu F)^{\nu}$$

$$\nu(\mu, F) \approx \frac{2\mu + 1 + \sqrt{4\mu^2 + 4\mu + 1 - 8\mu F}}{4\mu F}$$

Accurate to ≤ 0.01%

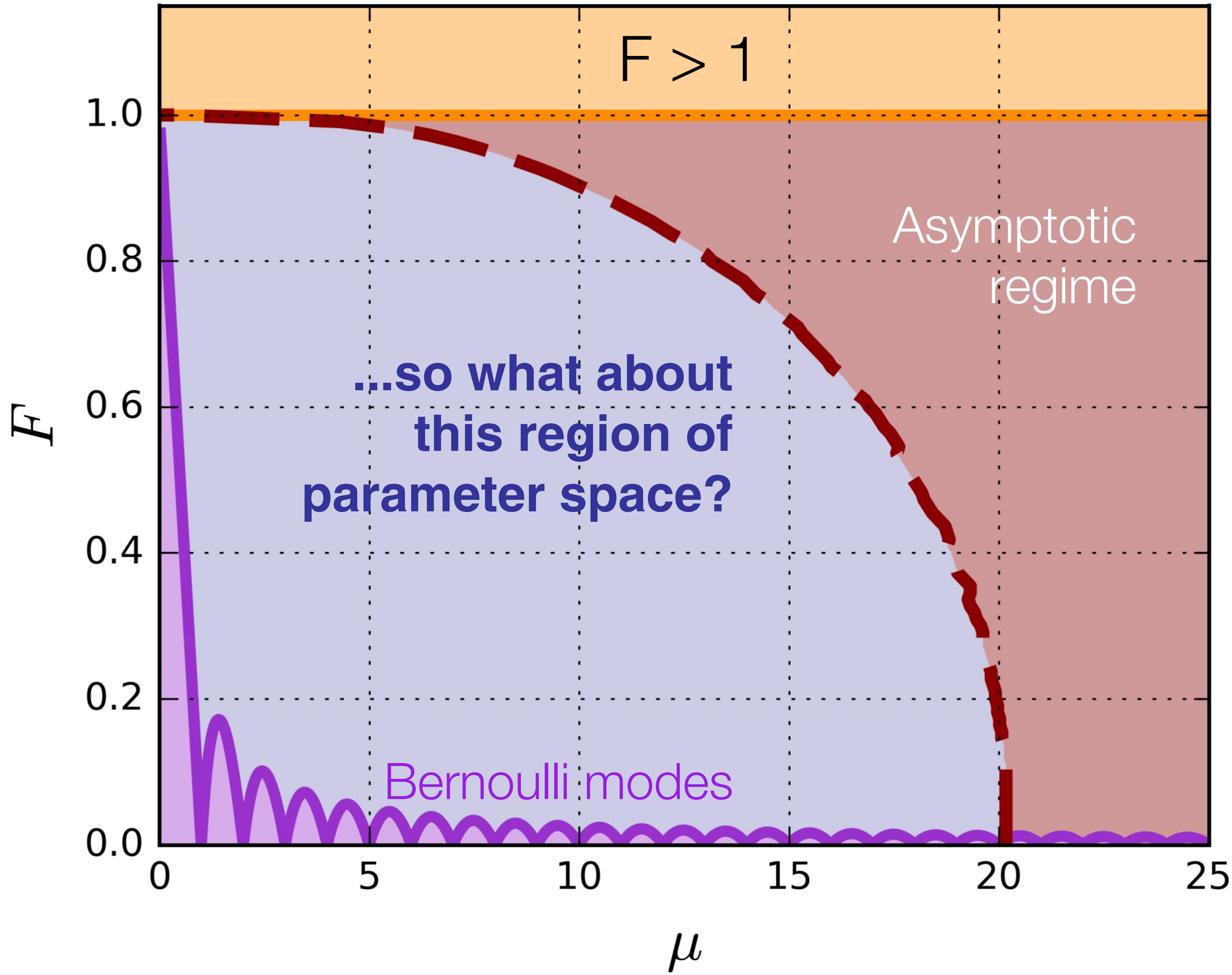


Fundamental lower limit of F are the “Bernoulli modes”

In this regime your distribution is a choice between two integers

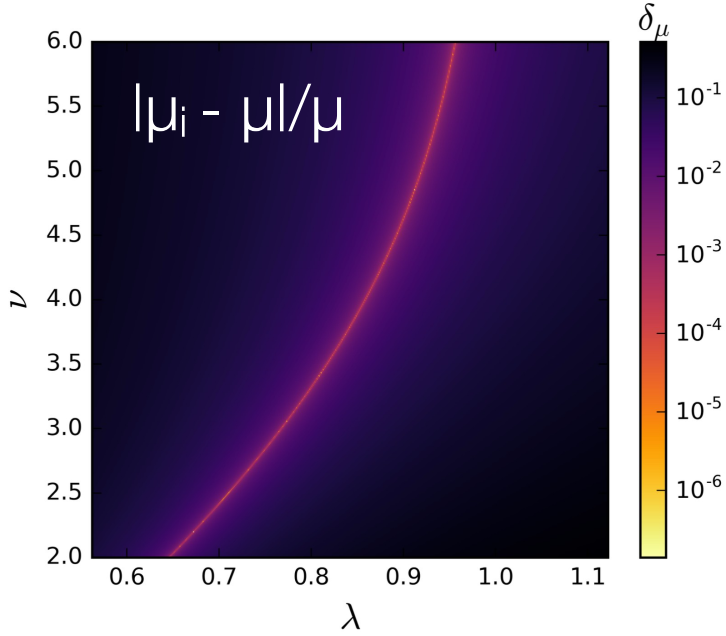
This parameter space is inaccessible to any discrete distribution

Using COM-Poisson



Minimization Algorithm

(Example for $\mu = 0.5$, $F = 0.6$)

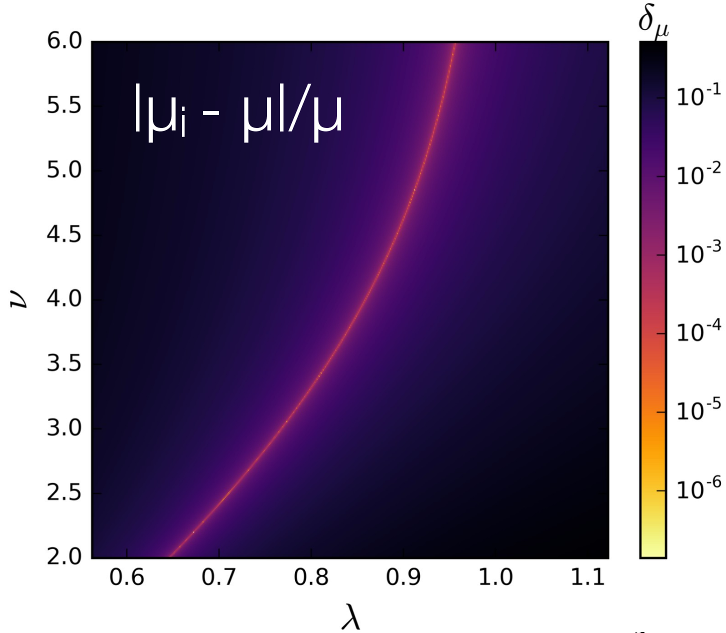


For a given **λ** and **ν**:

1. Calculate relative difference between the mean you want and the mean you get

Minimization Algorithm

(Example for $\mu = 0.5$, $F = 0.6$)



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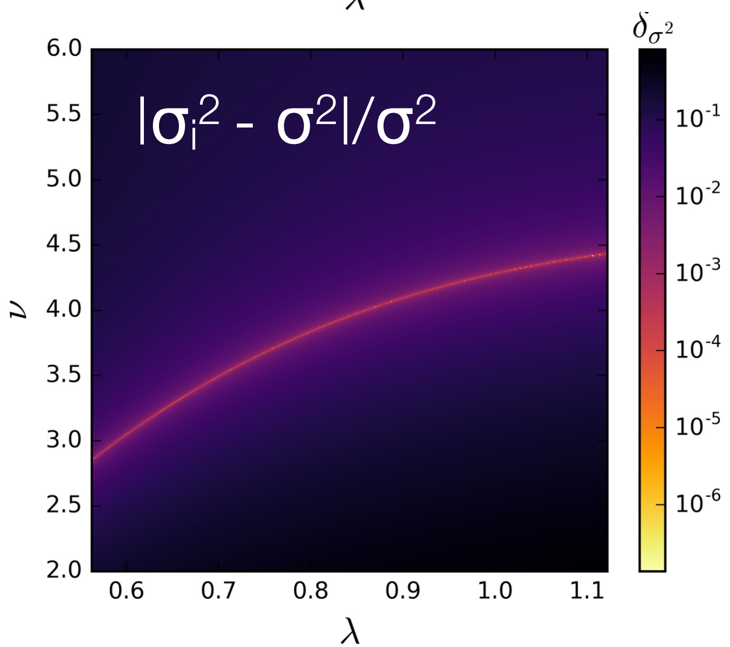
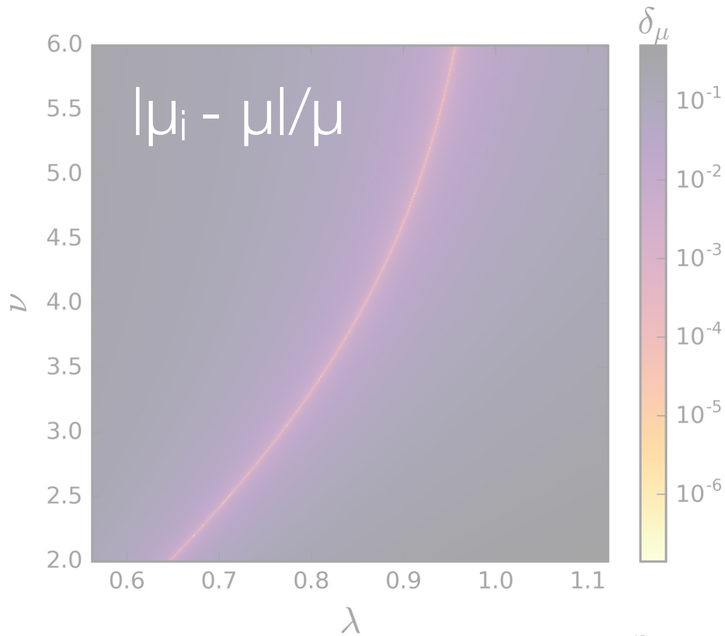
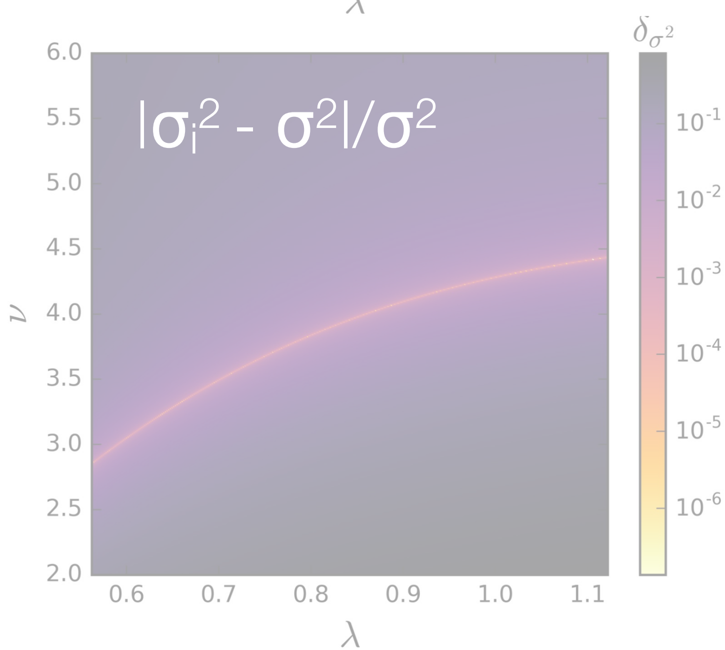


2. Calculate relative difference between the variance you want and the variance you get

Minimization Algorithm

(Example for μ = 0.5, F = 0.6)



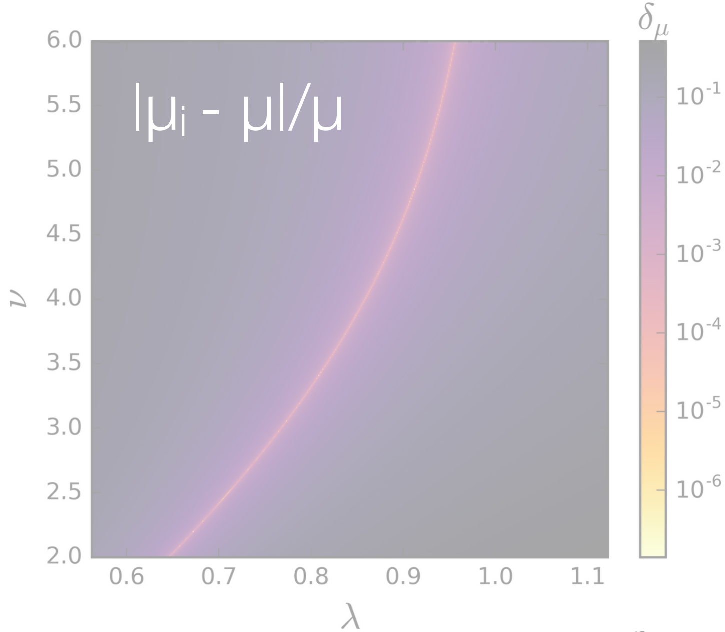


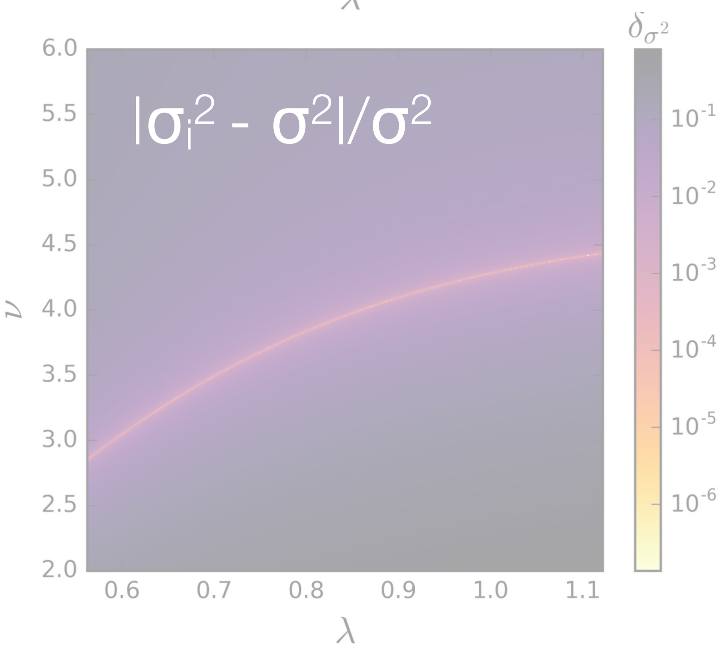
Take the weighted average of these two to combine them into one quantity...

$$X = \left(\left| w_1 \frac{\mu - \mu_i}{\mu} \right| + \left| w_2 \frac{\sigma^2 - \sigma_i^2}{\sigma^2} \right| \right)^p$$

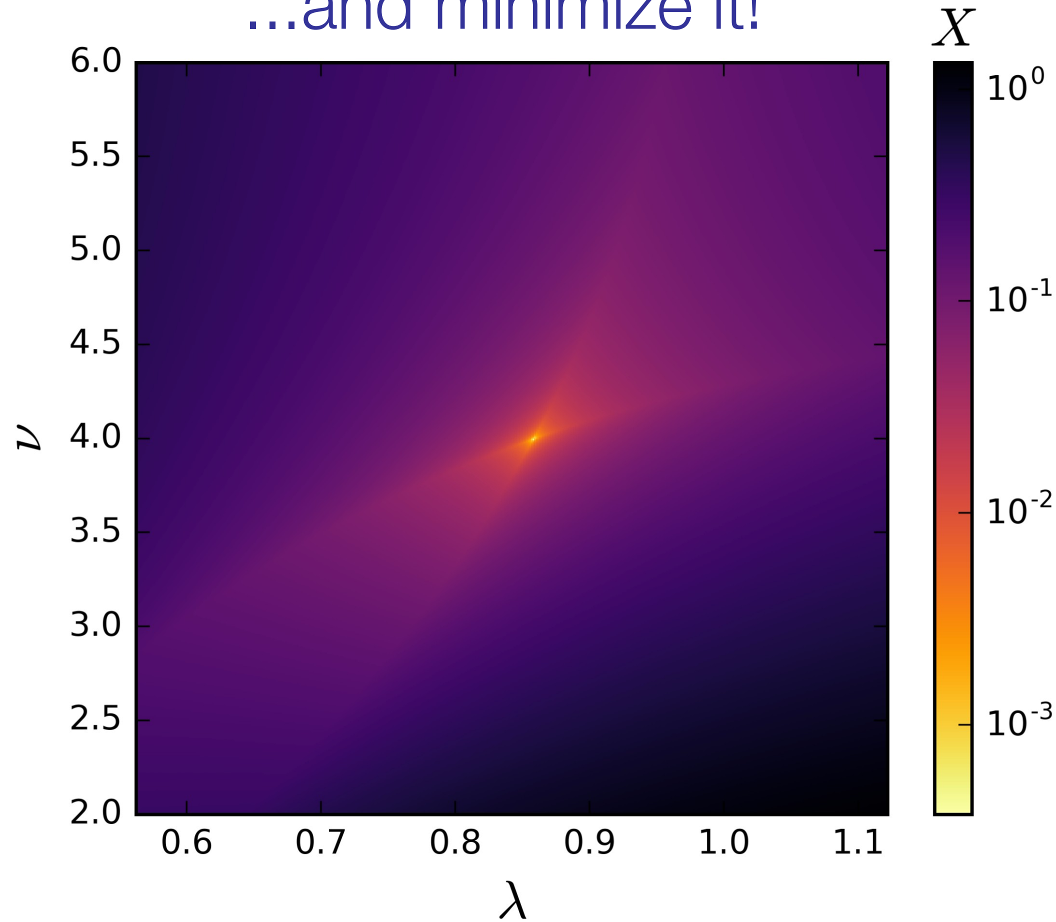
Minimization Algorithm

(Example for μ = 0.5, F = 0.6)





...and minimize it!



$$X = \left(\left| w_1 \frac{\mu - \mu_i}{\mu} \right| + \left| w_2 \frac{\sigma^2 - \sigma_i^2}{\sigma^2} \right| \right)^p$$

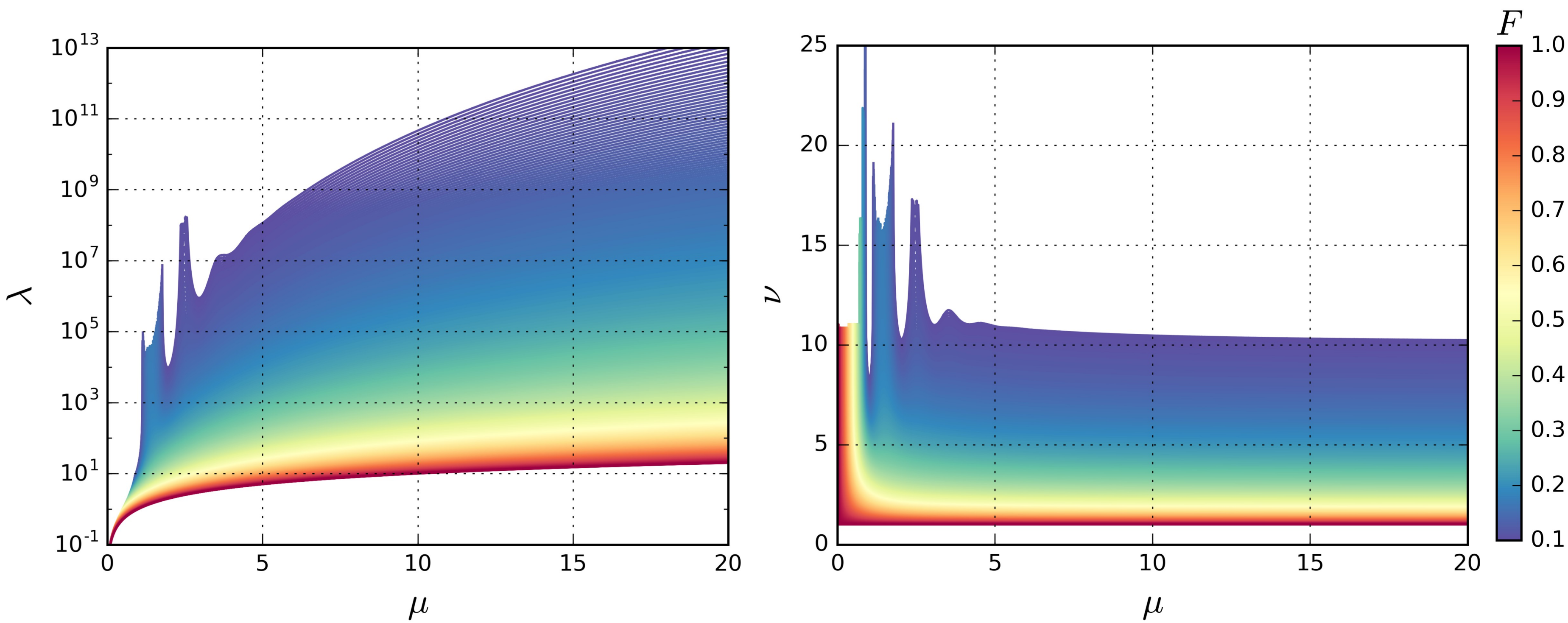
The Tables

The minimization algorithm is relatively slow

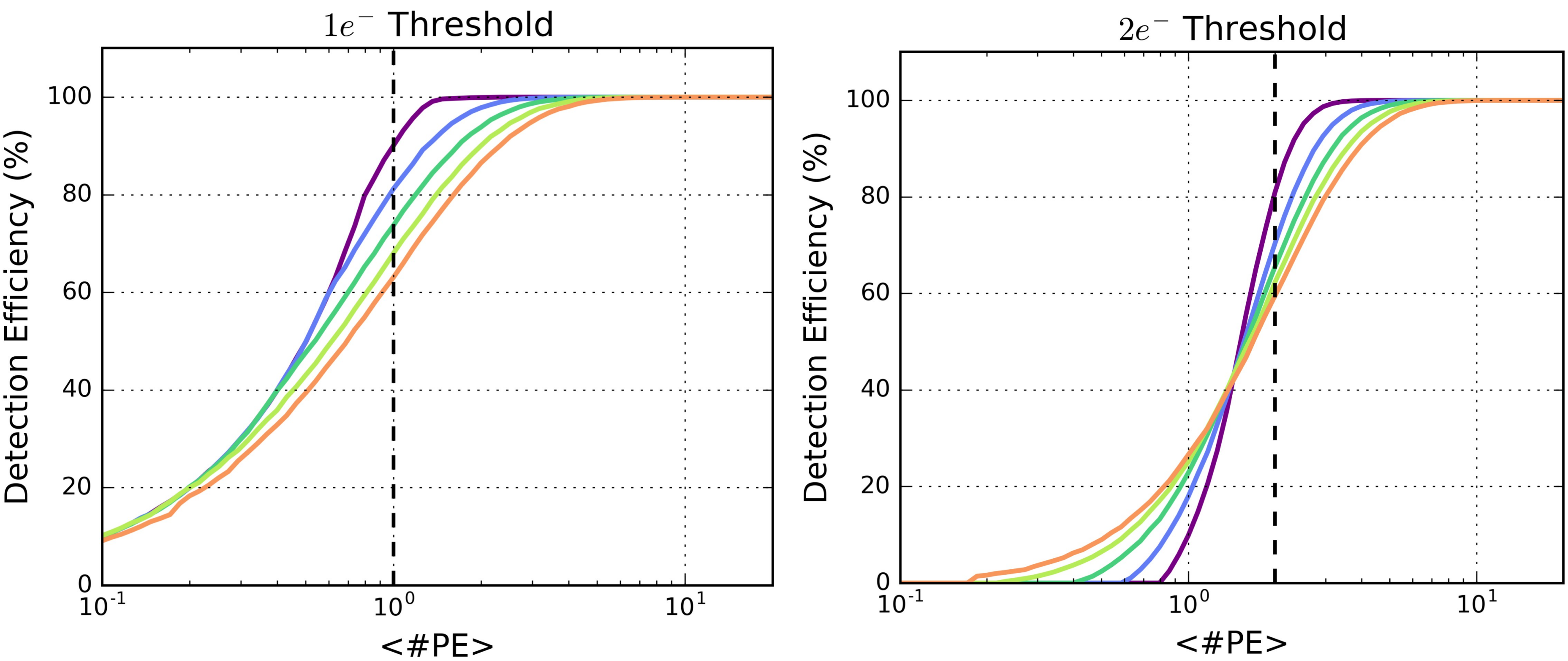
We have run it for a large range of values of μ/F and stored the results in look-up tables

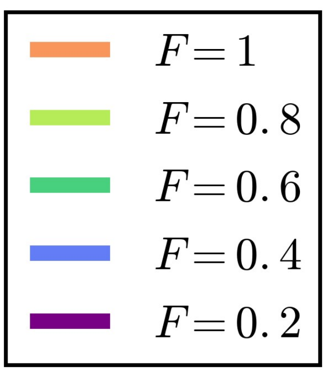
F \ μ	0	0.1	0.2	0.3	...	20
0.1						
0.2						
0.3						
.				λ		
.						
.						
1.0						

F \ μ	0	0.1	0.2	0.3	...	20
0.1						
0.2						
0.3						
.				ν		
.						
.						
1.0						



Impact of the Fano Factor

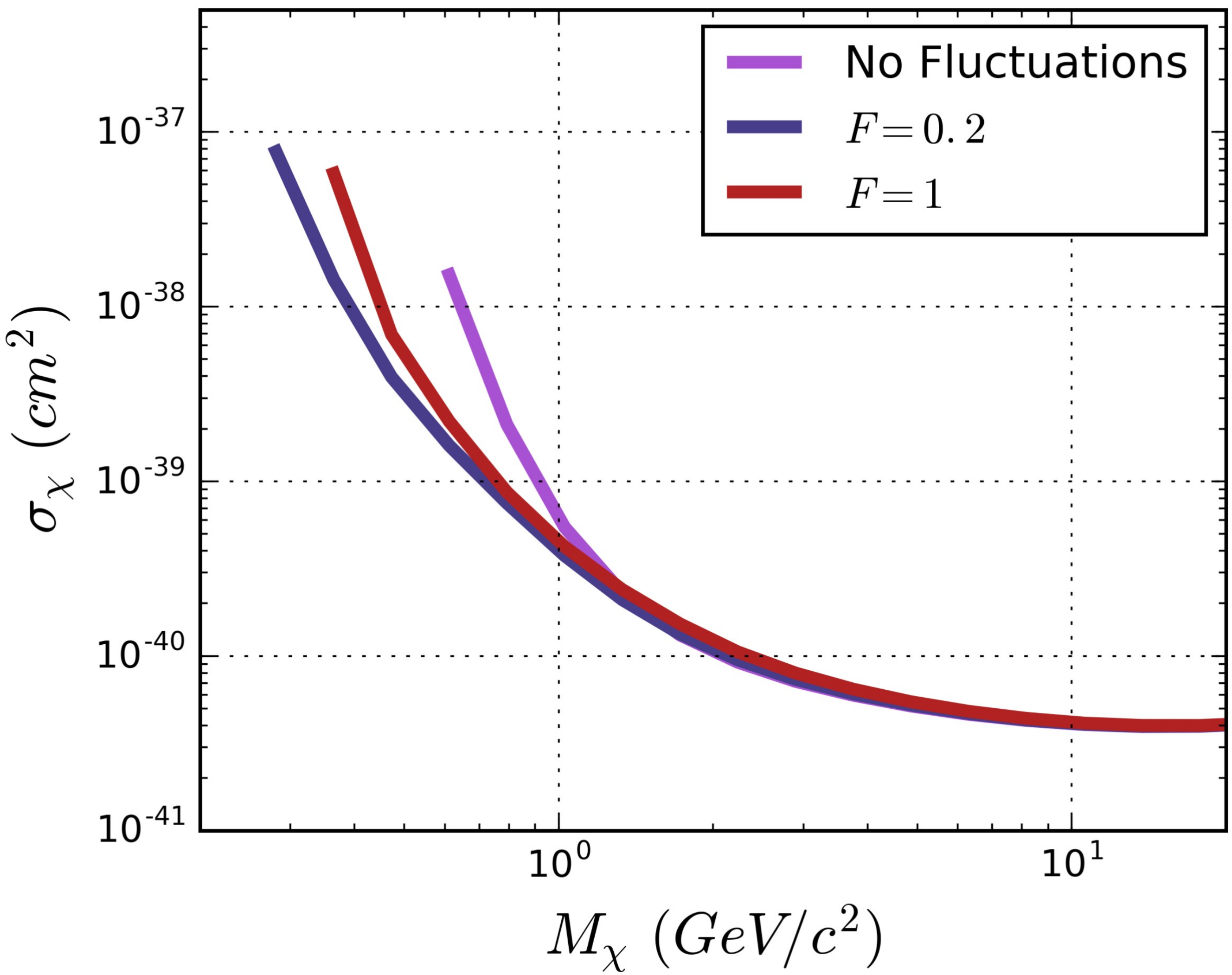




Impact of F can be understood by determining probability of “gaining” or “losing” events near energy threshold

Impact of the Fano Factor

Now we have a tool to simulate the effect of primary ionization statistics!



Hypothetical neon experiment, 1e- threshold

COM-Poisson + Gaussian (**σ = 10eV**) energy resolution

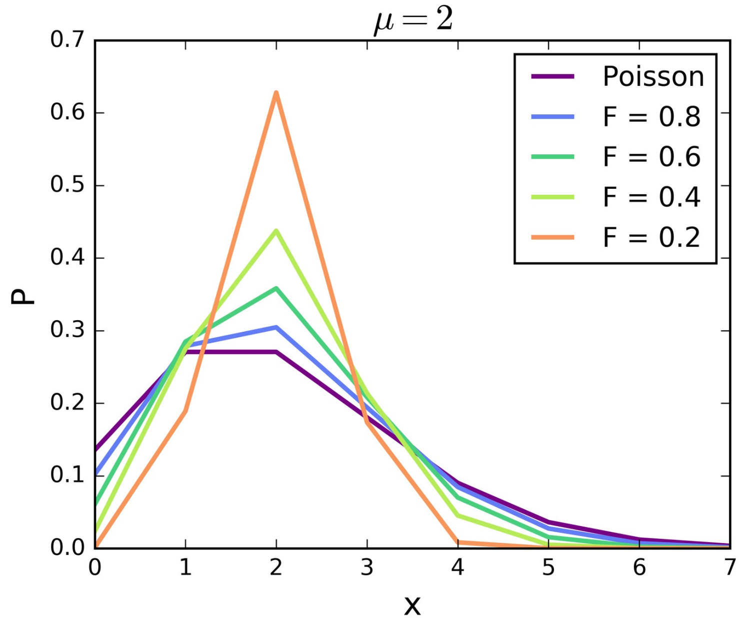
Conclusions

Including the Fano Factor in simulations of primary ionization is very important!

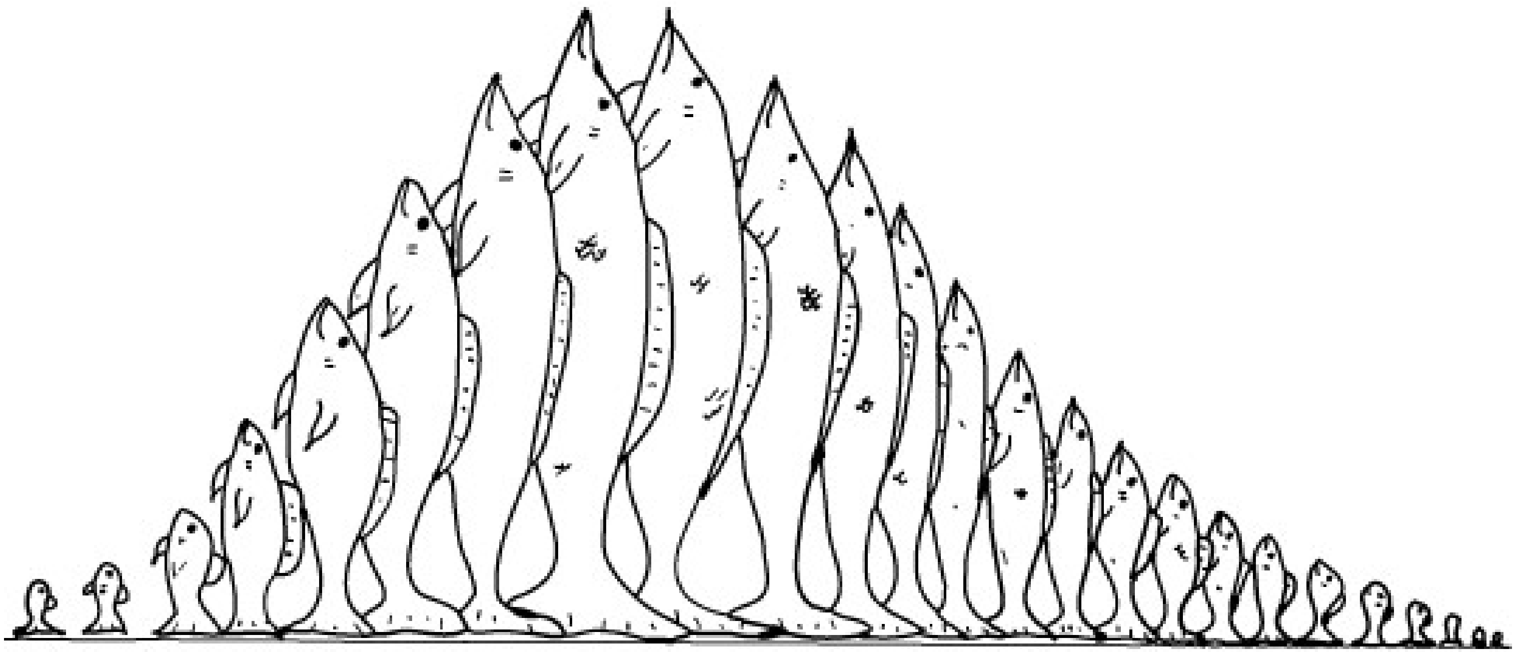
Finding a mathematical tool to do so is non-trivial

The COM-Poisson distribution is a possible solution, albeit not a very user friendly one

We plan to make these look-up tables and code publicly available!



Thank You!



$$P\{x = i\} = e^{-\lambda} \cdot \lambda^{i} / i!$$

NICO

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NEWS-G collaboration 2018

- **Queen's University Kingston – G Gerbier,** P di Stefano, R Martin, G Giroux, T Noble, D Dunrford, S Crawford, M Vidal, A Brossard, P Vazquez dS, Q Arnaud, K Dering, J Mc Donald, M Clark, M Chapellier, A Ronceray, P Gros, J Morrison, C Neyron
  - Copper vessel and gas set-up specifications, calibration, project management
  - Gas characterization, laser calibration, on smaller scale prototype
  - Simulations/Data analysis
- **IRFU** (Institut de Recherches sur les Lois fondamentales de l'Univers)**/CEA Saclay** -**I Giomataris**, M Gros, C Nones, I Katsioulas, T Papaevangelou, JP Bard, JP Mols, XF Navick,
  - Sensor/rod (low activity, optimization with 2 electrodes)
  - Electronics (low noise preamps, digitization, stream mode)
  - DAQ/soft
- **LSM** (Laboratoire Souterrain de **Modane**), IN2P3, U of Chambéry - F Piquemal, M Zampaolo, A DastgheibiFard
  - Low activity archeological lead
  - Coordination for lead/PE shielding and copper sphere
- **Thessaloniki University** – I Savvidis, A Leisos, S Tzamarias
  - Simulations, neutron calibration
  - Studies on sensor
- **LPSC (**Laboratoire de Physique Subatomique et Cosmologie) **Grenoble** - D Santos, JF Muraz, O Guillaudin
  - Quenching factor measurements at low energy with ion beams
- **Pacific National Northwest Lab**– E Hoppe, R Bunker
  - Low activity measurements, Copper electroforming
- **RMCC (**Royal Military College Canada**) Kingston** – D Kelly, E Corcoran
  - 37 Ar source production, sample analysis
- **SNOLAB –Sudbury** – P Gorel
  - Calibration system/slow control
- **University of Birmingham**– K Nikolopoulos, P Knights
  - Simulations, analysis, R&D

- **Associated labs : TRIUMF** - F Retiere,

Apr 2018

References

Alkhazov, G. D., Komar, A. P., & Vorob...ev, A. A. (1967). Ionization fluctuations and resolution of ionization chambers and semiconductor detectors. Nuclear Instruments and Methods, 48(1), 1-12. https://doi.org/10.1016/0029-554X(67)90455-7

F., S. K., Sharad, B., & Galit, S. (2011). The COM-Poisson model for count data: a survey of methods and applications. Applied Stochastic Models in Business and Industry, 28(2), 104-116. https://doi.org/10.1002/asmb.918

Grosswendt, B. (1984). Statistical fluctuations of the ionisation yield of low-energy electrons in He, Ne and Ar. Journal of Physics B: Atomic and Molecular Physics, 17(7), 1391. Retrieved from http://stacks.iop.org/0022-3700/17/i=7/a=022

Galit, S., P., M. T., B., K. J., Sharad, B., & Peter, B. (2004). A useful distribution for fitting discrete data: revival of the Conway-Maxwell-Poisson distribution. Journal of the Royal Statistical Society: Series C (Applied Statistics), 54(1), 127-142. https://doi.org/10.1111/j.1467-9876.2005.00474.x

Minka, T., Shmeuli, G., Kadane, J., Borle, S., & Boatwright, P. (2003). Computing with the COM-Poisson distribution. Department of Statistics. Retrieved from http://repository.cmu.edu/statistics/170

Arnaud, Q., Asner, D., Bard, J.-P., Brossard, A., Cai, B., Chapellier, M., ... Zampaolo, M. (2018). First results from the NEWS-G direct dark matter search experiment at the LSM. Astroparticle Physics, 97, 54-62. https://doi.org/10.1016/J.ASTROPARTPHYS.2017.10.009

Consul, P. C., & Jain, G. C. (1973). A Generalization of the Poisson Distribution. Technometrics, 15(4), 791-799. Retrieved from http://www.jstor.org/stable/1267389

Fano, U. (1946). On the Theory of Ionization Yield of Radiations in Different Substances. Phys. Rev., 70(1-2), 44-52. https://doi.org/10.1103/PhysRev.70.44

Fano, U. (1947). Ionization Yield of Radiations. II. The Fluctuations of the Number of Ions. Phys. Rev., 72(1), 26-29. https://doi.org/10.1103/PhysRev.72.26

Extra Slides

NEWS-G

Key attributes:

Simple design

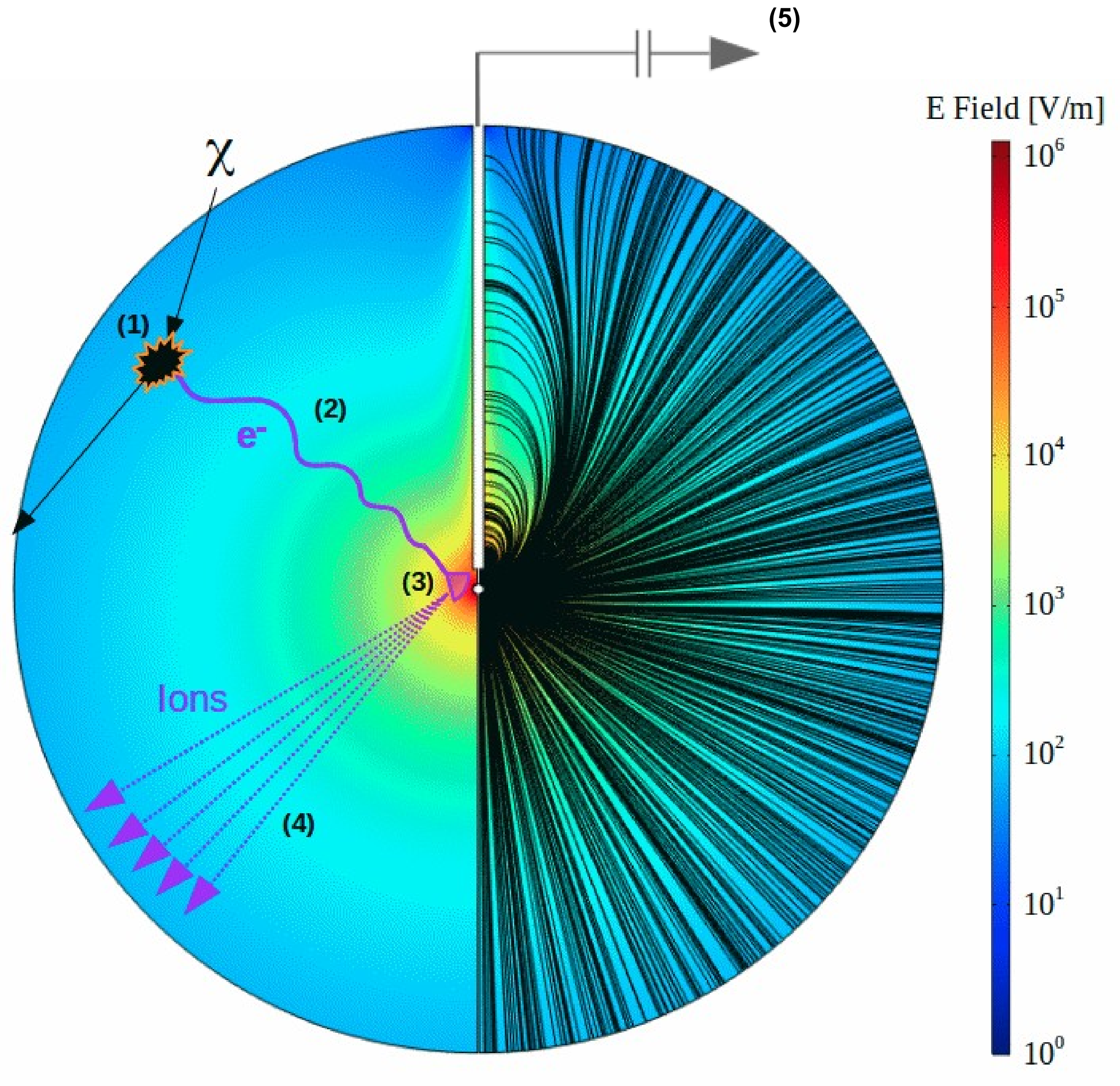
Single sensor

Gas easily changeable

1e⁻ energy threshold

Low A detector medium
→ Good for low mass WIMPs

NEWS-G



(1) Primary Ionization

Mean energy to create one pair in Ne :

$$w_e = 36eV/pair \quad w_n = \frac{w_e}{Q(E_r)} \approx 5w_e$$

(2) Drift of charges

Typical drift time surface → sensor : ~ 500 μs

(3) Avalanche of secondary e⁻/ion pairs

Amplification of signal through Townsend avalanche

(4) Signal formation

Current induced by the ions drifting away from anode

(5) Signal readout

Induced current integrated by a charge sensitive pre-amplifier and digitized at 2.08 MHz

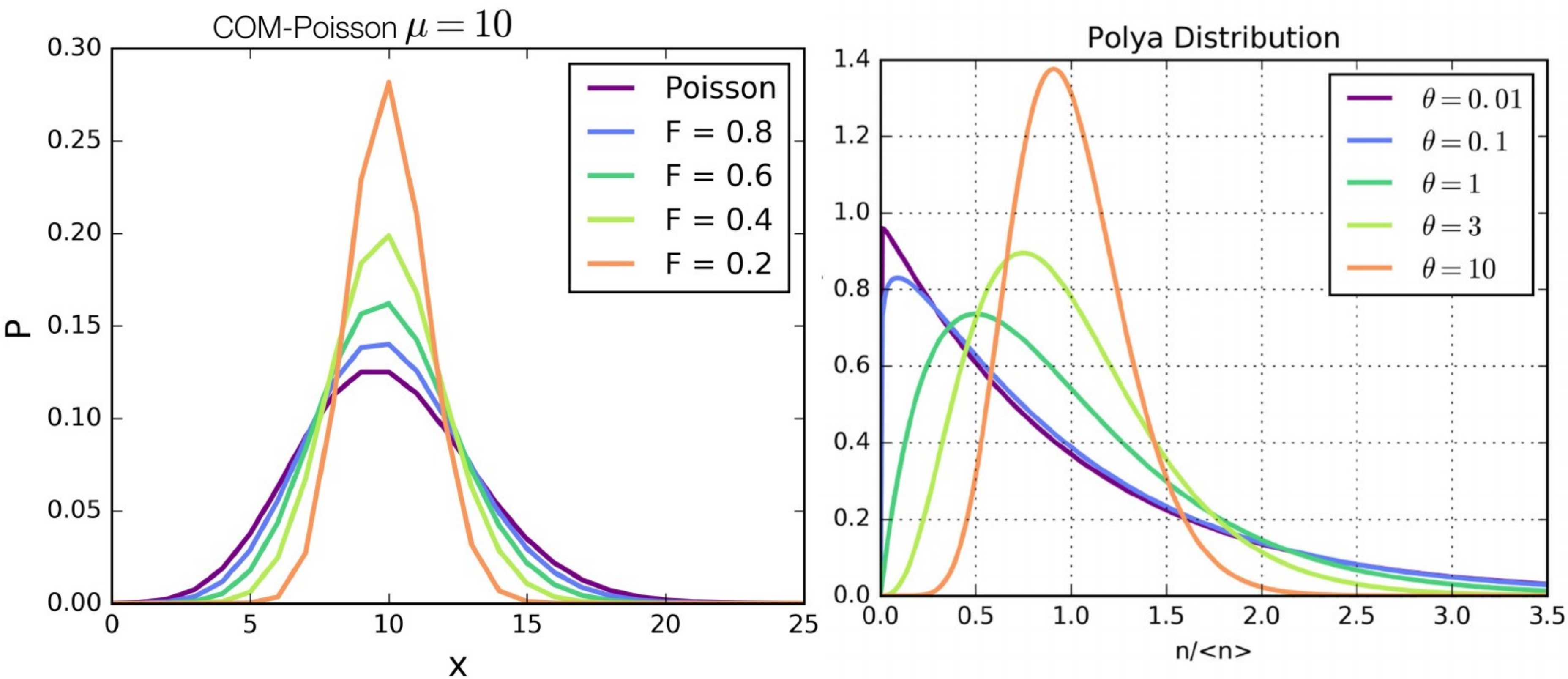
Math for mean and variance:

$$
\begin{aligned}
\mu &= \frac{\partial log Z(\lambda,\nu)}{\partial log \lambda} \\
&= \lambda \frac{\partial log Z(\lambda,\nu)}{\partial \lambda} \\
&= \frac{\lambda}{Z(\lambda,\nu)} \frac{\partial Z(\lambda,\nu)}{\partial \lambda} \\
&= \frac{\lambda}{Z(\lambda,\nu)} \frac{\partial}{\partial \lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu} \\
&= \frac{\lambda}{Z(\lambda,\nu)} \sum_{j=0}^{\infty} \frac{j\lambda^{j-1}}{(j!)^\nu} \\
&= \sum_{j=0}^{\infty} \frac{j\lambda^j}{(j!)^\nu Z(\lambda,\nu)}
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2 &= \frac{\partial \mu(\lambda,\nu)}{\partial log \lambda} = \lambda \frac{\partial \mu(\lambda,\nu)}{\partial \lambda} = \lambda \frac{\partial}{\partial \lambda} \sum_{j=0}^{\infty} \frac{j\lambda^j}{(j!)^\nu Z(\lambda,\nu)} \\
&= \lambda \left(Z^{-1} \sum_{j=0}^{\infty} \frac{j^2 \lambda^{j-1}}{(j!)^\nu} - Z^{-2} \frac{\partial Z}{\partial \lambda} \sum_{j=0}^{\infty} \frac{j\lambda^j}{(j!)^\nu} \right) \\
&= \sum_{j=0}^{\infty} \frac{j^2\lambda^j}{(j!)^\nu Z(\lambda,\nu)} - \lambda Z^{-2} \sum_{j=0}^{\infty} \frac{j\lambda^{j-1}}{(j!)^\nu} \sum_{j=0}^{\infty} \frac{j\lambda^j}{(j!)^\nu} \\
&= \sum_{j=0}^{\infty} \frac{j^2\lambda^j}{(j!)^\nu Z(\lambda,\nu)} - \left(\sum_{j=0}^{\infty} \frac{j\lambda^j}{(j!)^\nu Z(\lambda,\nu)} \right)^2 \\
&= \sum_{j=0}^{\infty} \frac{j^2\lambda^j}{(j!)^\nu Z(\lambda,\nu)} - \mu(\lambda,\nu)^2
\end{aligned}
$$

F vs θ

Larger F and smaller θ both make our energy resolution worse



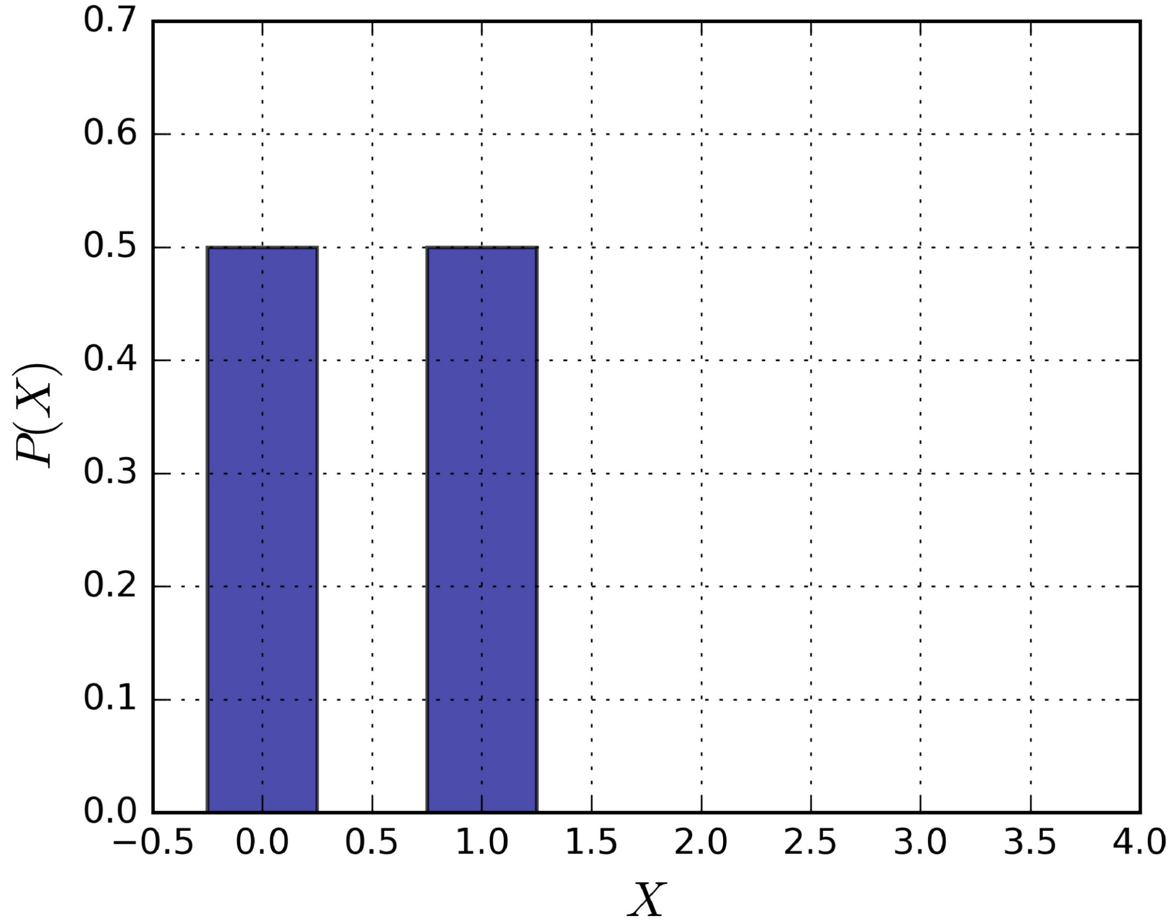
Bernoulli Modes

You can't always get the mean and variance you want:

With a very small variance you will have counts in only two bins

i.e. if you have a mean of 0.5 and variance of 0.25 you will have equal counts in the 0-bin and 1-bin

...In fact in this case you can't have a variance of less than 0.25



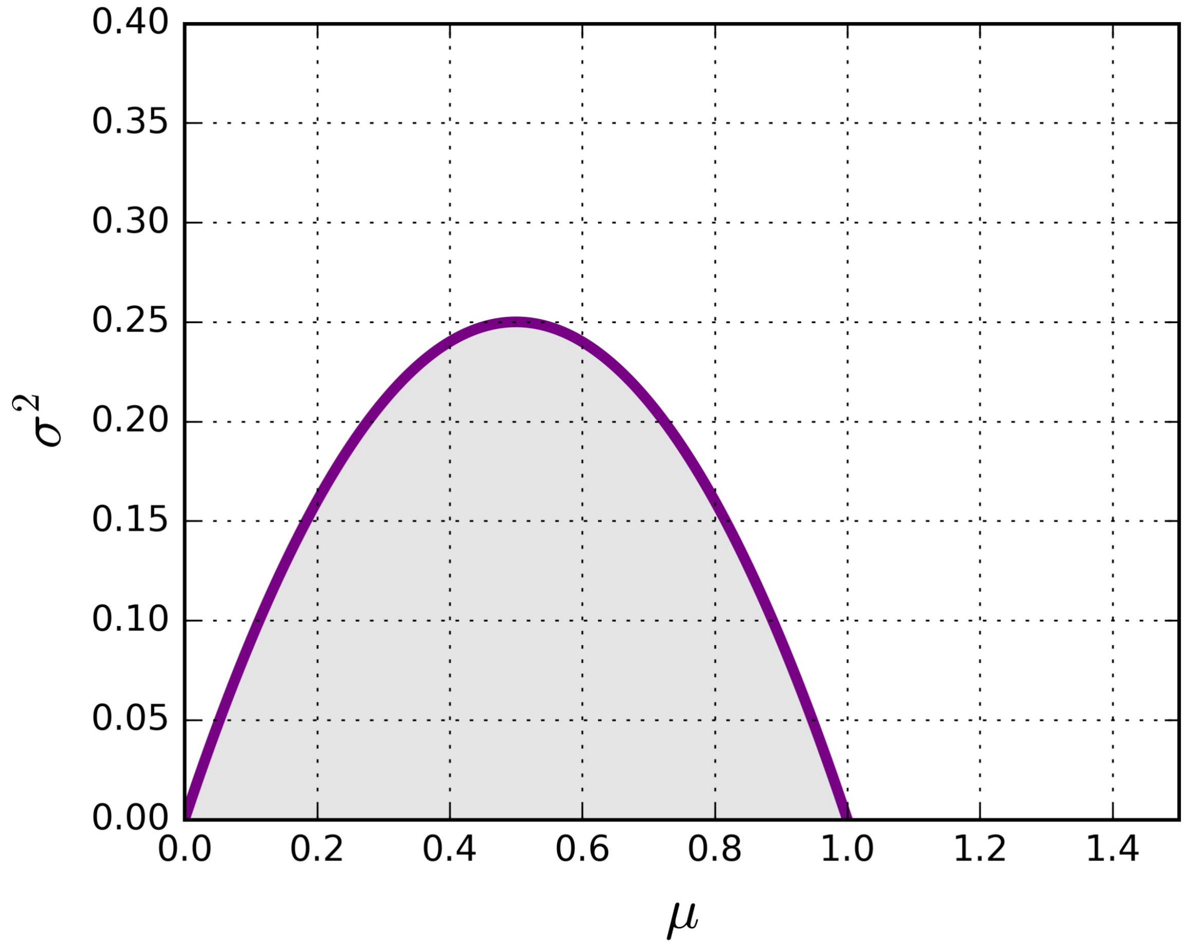
Bernoulli Modes

You can't always get the mean and variance you want:

If you change the mean slightly (larger or smaller), you can have a smaller variance but it is still restricted

The minimum variance you can have follows a parabola:

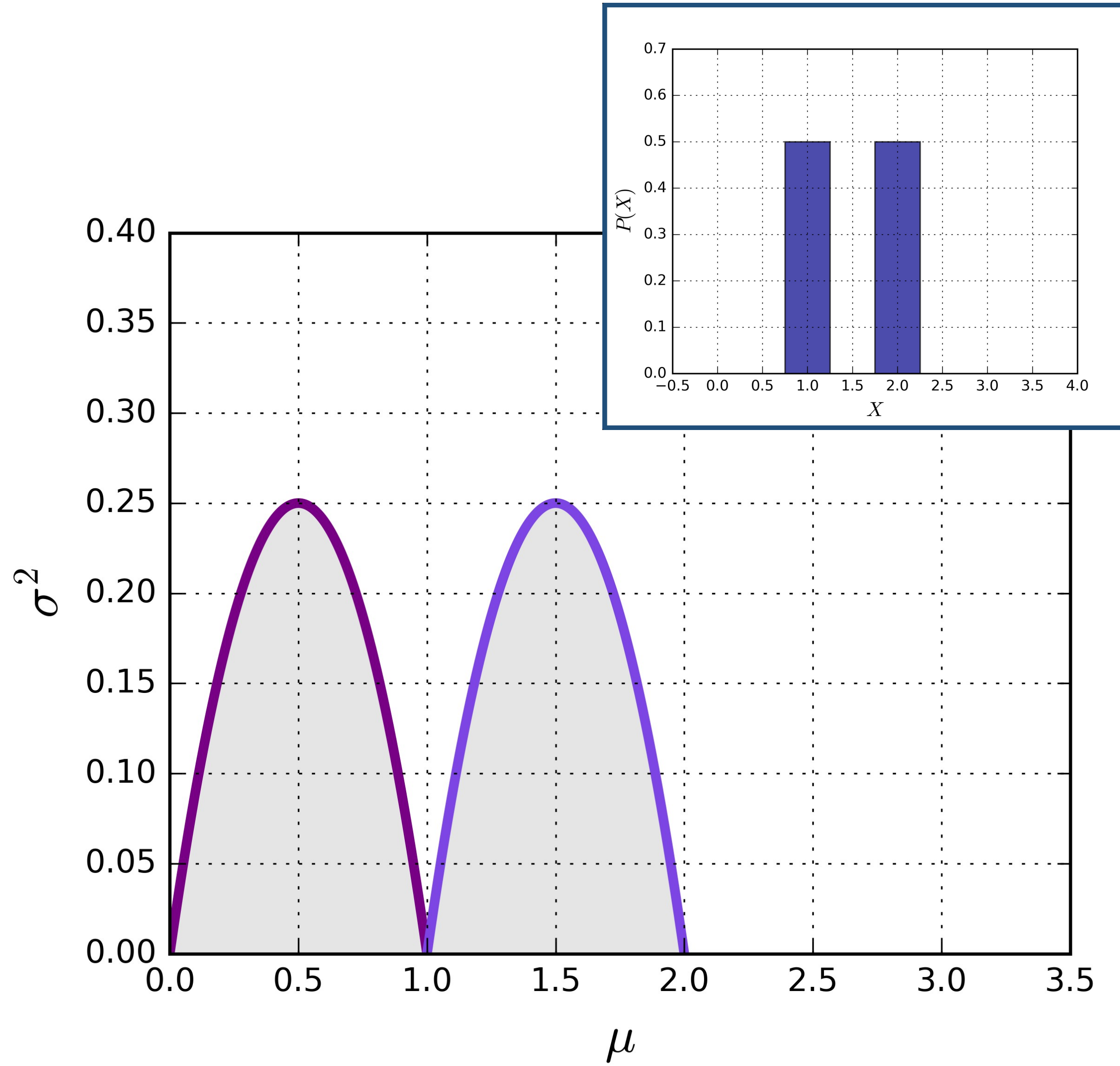
$$\sigma^2 = \mu(1 - \mu)$$



Bernoulli Modes

You can't always get the mean and variance you want:

The same argument applies if you have a mean between 1 and 2...

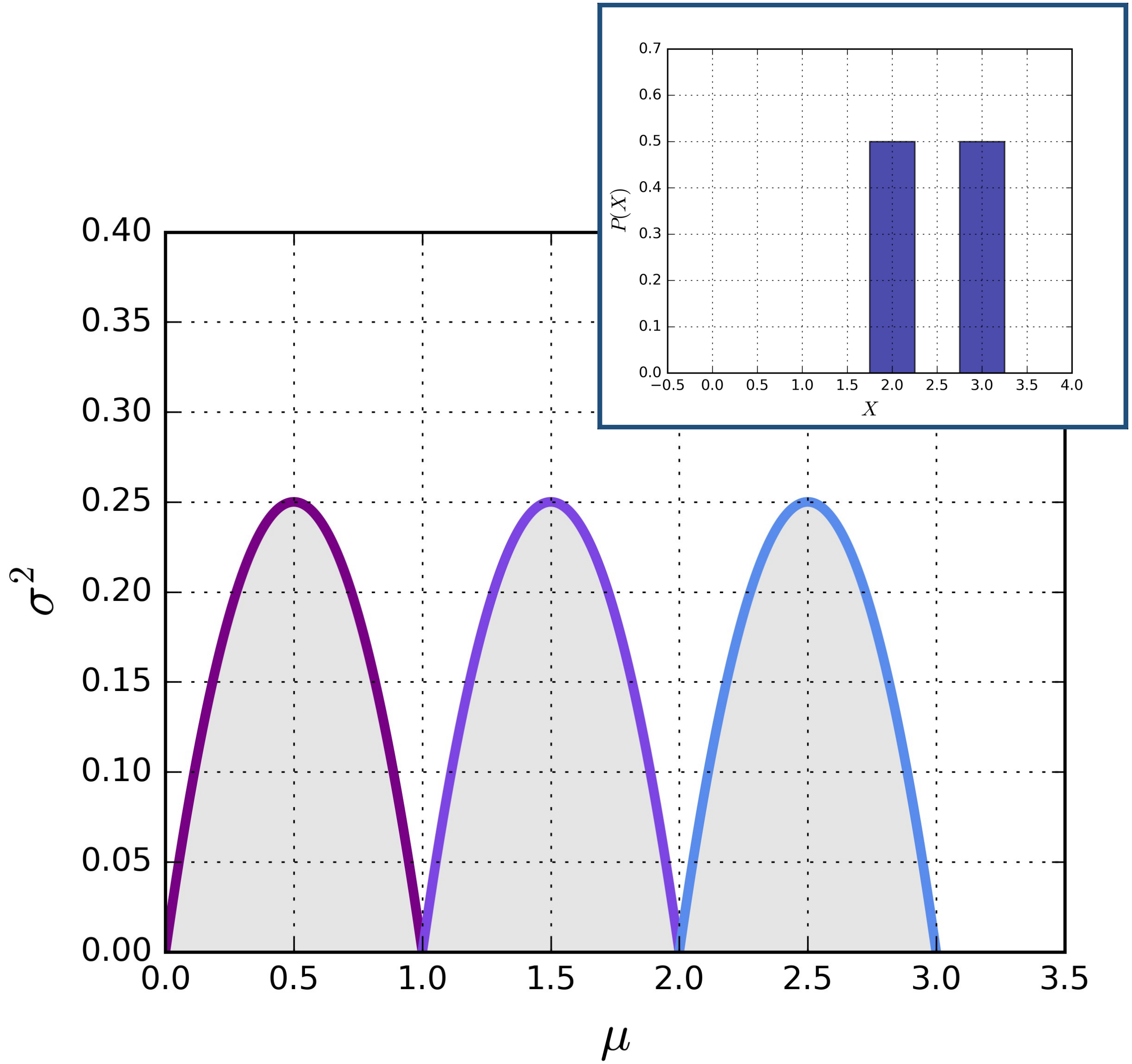


Bernoulli Modes

You can't always get the mean and variance you want:

The same argument applies if you have a mean between 1 and 2...

...between 2 and 3...



Bernoulli Modes

You can't always get the mean and variance you want:

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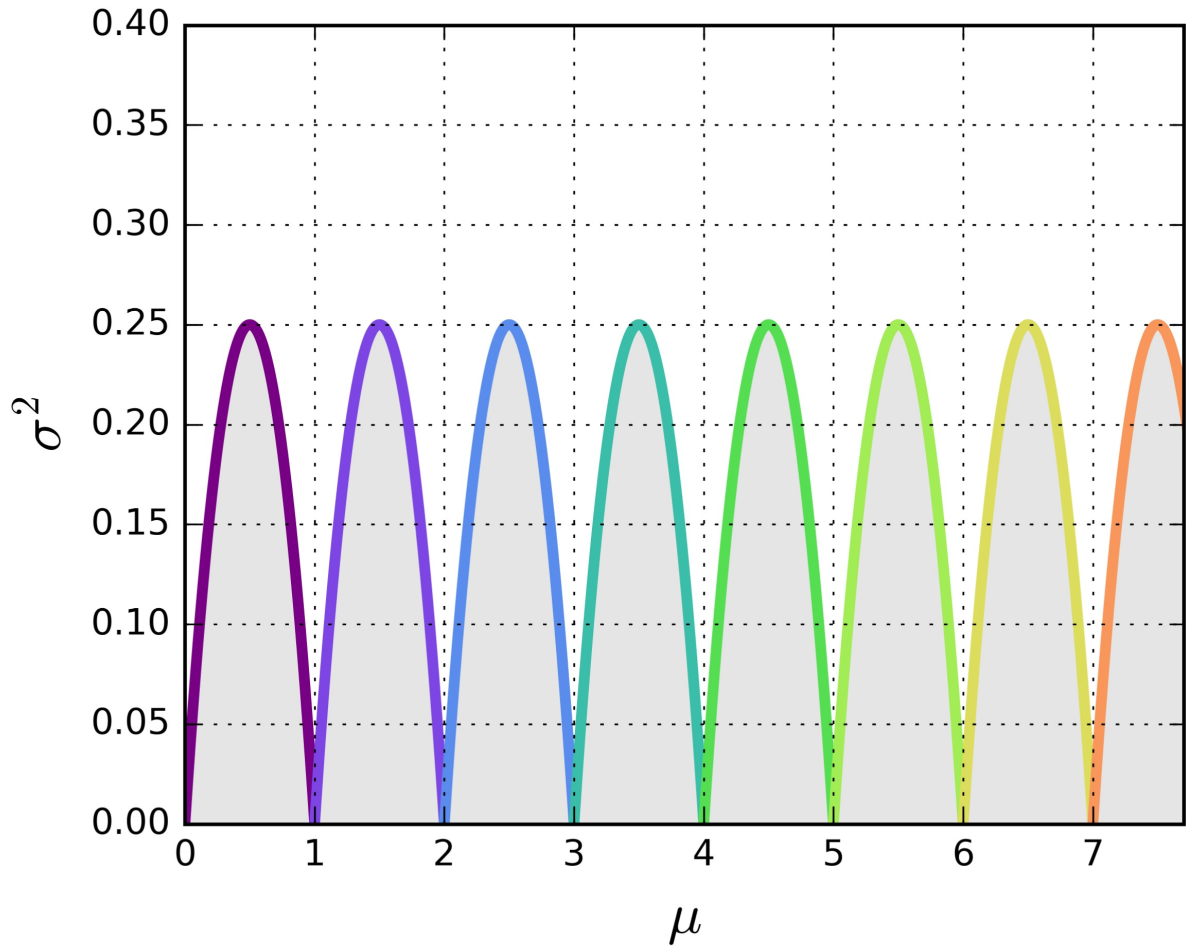
...between 2 and 3...

...and so on...

These are the "Bernoulli Modes", parameter space that is fundamentally inaccessible



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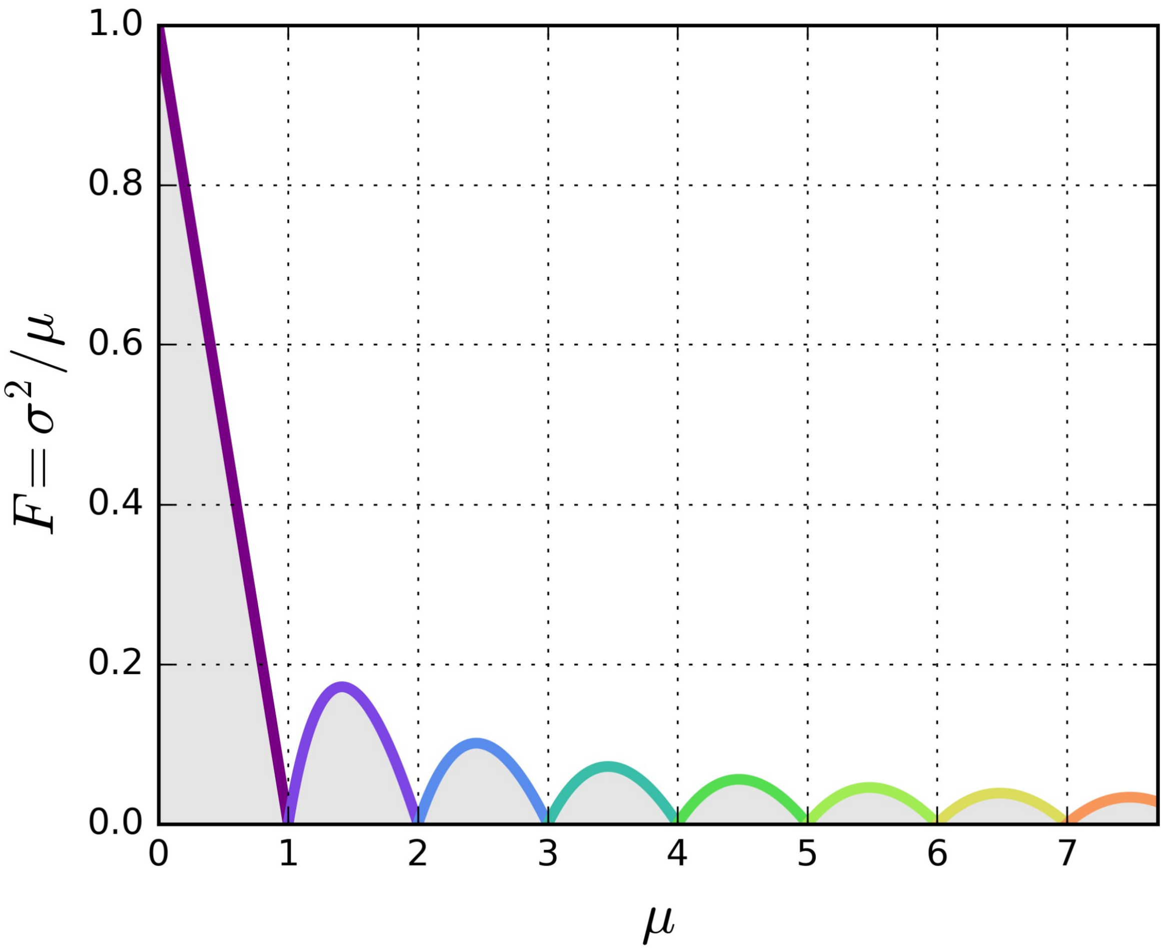
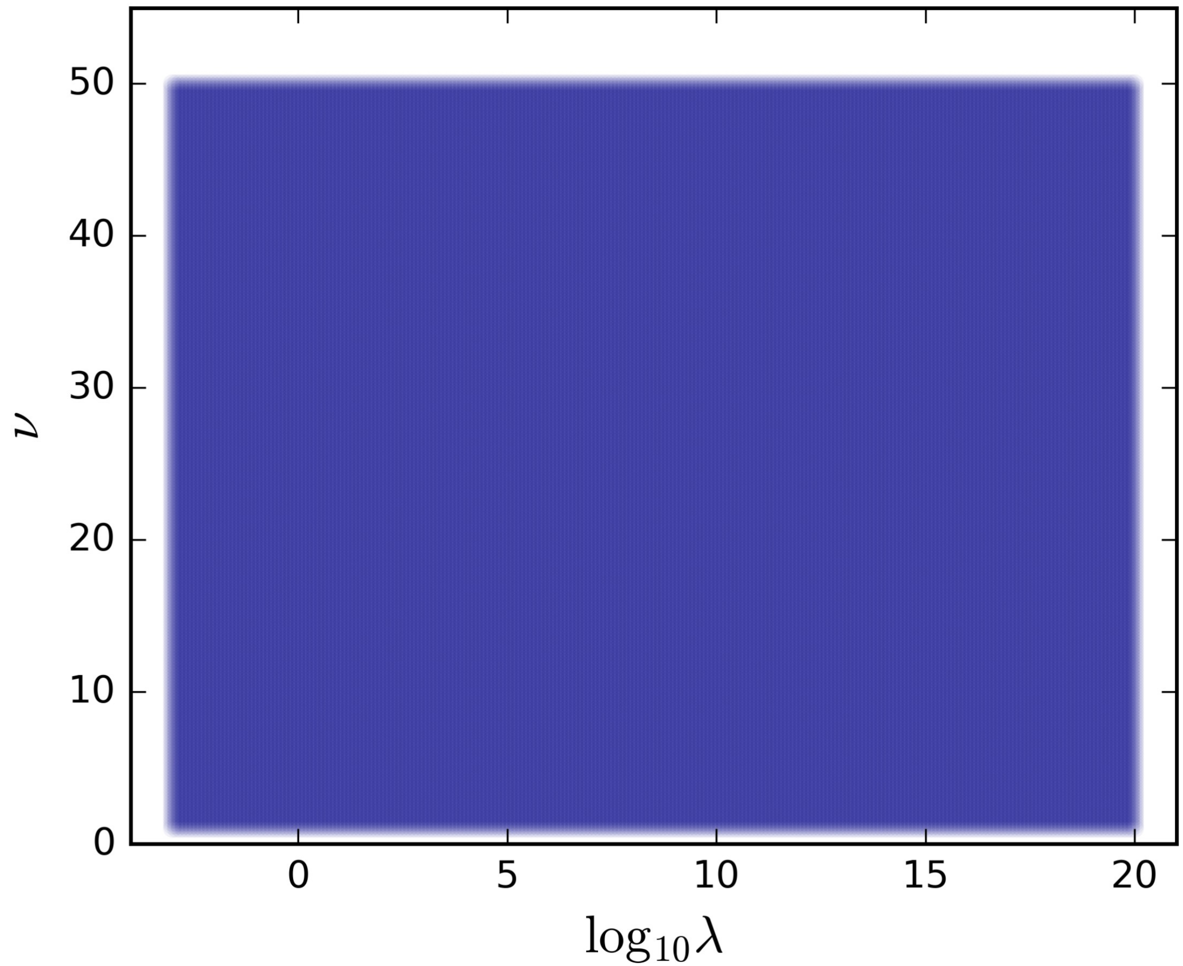


Bernoulli Modes

To prove that this is true for COM-Poisson, take a grid of points in λ and ν,

Then map those points to the mean and variance plane



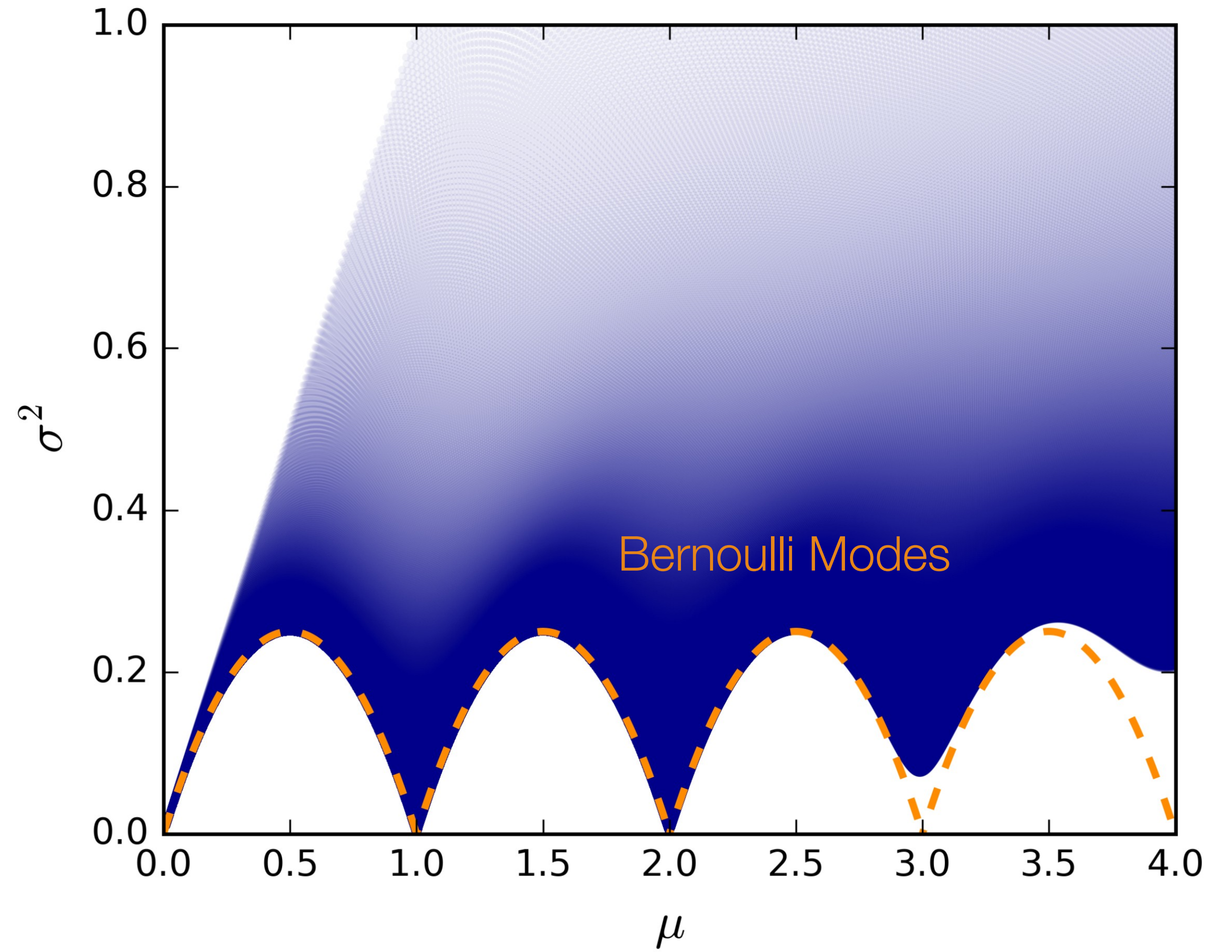
Bernoulli Modes

To prove that this is true for COM-Poisson, take a grid of points in λ and ν,

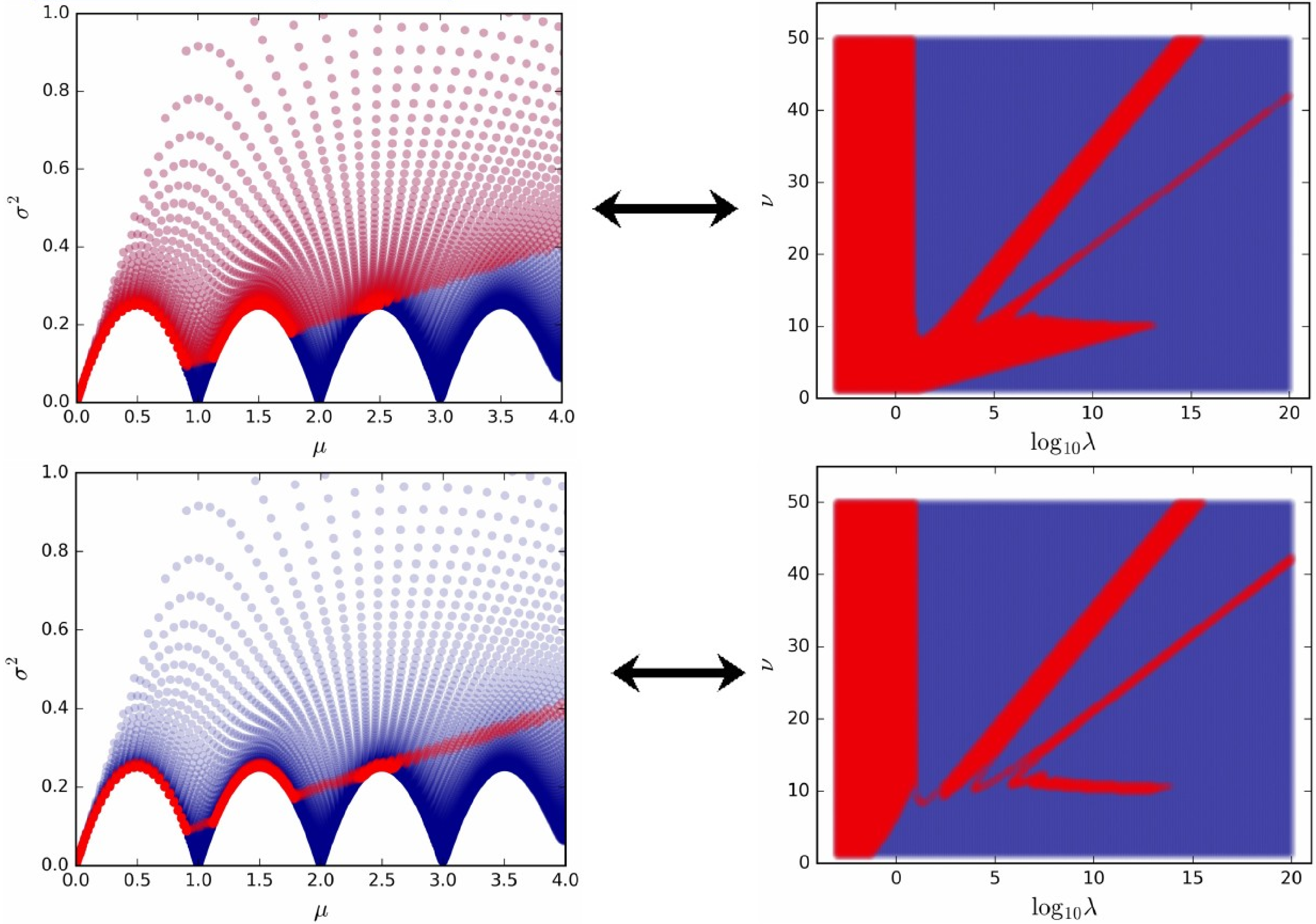
Then map those points to the mean and variance plane

The Bernoulli modes appear!

You cannot go into this forbidden parameter space!



Parameter Space?



Not Binomial

$$P(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

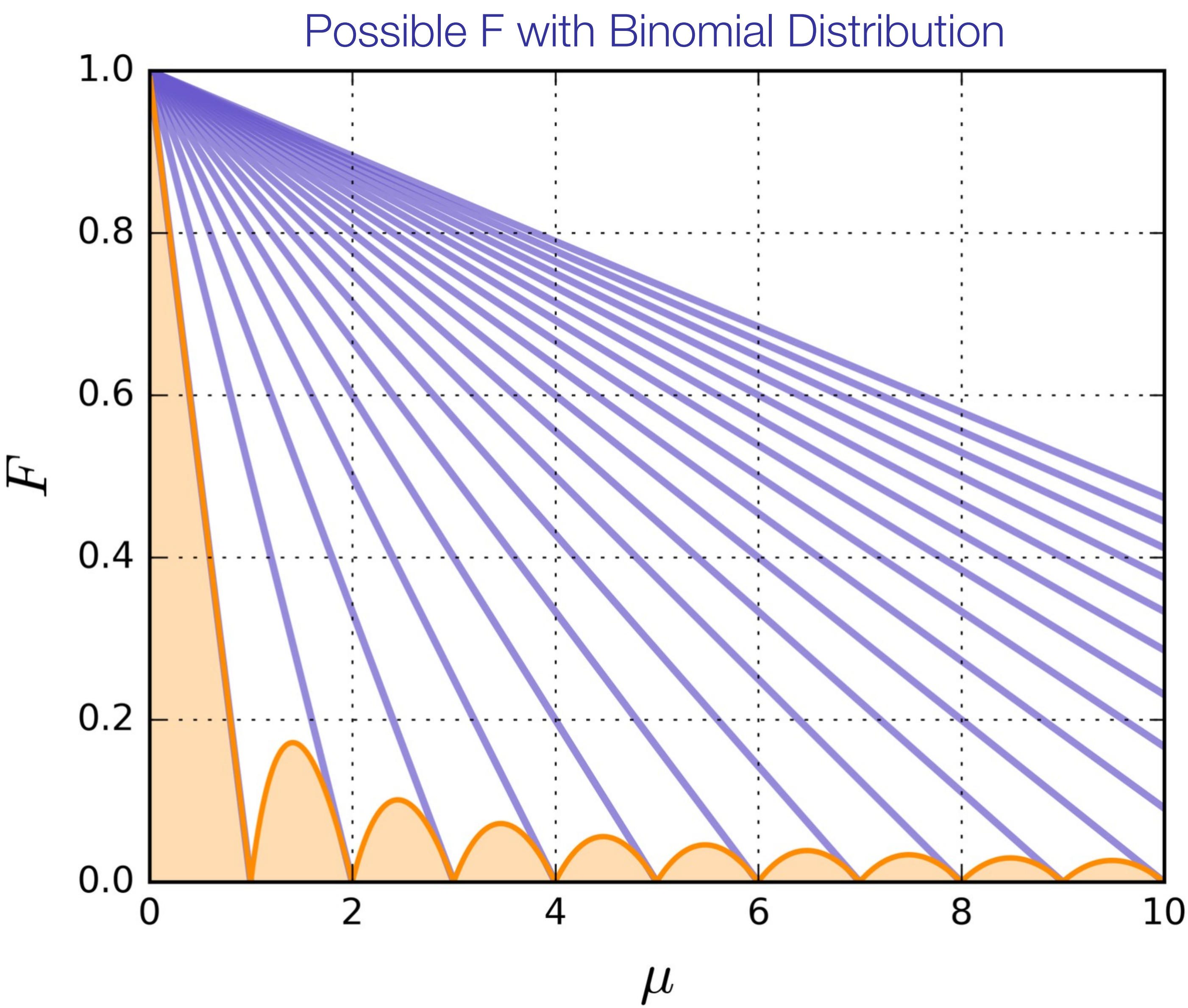
$$p \in [0,1] \qquad n \in \mathbb{N}_0$$

$$\mu = np \quad \sigma^2 = np(1-p)$$

$$\implies F = 1 - p$$

$$\therefore F = 1 - \frac{\mu}{n}$$

Not Binomial



$$F = 1 - \frac{\mu}{n}$$

Because n is not continuous, only certain values of F are possible for a given μ

F is known to vary only slightly with energy, so this is not an appropriate model

Asymptotic Regime

At larger values of mean, there is an asymptotic formula that gives us a closed form expression for the mean and variance! No need to use the minimization algorithm here!

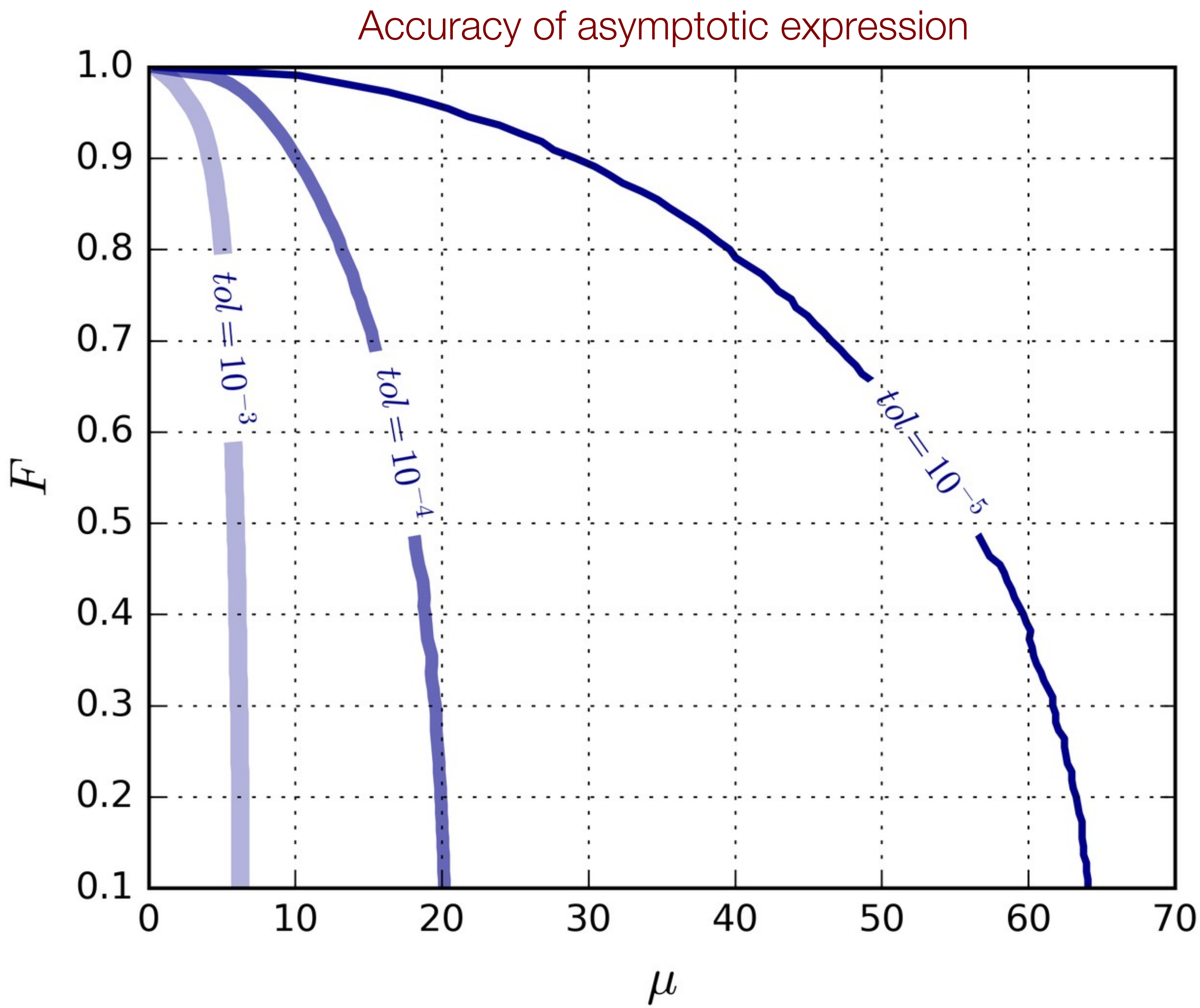
$$Z_2 = \frac{e^{\nu \lambda^{1/\nu}}}{\lambda^{\frac{\nu-1}{2\nu}} (2\pi)^{\frac{\nu-1}{2}} \sqrt{\nu}} \left(1 + \mathcal{O}\left(\lambda^{-1/\nu}\right)\right)$$

$$\mu \approx \lambda^{1/\nu} - \frac{\nu - 1}{2\nu} \qquad \sigma^2 \approx \frac{1}{\nu} \lambda^{1/\nu}$$

$$\lambda(\mu, F) \approx (\nu \mu F)^{\nu}$$

$$\nu(\mu, F) \approx \frac{2\mu + 1 + \sqrt{4\mu^2 + 4\mu + 1 - 8\mu F}}{4\mu F}$$

Asymptotic Regime



Nominally this approximation is valid when:

$$\lambda > 10^{\nu}$$

For us, it is valid to accuracy of 10^{-4} for all F above a μ of 20

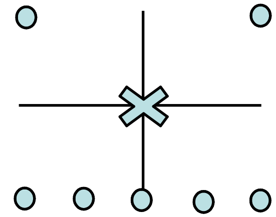
Design of Table

The goal is to guarantee accuracy to within a given distance of the Bernoulli modes

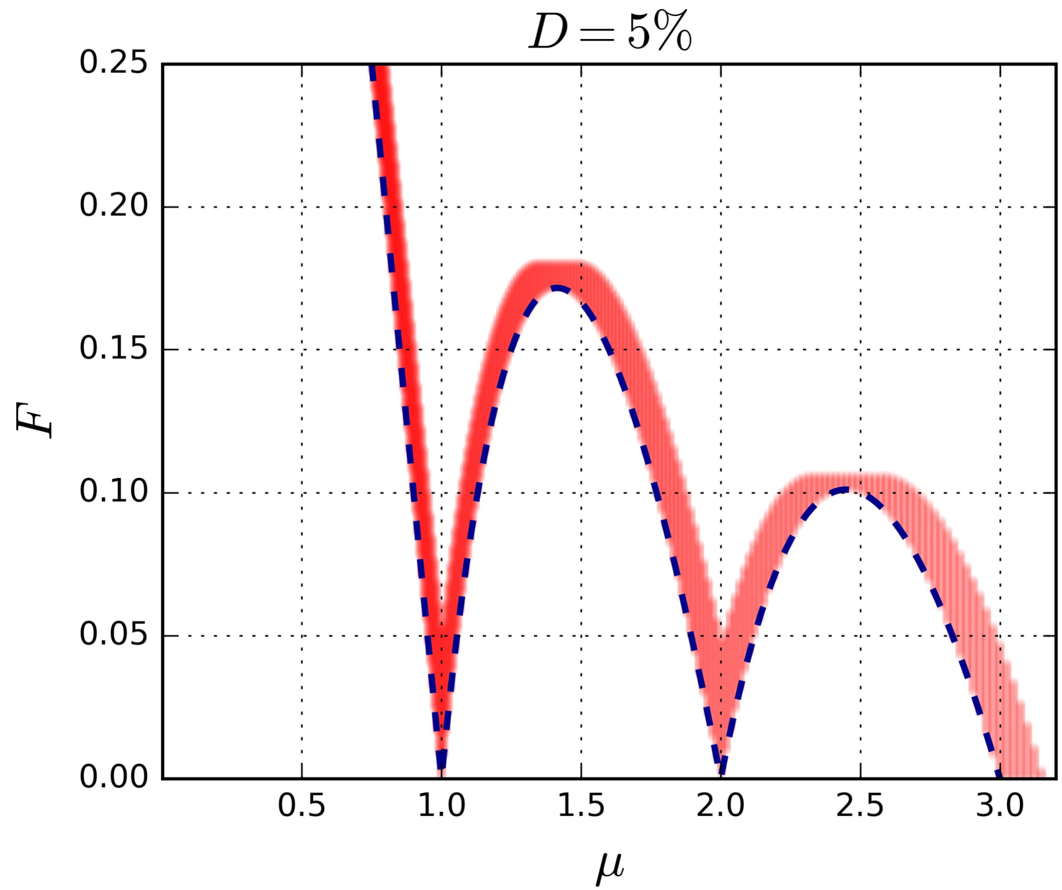
We linearly interpolate points, so we have to guarantee that linear interpolation is good to given distance from Bernoulli modes

Therefore we have to have some points within given distance of Bernoulli modes

Table point density such that each time a F-line crosses a Bernoulli mode, it is bounded by points within D = 0.1% of Bernoulli mode



If any  within Bernoulli mode and  not, then “point within D of Bernoulli mode”



Minimization Algorithm

1. For a given mean/F define an initial box λ/ν (based on asymptotic approximation and magic)

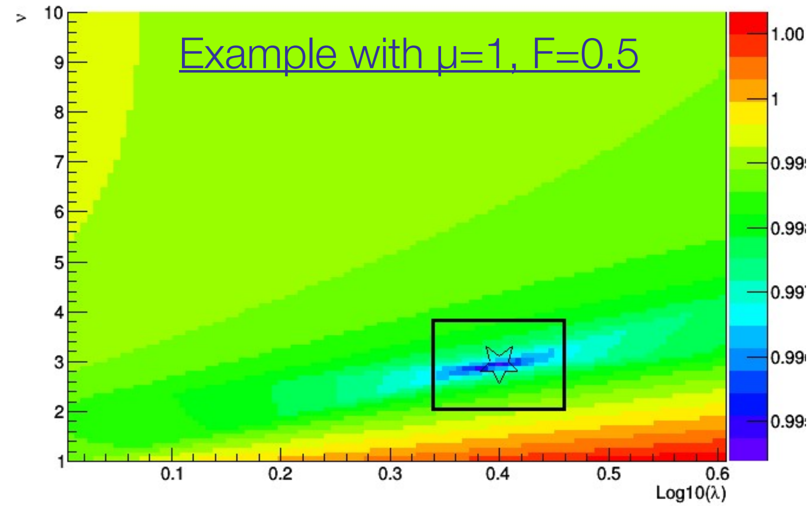
2. Perform a grid search to find min value of X:

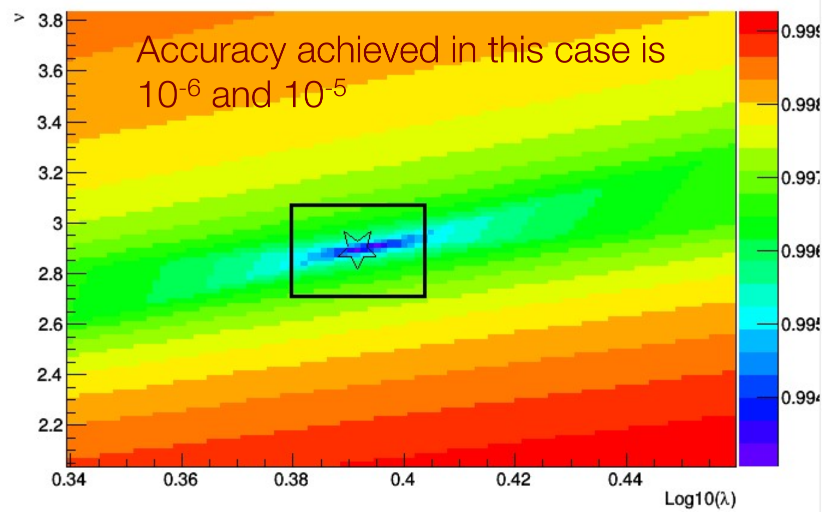
$$X = \left(\left| w_1 \frac{\mu - \mu_i}{\mu} \right| + \left| w_2 \frac{\sigma^2 - \sigma_i^2}{\sigma^2} \right| \right)^p$$

3. Perform optimization with minima of grid search as initial guess, smaller box

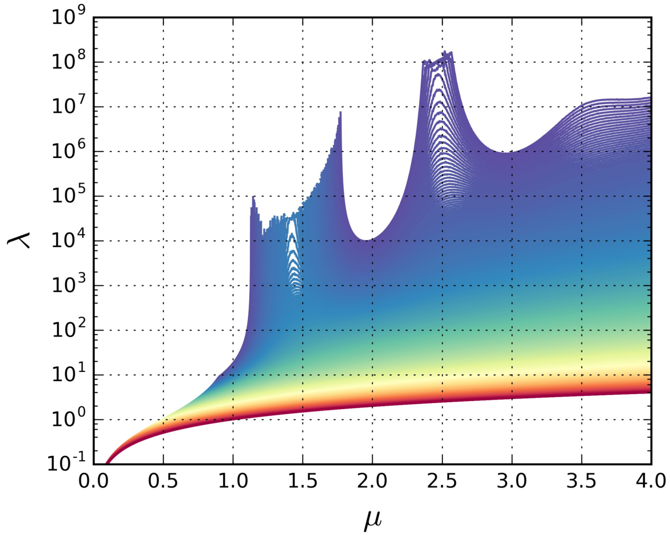
4. If not within tolerance (0.001 for mean and variance), try another grid search in smaller box and optimization in even smaller box

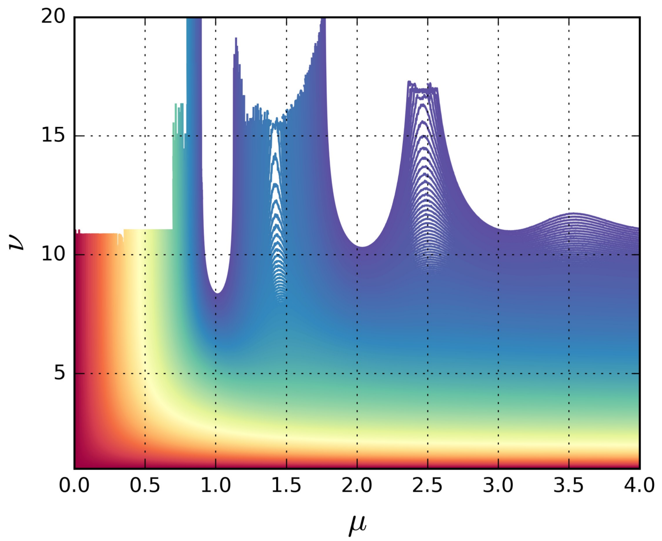
5. If this still doesn't work, repeat steps 1-4 for slightly perturbed values up to N (~25) times, keep best result

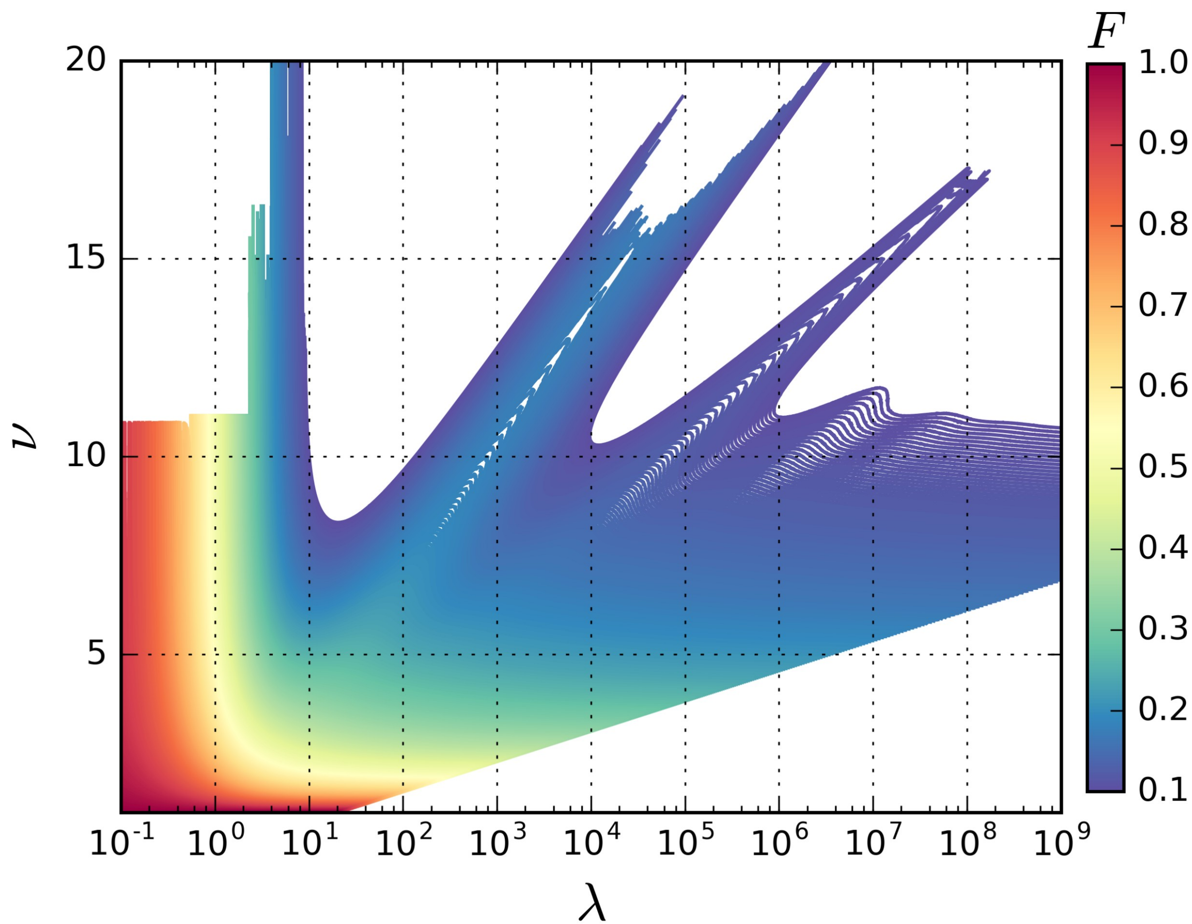


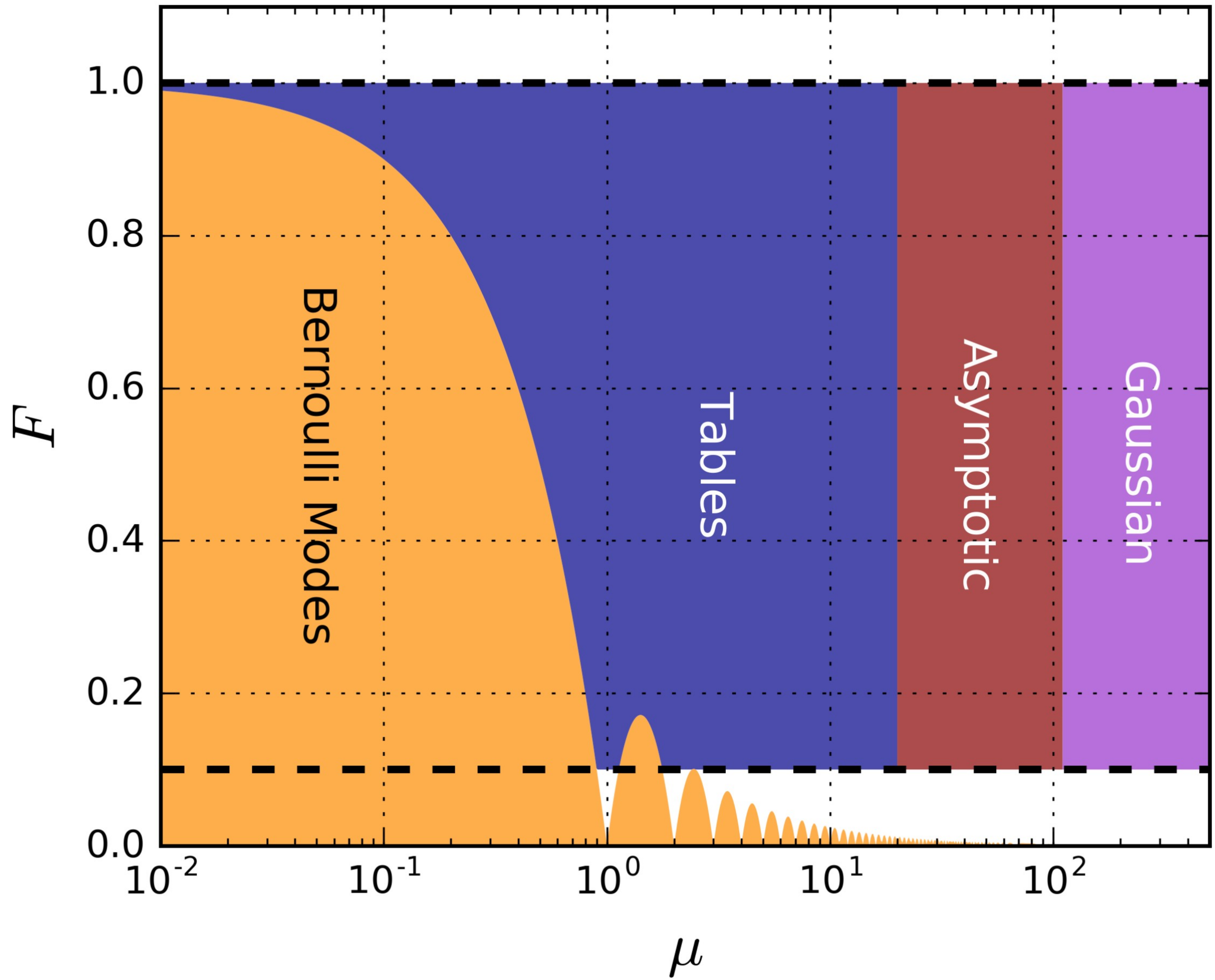


What does it look like?









1.0
0.8
0.6
0.4
0.2
0.0
F
Bernoulli Modes
Tables
Asymptotic
Gaussian
10^-2
10^-1
10^0
10^1
10^2
μ

How well does it work?