Quantum noise in excitable laser systems

Stochastic methods for driven-dissipative quantum optics

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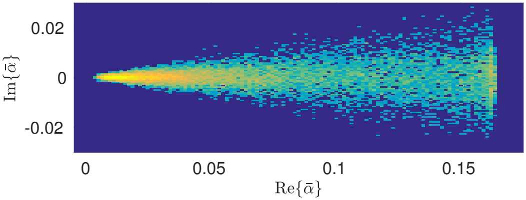
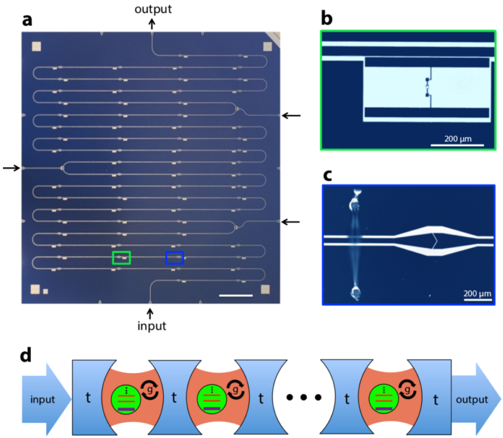


Outline

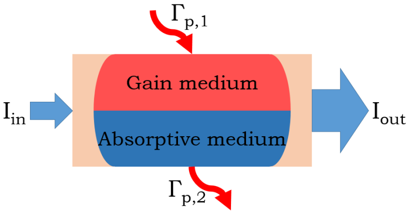
Motivation

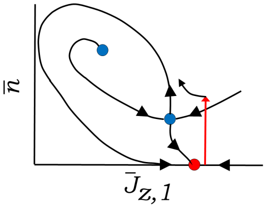
- Generally, develop tools to study large quantum systems
- Role of quantum noise
- Dynamical systems, driven-dissipative, no steady state
- Excitable laser systems

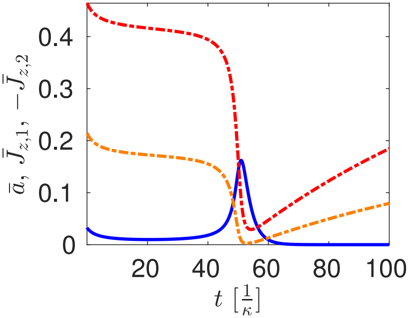


*Fitzpatrick et al. [2017]

Excitable systems







- Threshold response to input
- Optical: excitable lasers
- Interesting dynamical system
- Effect of quantum noise?

Fokker-Plank equations

- $\rho = \int d\alpha^2 d\alpha_*^2 P(\alpha, \alpha_*) \frac{|\alpha\rangle\langle(\alpha_*)^*|}{\langle(\alpha_*)^*|\alpha\rangle}$

$$\partial_t \rho = \mathcal{L}\rho \quad \Rightarrow \quad \partial_t P(\boldsymbol{\alpha}) = -\partial_{\alpha_i} A_i(\boldsymbol{\alpha}) P(\boldsymbol{\alpha}) + \partial_{\alpha_i}\partial_{\alpha_j} \frac{1}{2} D_{ij}(\boldsymbol{\alpha}) P(\boldsymbol{\alpha})$$

- $\mathbf{A}$: drift vector, evolution of mean
- $\mathbf{D}$: diffusion matrix, evolution of distribution width
- $\langle \hat{a}^{\dagger p} \hat{a}^q \rangle = \langle \alpha_*^p \alpha^q \rangle = \int d\boldsymbol{\alpha} P(\boldsymbol{\alpha}) \alpha_*^p \alpha^q$
- Linear $\rightarrow$ gaussian P; very difficult to solve in general

*Gilchrist et al. [1997], Carmichael [1999], Carmichael [2009]

Equivalent stochastic differential equations

- Map Fokker-Plank to equivalent stochastic differential equation
- Easy to simulate!!

$$\partial_t P(\boldsymbol{\alpha}) = -\partial_{\alpha_i} A_i(\boldsymbol{\alpha}) P(\boldsymbol{\alpha}) + \partial_{\alpha_i} \partial_{\alpha_j} \frac{1}{2} D_{ij}(\boldsymbol{\alpha}) P(\boldsymbol{\alpha})$$

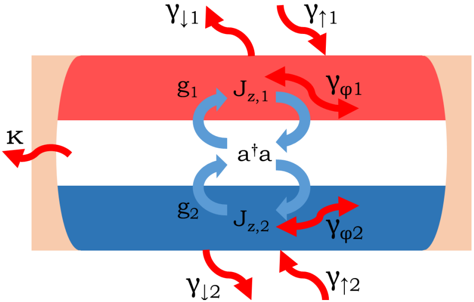
$$\Downarrow$$

$$d\boldsymbol{\alpha} = \mathbf{A}(\boldsymbol{\alpha})dt + \mathbf{B}^{(i)}(\boldsymbol{\alpha})d\mathbf{W}^{(i)}$$

- Noise matrix: $\mathbf{B}\mathbf{B}^T = \mathbf{D}$
- Normal-ordered expectation values: $\langle \hat{a}^{\dagger p} \hat{a}^q \rangle (t) = \langle \alpha_*^p \alpha^q \rangle (t)$

System of interest: single mode laser

$$\frac{d}{dt}\rho = -i\omega_c[\hat{a}^\dagger\hat{a},\rho] + \kappa\mathcal{D}[\hat{a}]\rho - i\frac{\omega_q}{2}[\hat{J}_z,\rho] + \frac{g}{\sqrt{N}}[\hat{a}^\dagger\hat{J}_- - \hat{a}\hat{J}_+,\rho]$$
$$+\sum_n \gamma_\downarrow\mathcal{D}[\hat{\sigma}_{-,n}]\rho + \gamma_\uparrow\mathcal{D}[\hat{\sigma}_{+,n}]\rho + \frac{\gamma_\phi}{2}\mathcal{D}[\hat{\sigma}_{n,z}]\rho.$$



- $\hat{J} = \sum_n^N \hat{\sigma}_n, \quad \Delta_q = \omega_q - \omega_c$
- $\gamma_{||} = \gamma_\uparrow + \gamma_\downarrow, \;\; \gamma_h = \gamma_{||} + 2\gamma_\phi$
- $\Gamma_p = \gamma_\uparrow - \gamma_\downarrow$

Transformation to phase space

$$\frac{d}{dt}\rho = -i\omega_c[\hat{a}^\dagger \hat{a}, \rho] + \kappa\mathcal{D}[\hat{a}]\rho - i\frac{\omega_q}{2}[\hat{J}_z, \rho] + \frac{g}{\sqrt{N}}[\hat{a}^\dagger \hat{J}_- - \hat{a}\hat{J}_+, \rho] + \sum_n \gamma_\downarrow \mathcal{D}[\hat{\sigma}_{-,n}]\rho + \gamma_\uparrow \mathcal{D}[\hat{\sigma}_{+,n}]\rho + \frac{\gamma_\phi}{2}\mathcal{D}[\hat{\sigma}_{n,z}]\rho.$$

$$\Downarrow$$

$$\frac{d}{dt}P(\boldsymbol{\alpha}) = L(\boldsymbol{\alpha}, \partial_{\boldsymbol{\alpha}})P(\boldsymbol{\alpha})$$

- $\{\hat{a}, \hat{a}^\dagger, \hat{J}_-, \hat{J}_+, \hat{J}_z\} \quad \rightarrow \quad \{\alpha, \alpha_*, v, v_*, m\} = \boldsymbol{\alpha}$
- Aside: Non-FP terms from discrete spin $\hat{J}_z$

*Carmichael [1999]

Scaled Fokker-Plank equation

- $\{\hat{a}, \hat{a}^\dagger, \hat{J}_-, \hat{J}_+, \hat{J}_z\} \quad \rightarrow \quad \{\alpha, \alpha_*, v, v_*, m\}$
- $\alpha = \sqrt{N}\bar{\alpha}, \quad v = N\bar{v}, \quad m = N\bar{m} \quad \rightarrow \quad \partial_m^n = N^{-n}\partial_{\bar{m}}$

$$\frac{d}{dt}P'(\bar{\boldsymbol{\alpha}}) = \left[-\partial_{\bar{\alpha}_i}A_i(\bar{\boldsymbol{\alpha}}) + \partial_{\bar{\alpha}_i}\partial_{\bar{\alpha}_j}\frac{N^{-1}}{2}D_{i,j}(\bar{\boldsymbol{\alpha}}) + O(N^{-2})\right]P'(\bar{\boldsymbol{\alpha}}, t)$$

$$\mathbf{A} = \left. \begin{pmatrix} -\frac{\kappa}{2}\bar{\alpha} + g\bar{v} \\ -\frac{\kappa}{2}\bar{\alpha}_* + g\bar{v}_* \\ -\left(i\Delta_q + \frac{\gamma_h}{2}\right)\bar{v} + g\bar{m}\bar{\alpha} \\ -\left(i\Delta_q + \frac{\gamma_h}{2}\right)\bar{v}_* + g\bar{m}\bar{\alpha}_* \\ \Gamma_p - \gamma_{||}\bar{m} - 2g(\bar{\alpha}\bar{v}_* + \bar{\alpha}_*\bar{v}) \end{pmatrix} \right\} = \text{Maxwell-Bloch Eqns.}$$

Scaled Fokker-Plank equation

- $\{\hat{a}, \hat{a}^\dagger, \hat{J}_-, \hat{J}_+, \hat{J}_z\} \quad \rightarrow \quad \{\alpha, \alpha_*, v, v_*, m\}$
- $\alpha = \sqrt{N}\bar{\alpha}, \quad v = N\bar{v}, \quad m = N\bar{m} \quad \rightarrow \quad \partial_m^n = N^{-n}\partial_{\bar{m}}$

$$\frac{d}{dt}P'(\bar{\boldsymbol{\alpha}}) = \left[-\partial_{\bar{\alpha}_i} A_i(\bar{\boldsymbol{\alpha}}) + \partial_{\bar{\alpha}_i}\partial_{\bar{\alpha}_j}\frac{N^{-1}}{2}D_{i,j}(\bar{\boldsymbol{\alpha}}) + O(N^{-2})\right] P'(\bar{\boldsymbol{\alpha}}, t)$$

$$\mathbf{D} = \begin{pmatrix} 2g\bar{\alpha}\bar{v} & \gamma_\uparrow + \gamma_\phi(1+\bar{m}) & -\gamma_\uparrow \bar{v} \\ \gamma_\uparrow + \gamma_\phi(1+\bar{m}) & 2g\bar{\alpha}_*\bar{v}_* & -\gamma_\uparrow \bar{v}_* \\ -\gamma_\uparrow \bar{v} & -\gamma_\uparrow \bar{v}_* & 2\left(\gamma_{||} - \Gamma_p \bar{m} - 2g(\bar{\alpha}\bar{v}_* + \bar{\alpha}_*\bar{v})\right) \end{pmatrix}$$

- In spin subspace
- Quantum noise correction, from 2nd order moments
- "Spontaneous emission noise"

Fluctuation expansion

- Decouple motion of macroscopic mean ($\bar{a}_i$) from fluctuations (d_i):
- Look at fluctuations about semiclassics
- $\bar{\alpha} = \bar{a}(t) + N^{-\frac{1}{4}} d \quad \bar{v} = \bar{J}_-(t) + N^{-\frac{1}{4}} \nu \quad \bar{m} = \bar{J}_z(t) + N^{-\frac{1}{4}} \mu$

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Semiclassics:

$$\left.\begin{aligned} \frac{d\bar{a}}{dt} &= -\frac{\kappa}{2}\bar{a} + g\bar{J}_- \\ \frac{d\bar{J}_-}{dt} &= -(i\Delta_q + \frac{\gamma_h}{2})\bar{J}_- + g\bar{J}_z\bar{a} \\ \frac{d\bar{J}_z}{dt} &= \Gamma_p - \gamma_{||}\bar{J}_z - 2g\left(\bar{J}_-^*\bar{a} + \bar{J}_-\bar{a}^*\right) \end{aligned}\right\} = \text{Maxwell-Bloch Eqns.}$$

Second order van Kampen's expansion

- $\bar{\alpha} = \bar{a}(t) + N^{-\frac{1}{4}} d \quad \bar{v} = \bar{J}_-(t) + N^{-\frac{1}{4}} \nu \quad \bar{m} = \bar{J}_z(t) + N^{-\frac{1}{4}} \mu$
- Retain nonlinear terms:

$$\frac{d}{dt} P'(\mathbf{d}) = \left[-\partial_{d_i} A_i(\mathbf{d}, t) + \partial_{d_i} \partial_{d_j} \frac{N^{-\frac{1}{2}}}{2} D_{i,j}(\mathbf{d}, t) + O(N^{-\frac{5}{4}}) \right] P'(\mathbf{d})$$

- Both $\mathbf{A}$ and $\mathbf{D}$ depend on fluctuations!
- Non-FP terms vanish as $O(N^{-\frac{5}{4}})$

Equivalent SDEs

$$\frac{d}{dt}P'(\mathbf{d}) = \Big[-\partial_{d_i}A_i(\mathbf{d},t) + \partial_{d_i}\partial_{d_j}\frac{N^{-\frac{1}{2}}}{2}D_{i,j}(\mathbf{d},t)\Big]P'(\mathbf{d})$$

$$\Downarrow$$

$$d\mathbf{d} = \mathbf{A}(\mathbf{d},t)dt + N^{-\frac{1}{4}}\mathbf{B}^{(i)}(\mathbf{d},t)d\mathbf{W}^{(i)}$$

$$d\mathrm{d} = \left(-\frac{\kappa}{2}\mathrm{d} + g\nu\right)dt + \sqrt{i\kappa_{\uparrow}}N^{-\frac{1}{4}}(dW_d - i dW_{d_*})$$

$$d\nu = \left(-\frac{\gamma_h}{2}\nu + g(\bar{J}_z\mathrm{d} + \bar{a}\mu + N^{-\frac{1}{4}}\mathrm{d}\mu)\right)dt + N^{-\frac{1}{4}}\left[\mathbf{B}^{(i)}d\mathbf{W}^{(i)}\right]_{\nu}$$

$$d\mu = \left(-\gamma_{||}\mu - 2g(\bar{J}_{-}{}^{*}\mathrm{d} + \bar{a}\nu_{*} + N^{-\frac{1}{4}}\mathrm{d}\nu_{*} + h.c.)\right)dt + N^{-\frac{1}{4}}\left[\mathbf{B}^{(i)}d\mathbf{W}^{(i)}\right]_{\mu}$$

- Noise matrices: $\mathbf{B}\mathbf{B}^T = \mathbf{D}(\bar{\mathbf{a}} + N^{-\frac{1}{4}}\mathbf{d})$
- Semiclassical variables, nonlinear contributions

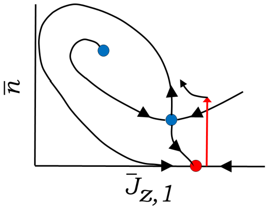
Calculating results

- First solve Maxwell-Bloch: $\frac{d}{dt}\bar{\mathbf{a}} = \mathbf{F}(\bar{\mathbf{a}})$
- Then non-linear SDEs: $d\mathbf{d} = \mathbf{A}(\mathbf{d}, t)dt + N^{-\frac{1}{4}}\mathbf{B}^{(i)}(\mathbf{d}, t)d\mathbf{W}^{(i)}$
- $\bar{\alpha} = \bar{a}(t) + N^{-\frac{1}{4}}d \quad \bar{v} = \bar{J}_{-}(t) + N^{-\frac{1}{4}}\nu \quad \bar{m} = \bar{J}_z(t) + N^{-\frac{1}{4}}\mu$
- $\langle\hat{a}\rangle = \sqrt{N}\langle\bar{\alpha}\rangle \quad \langle\hat{n}\rangle = N\langle\bar{\alpha}_*\bar{\alpha}\rangle \quad \langle\hat{J}_z\rangle = N\langle\bar{m}\rangle$

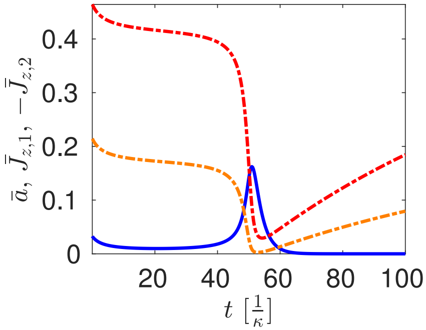
Excitable regime semiclassical results

Excitable regime: $6 < A < 7$

- One stable solution with $\bar{a} = 0$, two unstable
- Basin of stability, large excursions if escape



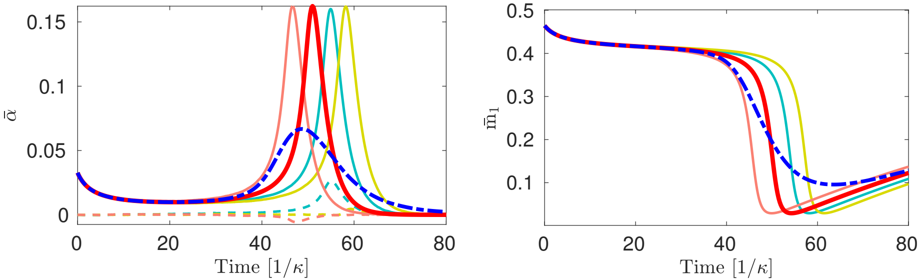
- Add quantum noise!
- Pulse shifting
- Noise-driven excitation
- $A = \frac{4|g_1|^2\Gamma_{p,1}}{\kappa\gamma_{||}\gamma_h}$



Parameters

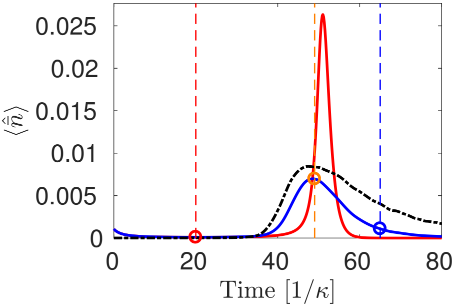
$|\frac{g_2}{g_1}|^2=2,\ \gamma_h=100\kappa,\ \gamma_{||}=10^{-2}\kappa,\ \frac{4|g_2|^2\Gamma_{p,2}}{\kappa\gamma_{||}\gamma_h}=-6$

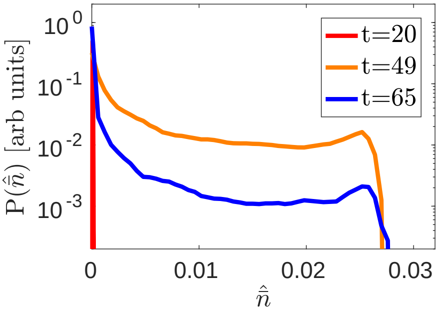
Fluctuation trajectories



- $A = 6.5$, $N = 10^9$, Initial pulse
- Noise changes escape time for each trajectory
- Dramatically broadens average response

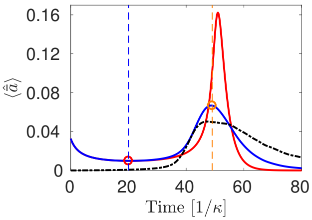
Pulse photon number

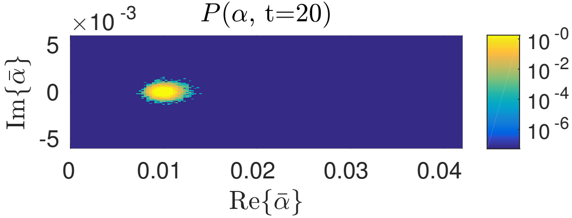


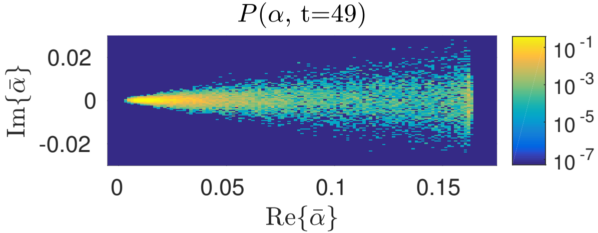


- Earlier pulse, lower mean photon number
- Large $\sigma_{\hat{n}}$ during pulse
- Bimodal number distribution

P-distributions

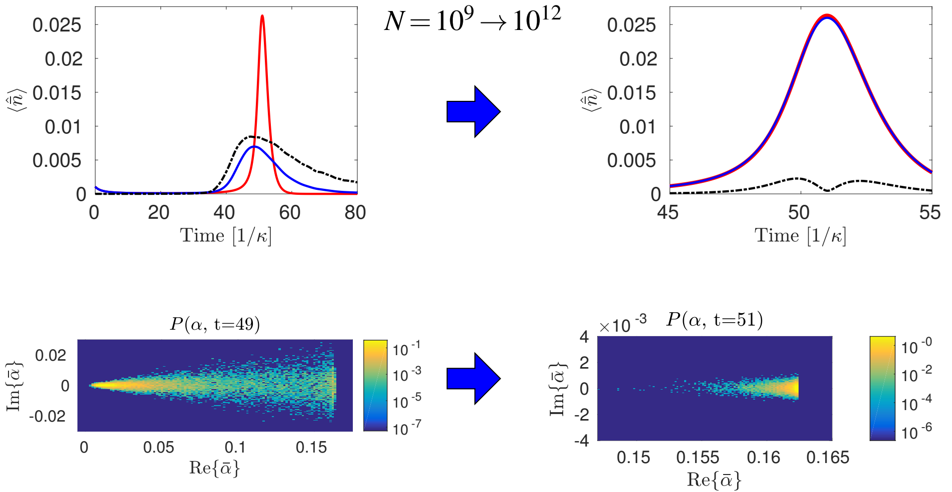




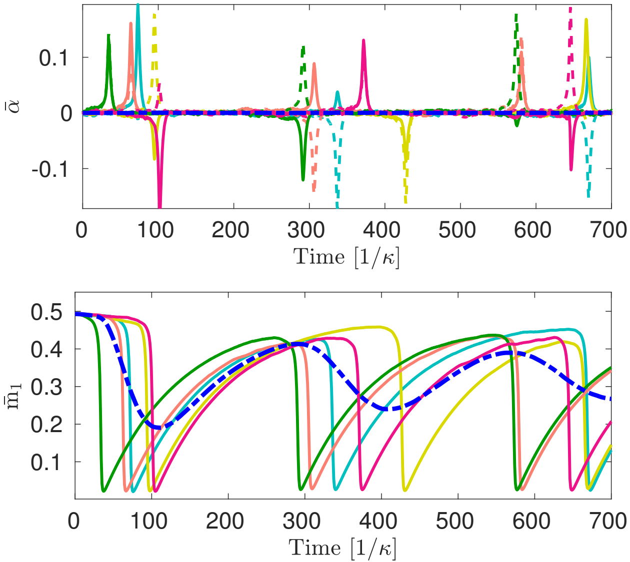


- Non-classical distribution
- Equivalent to Wigner function
- 15,000 trajectories

System-size scaling

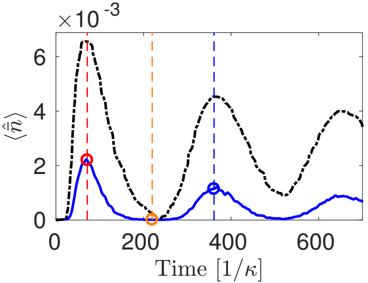


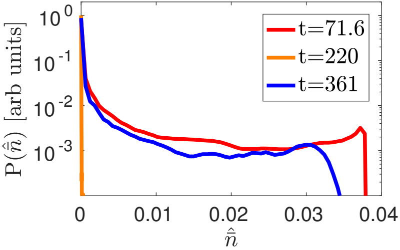
Noise-driven excitation

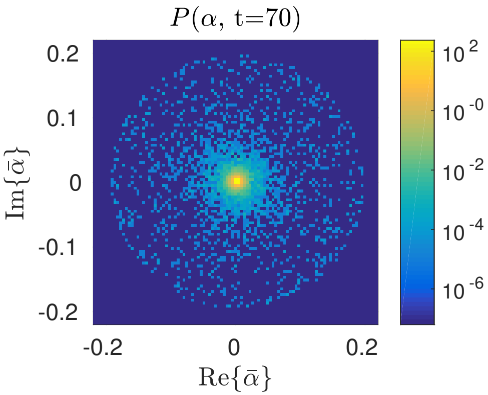


- $A = 6.9$, $N = 10^7$
- No input pulse
- periodic spiking due to noise!

Noise-driven excitation







Summary and Conclusions

- Study of quantum noise in driven-dissipative quantum systems
- Developed systemic phase-space approach
- Quantum noise modifies excitable response pulse
- Noise driven excitation, blurs transition to self-pulsing phase
- Highlights important role of quantum noise at mesoscopic scales

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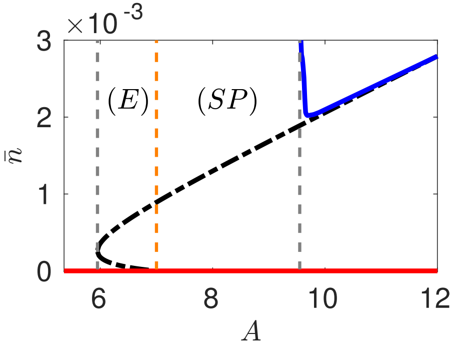
Phase diagram

$$\frac{d\bar{n}}{dt} = \left(-\kappa + \chi|g_1|^2\bar{J}_{z,1} + \chi|g_2|^2\bar{J}_{z,2}\right)\bar{n} \quad \frac{d\bar{J}_{z,i}}{dt} = \Gamma_{p,i} - \left(\gamma_{||} + 2\chi|g_i|^2\bar{n}\right)\bar{J}_{z,i}$$

- $\gamma_h \gg \kappa \gg \gamma_{||}, \quad g_2 > g_1$
- $\Gamma_{p,1} > 0 > \Gamma_{p,2}$
- 3 steady states
- excitable phase (E)
- self-pulsing phase (SP)
- $A = \frac{4|g_1|^2\Gamma_{p,1}}{\kappa\gamma_{||}\gamma_h}$

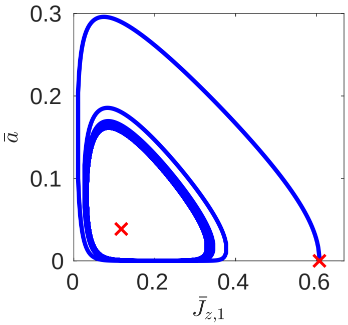


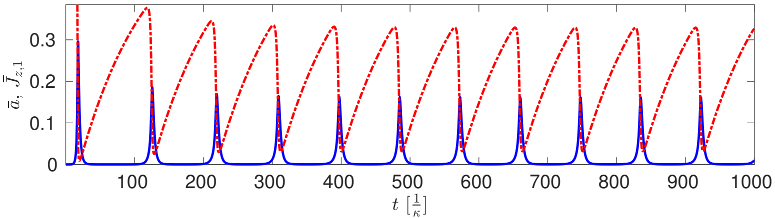
*Dubbeldam et al. [1999]

Semiclassical results

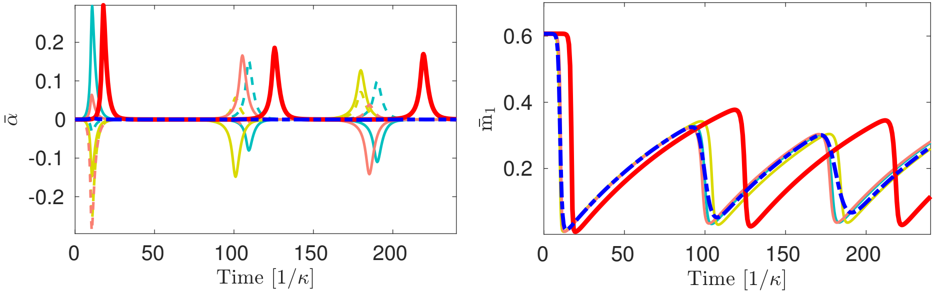
Self-Pulsing regime: $7 < A < 10$

- $\bar{a} = 0$ becomes unstable
- Orbit in phase space
- Output is pulse train



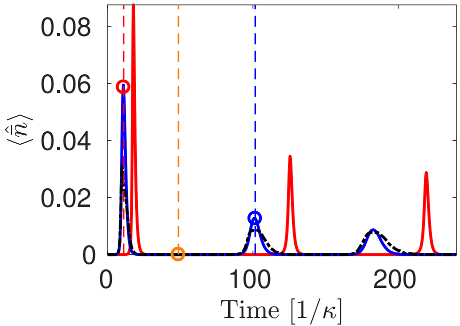


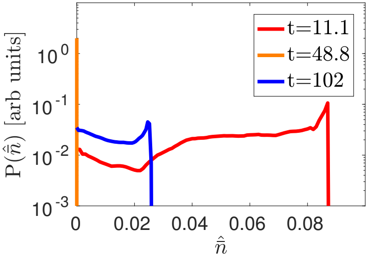
Trajectories

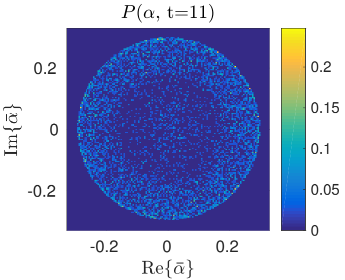


- $A = 8.5$, $N = 10^9$
- Pulse period shortened due to noise
- Field phase is random, $\langle \hat{\bar{a}} \rangle = 0$

Pulse photon number

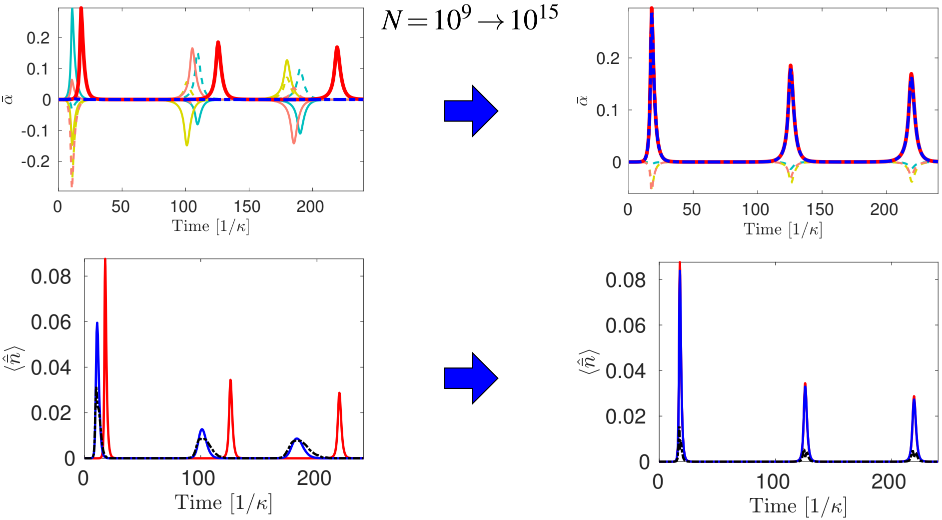






- Earlier pulse, mean broadens
- Bimodal number distribution

System size scaling



Quantum noise

Schrödinger: $\hat{H}|\psi\rangle = i\partial_t|\psi\rangle$

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Unitary, so what fluctuates?

- Correlations lead to noise during measurement
- HO: $\frac{d}{dt}\rho = \mathcal{L}\rho = -i\omega[\hat{a}^\dagger\hat{a}, \rho] + \kappa(\bar{n}_B + 1)\mathcal{D}[\hat{a}]\rho + \kappa\bar{n}_B\mathcal{D}[\hat{a}^\dagger]\rho$
- $\hat{q} = \sqrt{\frac{1}{2\omega}}(\hat{a} + \hat{a}^\dagger), \quad \Delta q\Delta p = \bar{n}_B + \frac{1}{2}$
- Quantum noise at $T = 0$, $\langle\hat{q}^2\rangle \neq \langle\hat{q}\rangle\langle\hat{q}\rangle$!
- Differences in Heisenberg and semiclassical equations
- Arises from higher-order correlations: $\langle\hat{a}^\dagger\hat{a}\rangle \neq \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle$

Positive-P Representation

- Write density matrix in off-diagonal basis of coherent states
- $\rho = \int d\alpha^2 \alpha_*^2 P(\alpha, \alpha_*) \frac{|\alpha\rangle\langle(\alpha_*)^*|}{\langle(\alpha_*)^*|\alpha\rangle}$
- $\hat{\Lambda}(\alpha, \alpha_*) = \frac{|\alpha\rangle\langle(\alpha_*)^*|}{\langle(\alpha_*)^*|\alpha\rangle} = \frac{e^{\alpha\hat{a}^\dagger}|0\rangle\langle 0|e^{\alpha_*\hat{a}}}{e^{\alpha\alpha_*}}$
- $\mathcal{L}\rho = -i[\hat{H}, \rho] + \sum_i \gamma_i \mathcal{D}[\hat{A}_i]\rho$
- $\partial_t \rho = \mathcal{L}\rho \quad \rightarrow \quad \int d\boldsymbol{\alpha} \partial_t P(\boldsymbol{\alpha})\hat{\Lambda}(\boldsymbol{\alpha}) = \int d\boldsymbol{\alpha} P(\boldsymbol{\alpha})\mathcal{L}[\hat{\Lambda}(\boldsymbol{\alpha})]$
- $\int d\boldsymbol{\alpha} P(\boldsymbol{\alpha})\mathcal{L}[\hat{\Lambda}(\boldsymbol{\alpha})] \Rightarrow \partial_t P(\boldsymbol{\alpha}) = L(\boldsymbol{\alpha}, \partial_{\boldsymbol{\alpha}})P(\boldsymbol{\alpha})$

Phase space transformation

- $\int d\boldsymbol{\alpha} \partial_t P(\boldsymbol{\alpha}) \hat{\Lambda}(\boldsymbol{\alpha}) = \int d\boldsymbol{\alpha} P(\boldsymbol{\alpha}) \mathcal{L}[\hat{\Lambda}(\boldsymbol{\alpha})]$
- Evaluate $\mathcal{L}[\hat{\Lambda}(\boldsymbol{\alpha})]$
- $\hat{a}\hat{\Lambda} = \alpha\hat{\Lambda} \qquad \hat{\Lambda}\hat{a}^\dagger = \alpha_*\hat{\Lambda} \qquad \hat{a}^\dagger\hat{\Lambda} = (\partial_\alpha + \alpha_*)\hat{\Lambda} \qquad \hat{\Lambda}\hat{a} = (\partial_{\alpha_*} + \alpha)\hat{\Lambda}$
- Result is a phase space equation
- $\int d\boldsymbol{\alpha} \partial_t P(\boldsymbol{\alpha}) \hat{\Lambda}(\boldsymbol{\alpha}) = \int d\boldsymbol{\alpha} P(\boldsymbol{\alpha}) L^+(\boldsymbol{\alpha}, \partial_{\boldsymbol{\alpha}}) \hat{\Lambda}(\boldsymbol{\alpha})$
- Integrate by parts
- $\partial_t P(\boldsymbol{\alpha}) = L(\boldsymbol{\alpha}, \partial_{\boldsymbol{\alpha}}) P(\boldsymbol{\alpha})$

Characteristic function

- $\chi(\zeta) = \mathrm{tr}\{\rho e^{i\zeta_* \hat{J}_+} e^{i\eta \hat{J}_z} e^{i\zeta \hat{J}_-}\} \equiv \mathrm{tr}\{\rho e^{i\boldsymbol{\zeta}\cdot\hat{\mathbf{J}}}\}$
- $\chi(\boldsymbol{\zeta}) = \int d\mathbf{v}\, \mathrm{tr}\{P(\mathbf{v})\hat{\Lambda} e^{i\boldsymbol{\zeta}\cdot\hat{\mathbf{J}}}\} = \int d\mathbf{v} P(\mathbf{v}) e^{i\boldsymbol{\zeta}\cdot\mathbf{v}}$
- $P((\mathbf{v}) = \int \frac{d\boldsymbol{\zeta}}{(2\pi)^4}\chi(\boldsymbol{\zeta})e^{-i\boldsymbol{\zeta}\cdot\mathbf{v}}$
- $\frac{d}{dt}\chi(\boldsymbol{\zeta}) = \mathrm{tr}\{\mathcal{L}\rho e^{i\boldsymbol{\zeta}\cdot\hat{\mathbf{J}}}\} = D(\boldsymbol{\zeta}, \partial_{\boldsymbol{\zeta}})\chi(\boldsymbol{\zeta})$
- $\frac{d}{dt}\chi(\boldsymbol{\zeta}) = \int d\mathbf{v} \frac{d}{dt} P(\mathbf{v}) e^{i\mathbf{v}\cdot\zeta} = \int d\mathbf{v} P(\mathbf{v}) D(\boldsymbol{\zeta}, \partial_{\boldsymbol{\zeta}}) e^{i\mathbf{v}\cdot\zeta}$
- $\int d\mathbf{v} e^{i\mathbf{v}\cdot\zeta} \frac{d}{dt} P(\mathbf{v}) = \int d\mathbf{v} e^{i\mathbf{v}\cdot\zeta} L(\mathbf{v}, \partial_{\mathbf{v}}) P(\mathbf{v})$

Single mode laser Chapman-Kolmogorov equation

$$\frac{d}{dt}P(\boldsymbol{\alpha},m)=\Big(-\partial_\alpha A_i(\boldsymbol{\alpha},m)+\partial_{\alpha_i}\partial_{\alpha_j}\frac{1}{2}D_{ij}(\boldsymbol{\alpha},m)+M(\boldsymbol{\alpha},m)\Big)P(\boldsymbol{\alpha},m)$$

$$\begin{pmatrix}A_\alpha\\A_v\end{pmatrix}=\begin{pmatrix}-\frac{\kappa}{2}\alpha+\frac{g}{\sqrt{N}}v\\-\left(i\Delta_q+\frac{\gamma_h}{2}\right)v+\frac{g}{\sqrt{N}}m\alpha\end{pmatrix}$$

$$\mathbf{D}=\begin{pmatrix}D_{vv}&D_{vv_*}\\D_{v_*v}&D_{v_*v_*}\end{pmatrix}=\begin{pmatrix}2\frac{g}{\sqrt{N}}mv&\gamma_\uparrow+\gamma_\phi(N+m)\\\gamma_\uparrow+\gamma_\phi(N+m)&2\frac{g}{\sqrt{N}}mv_*\end{pmatrix}$$

$$M(\boldsymbol{\alpha},m)=-\frac{g}{\sqrt{N}}\sum_{n=1}^{\infty}\frac{(-2\partial_m)^n}{n!}(\alpha v_*+\alpha_*v)+\frac{\gamma_\downarrow}{2}\sum_{n=1}^{\infty}\frac{(2\partial_m)^n}{n!}(N+m)+\gamma_\phi\partial_v\partial_{v_*}\sum_{n=1}^{\infty}\frac{(-2\partial_m)^n}{n!}(N+m)$$
$$+\frac{\gamma_\uparrow}{2}\left(\partial_v^2\partial_{v_*}v+\partial_v\partial_{v_*}^2v_*+\sum_{n=1}^{\infty}\frac{(-2\partial_m)^n}{n!}\left(N-m+\partial_vv+\partial_{v_*}v_*+\partial_v^2\partial_{v_*}^2(N+m)\right)\right)$$

Non–Fokker-Plank terms

$$\frac{d}{dt}\bar{P}(\bar{\boldsymbol{\alpha}}) = \Big[-\partial_{\bar{\alpha}_i}A_i(\bar{\boldsymbol{\alpha}}) + \partial_{\bar{\alpha}_i}\partial_{\bar{\alpha}_j}\frac{N^{-1}}{2}D_{i,j}(\bar{\boldsymbol{\alpha}}) + \sum_{k=3}^{\infty}\frac{(-\partial_{\bar{\boldsymbol{\alpha}}})^k}{N^{k-1}k!}L_k\Big]\bar{P}(\bar{\boldsymbol{\alpha}},t)$$

$$\frac{N^{-2}}{8}\partial^3_{\bar{\boldsymbol{\alpha}}}L_k = \frac{N^{-2}}{2}\Big(\partial_{\bar{v}}\partial_{\bar{v}_*}\partial_{\bar{m}}2\gamma_\phi(1+\bar{m}) - \partial_{\bar{v}}(2\partial^2_{\bar{m}} + \partial_{\bar{v}}\partial_{\bar{v}_*})\gamma_\uparrow\bar{v} + h.c.$$
$$+ \partial^3_{\bar{m}}\frac{4}{3}(\Gamma_p - \gamma_{||}\bar{m} + 2g(\bar{\alpha}\bar{v}_* + h.c.)\Big)$$

- Fluctuation expansion: $N^{-2} \rightarrow N^{3q-2}$, $N^{-3} \rightarrow N^{4q-3}$, etc

Fluctuation expansion

- Scaled FP fails for higher moments, powers of N hidden in $O(N^{-2})$ term
- decouple motion of macroscopic mean ($\bar{a}_i$) from fluctuations (d_i):
- $\bar{\alpha} = \bar{a}(t) + N^{-q} d \quad \bar{v} = \bar{J}_-(t) + N^{-q}\nu \quad \bar{m} = \bar{J}_z(t) + N^{-q}\mu \quad \partial_{\bar{\alpha}_i} \to N^q \partial_{d_i}$

$$\frac{d}{dt}\bar{P}(\mathbf{d},t) = \Big[N^q \left(\frac{d}{dt}\bar{a}_i - A_i(\bar{\mathbf{a}})\right)\partial_{d_i} - \partial_{d_i}\left(\sum_n \frac{(d_j \partial_{\bar{a}_j})^n}{N^{\frac{n}{2}-q} n!} A_i(\bar{\mathbf{a}})\right) + \partial_{d_i}\partial_{d_j}\frac{N^{2q-1}}{2} D_{i,j}(\bar{\mathbf{a}}(t) + N^{-q}\mathbf{d}) + O(N^{3q-2})\Big]\bar{P}(\mathbf{d},t)$$

- Choose $q = 1/2$

$q = \frac{1}{4}$ Second order van Kampen's expansion

$$\frac{d}{dt}\bar{P}(\mathbf{d}) = \Big[- \partial_{d_i} A_i(\mathbf{d}, t) + \partial_{d_i}\partial_{d_j} \frac{N^{-\frac{1}{2}}}{2} D_{i,j}(\mathbf{d}, t) \Big] \bar{P}(\mathbf{d})$$

$$\mathbf{A} = \begin{pmatrix} -\frac{\kappa}{2} d + g\nu \\ -\frac{\kappa}{2} d_* + g\nu_* \\ -\left(i\Delta_q + \frac{\gamma_h}{2}\right)\nu + g(\bar{J}_{z,} d + \bar{a}\mu + N^{-\frac{1}{4}}\mu d) \\ -\left(-i\Delta_q + \frac{\gamma_h}{2}\right)\nu_* + g(\bar{J}_z d_* + \bar{a}^\dagger \mu + N^{-\frac{1}{4}}\mu d_*) \\ -\gamma_{||}\mu - 2g(\bar{J}_+ d + \bar{a}\nu_* + N^{-\frac{1}{4}}\nu_* d + h.c.) \end{pmatrix}$$

$$\mathbf{D}^\nu = \begin{pmatrix} 2g(\bar{J}_- + N^{-\frac{1}{4}}\nu)(\bar{a} + N^{-\frac{1}{4}} d) & \gamma_\uparrow + \gamma_\phi(1 + \bar{J}_z + N^{-\frac{1}{4}}\mu) & -\gamma_\uparrow(\bar{J}_- + N^{-\frac{1}{4}}\nu) \\ \gamma_\uparrow + \gamma_\phi(1 + \bar{J}_z + N^{-\frac{1}{4}}\mu) & 2g(\bar{J}_+ + N^{-\frac{1}{4}}\nu_*)(\bar{a}^\dagger + N^{-\frac{1}{4}} d_*) & -\gamma_\uparrow(\bar{J}_+ + N^{-\frac{1}{4}}\nu_*) \\ -\gamma_\uparrow(\bar{J}_- + N^{-\frac{1}{4}}\nu) & -\gamma_\uparrow(\bar{J}_+ + N^{-\frac{1}{4}}\nu_*) & 2\big(\gamma_{||} - \Gamma_p(\bar{J}_z + N^{-\frac{1}{4}}\mu) - 2g(\bar{J}_+ + N^{-\frac{1}{4}}\nu_*)(\bar{a} + N^{-\frac{1}{4}} d) + h.c\big)\big) \end{pmatrix}$$

Noise Matrices

- $\bar{\alpha} = \bar{a}(t) + N^{-q}d \quad \bar{v} = \bar{J}_-(t) + N^{-q}\nu \quad \bar{m} = \bar{J}_z(t) + N^{-q}\mu$

$$\mathbf{B}_i^{\gamma} = \begin{pmatrix} \sqrt{\frac{i(\gamma_{\uparrow,i}+\gamma_{\phi}(1+\bar{m})}{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} & 0 \\ 0 & \sqrt{2}\sqrt{\gamma_{||} - (\gamma_{\uparrow,i} - \gamma_{\downarrow,i})\bar{m} - 2g_i(\bar{v}_*\bar{\alpha} + h.c)} \end{pmatrix}$$

$$\mathbf{B}_i^{g} = \sqrt{2g_i} \begin{pmatrix} \sqrt{\bar{v}\bar{\alpha}} & 0 & 0 \\ 0 & \sqrt{\bar{v}_*\bar{\alpha}_*} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B}_i^{-} = \sqrt{\frac{i\gamma_{\uparrow,i}\bar{v}}{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{pmatrix} \quad \mathbf{B}_i^{+} = \sqrt{\frac{i\gamma_{\uparrow,i}\bar{v}_*}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix}$$

Comparison with exact Heisenberg equations

- Scale and displace operator Heisenberg equations
- $\frac{\hat{a}}{\sqrt{N}} = \bar{a} + N^{-\frac{1}{4}}\hat{d} \quad \frac{\hat{J_-}}{N} = \bar{J_-} + N^{-\frac{1}{4}}\hat{\nu} \quad \frac{\hat{J_z}}{N} = \bar{J_z} + N^{-\frac{1}{4}}\hat{\mu}$
- Exact agreement for all 1st and 2nd order moments with Ito SDEs
- $\frac{d}{dt}\langle\hat{\mathrm{d}}\rangle = -\frac{\kappa}{2}\langle\hat{\mathrm{d}}\rangle + g\langle\hat{\nu}\rangle$
- $\frac{d}{dt}\langle\hat{\mathrm{d}}^\dagger\hat{\mathrm{d}}\rangle = N^{-\frac{1}{2}}\kappa_\uparrow - \kappa\langle\hat{\mathrm{d}}^\dagger\hat{\mathrm{d}}\rangle + g(\langle\hat{\mathrm{d}}\hat{\nu}^\dagger\rangle + h.c.)$
- $\frac{d}{dt}\langle\hat{\nu}\rangle = -\frac{\gamma_h}{2}\langle\hat{\nu}\rangle + g(\bar{J}_z\langle\hat{\mathrm{d}}\rangle + \bar{a}\langle\hat{\mu}\rangle + N^{-\frac{1}{4}}\langle\hat{\mathrm{d}}\hat{\mu}\rangle)$
- $\frac{d}{dt}\langle\hat{\nu}^\dagger\hat{\nu}\rangle = N^{-\frac{1}{2}}\gamma_\uparrow + N^{-\frac{1}{2}}\gamma_\phi(1 + \bar{J}_z + N^{-\frac{1}{4}}\langle\hat{\mu}\rangle) - \gamma_h\langle\hat{\nu}^\dagger\hat{\nu}\rangle$
  $+g(\bar{J}_z\langle \mathrm{d}\hat{\nu}^\dagger\rangle + \bar{\mathrm{a}}^*\langle\hat{\nu}^\dagger\hat{\mu}\rangle + \mathrm{N}^{-\frac{1}{4}}\langle\hat{\mathrm{d}}\hat{\nu}^\dagger\hat{\mu}\rangle + \mathrm{h.c.})$
- Differences in 3rd order moments of $\mathcal{O}(N^{-\frac{5}{2}})$
- Results in $\mathcal{O}(N^{-2})$ correction to $\bar{m}$

Numerics

Ito calculus: $\int_{t_0}^{t} f(x(t'))dW(t') = \lim_{n\to\infty} \sum_i^n f(x(t_{i-1}))\,(W(t_i) - W(t_{i-1}))$

- $\langle \int_{t_0}^{t} f(x(t'))dW(t') \rangle = 0$
- Explicit Euler:
  $\mathbf{x}(t_i) = \mathbf{x}(t_{i-1}) + \mathbf{A}(\mathbf{x}(t_{i-1}), t_{i-1})\Delta t + \mathbf{B}(\mathbf{x}(t_{i-1}), t_{i-1})\sqrt{\Delta t}\phi_{i-1}$

Stratonovich calculus:

$\int_{t_0}^{t} f(x(t'))dW(t') = \lim_{n\to\infty} \sum_i^n f(x(\frac{t_i+t_{i-1}}{2}))\,(W(t_i) - W(t_{i-1}))$

- $\langle \int_{t_0}^{t} f(x(t'))dW(t') \rangle \neq 0$
- $A_i^{\text{strat}} = A_i^{\text{Ito}} - \frac{1}{2}\sum_{jk} B_{kj}\partial_{x_k} B_{ij}$
- Semi-Implicit:
  $\mathbf{x}(t_i) = \mathbf{x}(t_{i-1}) + \mathbf{A}(\frac{\mathbf{x}(t_i)+\mathbf{x}(t_{i-1})}{2}, t_{i-1})\Delta t + \mathbf{B}(\frac{\mathbf{x}(t_i)+\mathbf{x}(t_{i-1})}{2}, t_{i-1})\sqrt{\Delta t}\phi_{i-1}$

Ito's formula

Ito calculus

- Wiener increment: $dW_i^{(k)} dW_j^{(l)} = \delta_{ij}\delta_{kl}dt, \quad \langle W \rangle = 0$
- $\int_{t_0}^{t} f(x(t'))dW(t') = \lim_{n\to\infty} \sum_i^n f(x(t_{i-1}))\left(W(t_i) - W(t_{i-1})\right)$
- $\langle \int_{t_0}^{t} f(x(t'))dW(t') \rangle = 0$

$$d\mathbf{d} = \mathbf{A}(\mathbf{d},t)dt + \mathbf{B}^{(i)}(\mathbf{d},t)d\mathbf{W}^{(i)}$$

$$df(\mathbf{x}) = \left(A_i(\mathbf{x})\partial_{x_i} f(\mathbf{x}) + \frac{1}{2}\left[\mathbf{B}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\right]_{ij}\partial_{x_i}\partial_{x_j} f(\mathbf{x})\right)dt + B_{ij}(\mathbf{x})\partial_{x_i} f(\mathbf{x})dW_j$$