Coleman-Weinberg Mechanism in a Gravitational Weyl invariant theory

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Symmetry and Conformal Symmetry

- Symmetry has proved to be a fruitful principle in Physics.

  Besides aesthetic appeal it is also useful: guides us in constructing our models e.g. Lorentz and gauge invariance are at the heart of the standard model.

- Spontaneous symmetry breaking (SSB): breaks part of the symmetry without destroying renormalizability. Not true if the symmetry were broken by hand.

- Conformal symmetry: local scale invariance
  - 15 parameter conformal group. Poincare is a subgroup.
  - (classical) action of standard model already scale invariant (except for the Higgs mass term).
  - Conformal (Weyl) invariance requires gravity.

- Recent paper by C.T. Hill : Is the Higgs Boson Associated with Coleman-Weinberg Dynamical Symmetry Breaking? They start with classical scale invariant action.

1) How does one introduce a length scale in a (classically) scale invariant theory?

add quantum corrections $\Rightarrow$ renormalization scale M

2) How does one generate SSB when there is no mass term?

It is radiatively induced after quantum corrections
(dimensional transmutation: Coleman-Weinberg scenario)

The VEV is obtained from the effective potential after one-loop quantum corrections.

We will apply this to the magnetic monopole conformally coupled to gravity. This will build on previous classical work (Paranjape et al, 2006).

Action for the conformally coupled magnetic monopole

$$S = \int d^4x\,\sqrt{-g}\left(C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} - \frac{1}{4e^2}\,F^{\,a}_{\mu\nu}\,F^{\mu\nu}_{a} + D_\mu\phi^a\,D^\mu\phi_a + \frac{1}{6}\,R\,\phi^a\phi_a - \frac{\lambda}{4!}(\phi^a\,\phi_a)^2\right)$$

No "mass term" $\mu^2\phi^2$. "Replaced by" term $R\,\phi^2$

Invariant under the conformal transformation

$$g_{\mu\nu} \to \Omega^2(x)\,g_{\mu\nu} \text{ and } \phi^a \to \Omega^{-1}(x)\phi^a$$

Spontaneous symmetry breaking (SSB) via gravitation.

$$\phi_0^2 = \frac{2R}{\lambda}$$

AdS background, R=positive constant in our convention

Magnetic monopole in AdS space (Paranjape et al, 2006).

Quantum Corrections and Trace Anomaly

$W = W_{div} + W_{ren}$ (divergent and finite part of the effective action)

One loop divergent part of effective action (local part)

$$W_{div} = \frac{1}{n-4} \int d^4x \sqrt{g} \;\; \mathrm{tr}\, \hat{a}_2(x,x) \qquad (n \to 4)$$

Trace is no longer zero (trace anomaly)

$$< T^{\mu}_{\mu} > = \frac{a_2(x)}{16\pi^2} - [E]$$
$$= \frac{1}{16\pi^2} \left\{ \frac{1}{60} \left(R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - R_{\mu\nu} R^{\mu\nu} + \Box R \right) - \frac{1}{6} F^2 + \frac{11}{3} \frac{\lambda^2}{4!} \phi^4 \right\} - [E]$$

where composite operator $$[E] \equiv \frac{\phi^a}{\sqrt{-g}} \frac{\delta W_{ren}}{\delta \phi^a}$$

Asymptotic value of EMT: the spacetime is AdS space [SO(2,3) symmetry]

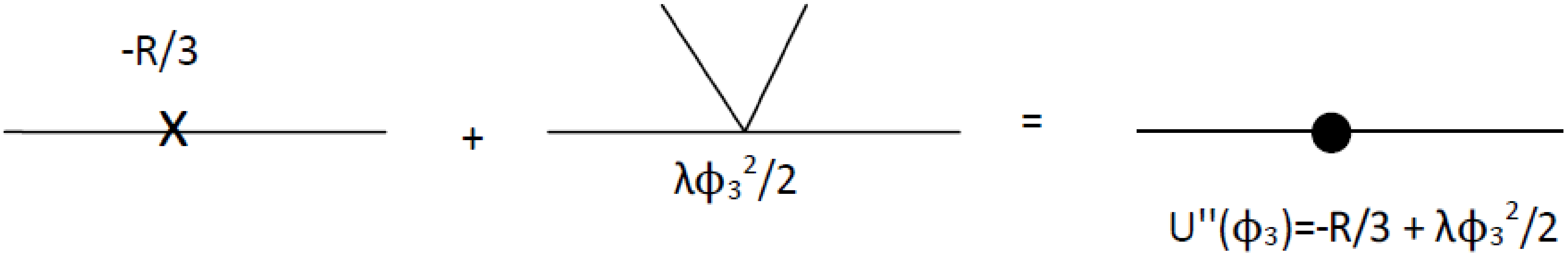
$$< T_{\mu\nu} > \equiv \frac{2}{\sqrt{-g}} \frac{\delta W_{ren}}{\delta g^{\mu\nu}} = \frac{1}{4} g_{\mu\nu} < T^{\mu}_{\mu} >$$

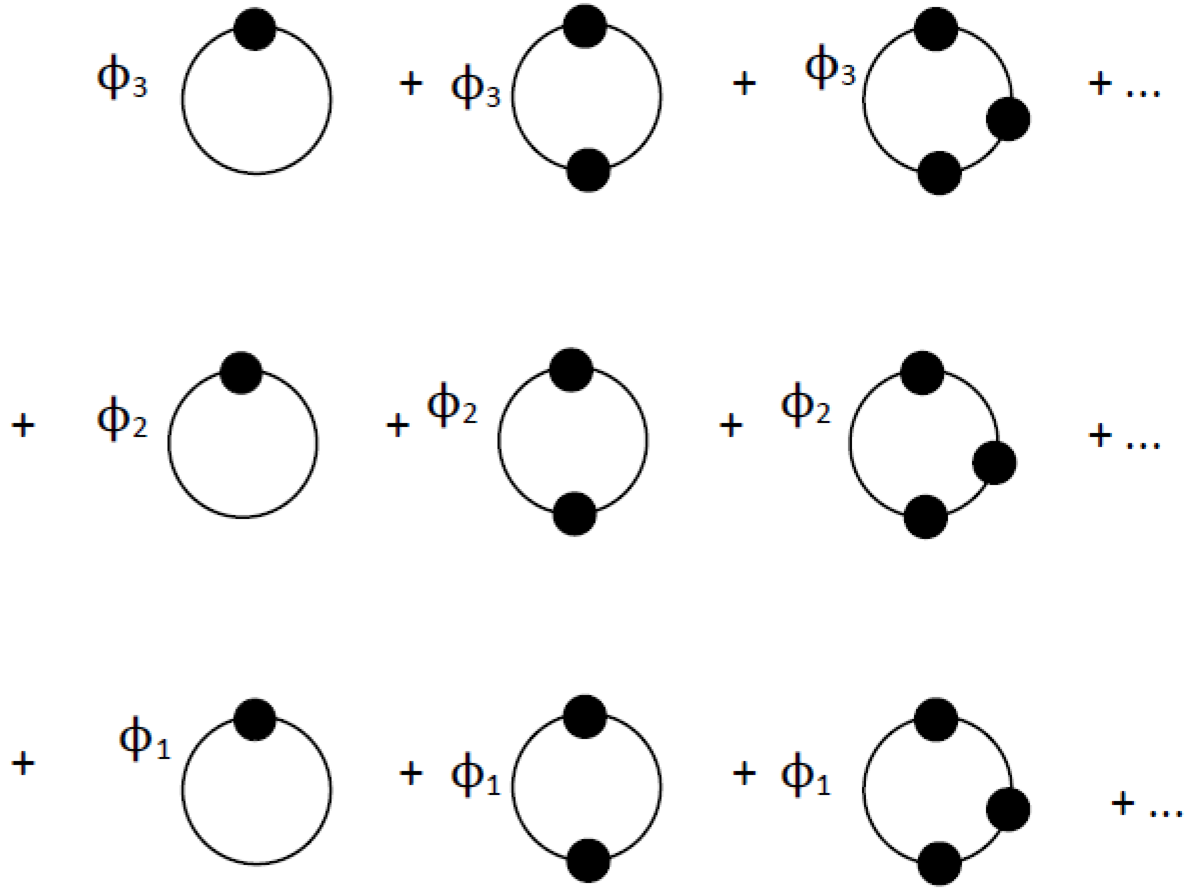
Feynman Diagrams for a Triplet of Scalar fields
R ϕ^2 and $\lambda\phi^4$ interactions

The classical potential is given by

$$U = \frac{\lambda}{4!}\phi^4 - \frac{1}{6}R\,\phi^2$$

There are two vertices which can be combined into a single vertex:





λ -> λ/3

λ -> λ/3

$$\phi_c(x) = \left. \frac{< 0|\phi_3(x)|0 >}{< 0|0 >} \right|_J .$$

propagator is massless

$$V = i \int \frac{d^4k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n} \left[\frac{-R/3 + \lambda\phi_c^2/2}{k^2 + i\epsilon} \right]^n$$

$$= \frac{1}{16\pi^2} \int_0^{\Lambda} k_E^3 \ln \left[1 + \frac{-R/3 + \lambda\phi_c^2/2}{k_E^2} \right]$$

The Renormalization Scale M

$$V_1 = V + 2\,V[\lambda \to \lambda/3]$$

V_1 divergent: add counterterms

$$V_{tot} = \frac{\lambda}{4!}\,\phi_c^4 - \frac{1}{6}R\,\phi_c^2 + V_1 + A\phi_c^2 + B\phi_c^4$$

Apply renormalization conditions to find constants A and B

$$\lambda = \left.\frac{d^4V_{tot}}{d\phi_c^4}\right|_{\phi_c=M} \quad \text{and} \quad -\frac{R}{3} = \left.\frac{d^2V_{tot}}{d\phi_c^2}\right|_{\phi_c=0} .$$

λ=λ(M) ⇒ running coupling constant

Effective Potential and Dimensional Transmutation

$$V_{eff} = \frac{\lambda}{4!}\,\phi_c^4 - \frac{1}{6}R\,\phi_c^2 + \frac{1}{1152\pi^2}\Bigg[18\left(-\frac{R}{3}+\frac{\lambda\phi_c^2}{2}\right)^2 \ln\left[\frac{2R-3\lambda\phi_c^2}{2R}\right] + 36\left(-\frac{R}{3}+\frac{\lambda\phi_c^2}{6}\right)^2 \ln\left[\frac{2R-\lambda\phi_c^2}{2R}\right] + 5R\,\lambda\,\phi_c^2 + \lambda^2\phi_c^4\left(\frac{9}{2}\ln\left[\frac{2R}{2R-3\lambda M^2}\right] + \ln\left[\frac{2R}{2R-\lambda M^2}\right]\right)\Bigg] + \Delta$$

Let v =minima of $V_{eff.}$ Free to choose M=v.

We will see that the gravity equations yield $R_{AdS} = \sqrt{110}\,\lambda\,v^2$

$$\left.\frac{dV_{eff}}{d\phi_c}\right|_{\substack{\phi_c=v \\ R=\sqrt{110}\lambda\,v^2}} = 0\,.$$

Yields a numerical value for λ. No longer free parameter.

dimensionless λ traded for dimensionful VEV.

Equations of Motions for Magnetic Monopole

Spherical Symmetry

metric: $ds^2 = N(r)\,dt^2 - \psi(r)dr^2 - r^2\,(d\theta^2 + \sin^2(\theta)\,d\phi^2)$

scalar: $\phi^a(r) = f(r)\,\frac{r^a}{r} = f(r)[\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta]$

gauge: $A^\mu_a = q(r)\,\xi^\mu_a$ where ξ^μ_a are the Killing vectors for $SO(3)$, namely

$\xi^\mu_1 = [0, 0, \cos\varphi, -\sin\varphi\cot\theta], \xi^\mu_2 = [0, 0, -\sin\varphi, -\cos\varphi\cot\theta]$ and $\xi^\mu_3 = [0, 0, 0, 1$

Metric fields: N(r), ψ(r).

Scalar field: f(r)

gauge field: $a(r)=1+r^2\,q(r)$

Variation with respect to N(r) yields

$$\frac{\sqrt{N\psi}\,r^2}{2N}(\alpha\,R^2+\beta\widetilde{R}_j\widetilde{R}^j)+\sqrt{N\psi}\,r^2\Big(2\,\alpha\,R\frac{\partial R}{\partial N}+2\,\beta\,\widetilde{R}_j\frac{\partial\widetilde{R}^j}{\partial N}\Big)$$

$$-\Big[\sqrt{N\psi}\,r^2(2\,\alpha\,R\frac{\partial R}{\partial N'}+2\,\beta\,\widetilde{R}_j\frac{\partial\widetilde{R}^j}{\partial N'})\Big]'+\Big[\sqrt{N\psi}\,r^2\,(2\,\alpha\,R\frac{\partial R}{\partial N''}+2\,\beta\,\widetilde{R}_j\frac{\partial\widetilde{R}^j}{\partial N''})\Big]''$$

$$+\frac{1}{6\,\sqrt{N\,\psi}}(f^2-f^2\psi-f^2r\frac{\psi'}{\psi}-r^2ff'\frac{\psi'}{\psi}+4rff'+2r^2ff'')-\frac{\lambda\,r^2f^4\psi}{48\sqrt{N\psi}}$$

$$-\frac{((a^2-1)^2\psi+2r^2a'^2)}{4\,e^2\,r^2\sqrt{N\psi}}-\frac{a^2f^2\psi}{\sqrt{N\psi}}-\frac{r^2f'^2}{6\sqrt{N\psi}}=\frac{\sqrt{N\,\psi}\,r^2}{2}\langle T^{tt}\rangle \qquad ($$

Variation with respect to ψ(r) yields

$$\frac{\sqrt{N\psi}\,r^2}{2\psi}(\alpha\,R^2+\beta\widetilde{R}_j\widetilde{R}^j)+\sqrt{N\psi}\,r^2\Big(2\,\alpha\,R\frac{\partial R}{\partial\psi}+2\,\beta\,\widetilde{R}_j\frac{\partial\widetilde{R}^j}{\partial\psi}\Big)$$

$$-\Big[\sqrt{N\psi}\,r^2(2\,\alpha\,R\frac{\partial R}{\partial\psi'}+2\,\beta\,\widetilde{R}_j\frac{\partial\widetilde{R}^j}{\partial\psi'})\Big]'+\Big[\sqrt{N\psi}\,r^2\,(2\,\alpha\,R\frac{\partial R}{\partial\psi''}+2\,\beta\,\widetilde{R}_j\frac{\partial\widetilde{R}^j}{\partial\psi''})\Big]''$$

$$-\frac{\lambda\,r^2f^4N}{48\,\sqrt{N\psi}}-\frac{N\,(a^2-1)^2}{4e_R^2\,r^2\,\sqrt{N\psi}}+\frac{N^2\,a'^2}{2\,e^2\,(N\psi)^{3/2}}+\frac{N^2r^2f'^2}{2\,(N\psi)^{3/2}}$$

$$+\frac{r\,f\,f'\,(4N^2+rN'N)}{6\,(N\psi)^{3/2}}+\frac{f^2\,(N^2-N^2\psi-6a^2N^2\psi+rNN')}{6\,(N\psi)^{3/2}}$$

$$=-\frac{\sqrt{N\,\psi}\,r^2}{2}\langle T^{rr}\rangle\,.$$

Variation with respect to f(r) yields

$$-\frac{4a^2 f^2}{r^2} + \frac{f^2 R}{3} - \frac{\lambda}{6} f^4 + \frac{2f\, f''}{\psi} + \frac{f\, f'}{\psi}\Big(\frac{4}{r} + \frac{N'}{N} - \frac{\psi'}{\psi}\Big) = -[E]$$

Variation with respect to a(r) yields

$$2\, a\, (a^2 - 1) + 4\, a\, e^2\, f^2\, r^2 - \frac{2}{\psi}\, a''\, r^2 + \frac{a'\, r^2}{\psi}\Big(\frac{\psi'}{\psi} - \frac{N'}{N}\Big) = 0\,.$$

Analytical solution in the asymptotic regime
Ricci scalar of AdS space determined solely by the VEV

AdS space: $$ds^2 = (1 + k\, r^2)\, dt^2 - \frac{dr^2}{1 + k\, r^2} - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

gravity equations: $$\frac{v^2}{2} - \frac{\lambda v^4}{48\, k} = \frac{1}{8\, k}\Big(- \frac{1}{80}\frac{k^2}{\pi^2} + \frac{11}{1152}\frac{\lambda^2}{\pi^2} v^4 - [E]_0 \Big)$$

scalar equation: $$4\, k\, v^2 - \frac{\lambda v^4}{6} = -[E]_0$$

Solution: $$k = \frac{\sqrt{110}}{12} \lambda\, v^2$$

R_{AdS} = 12 k= $\sqrt{110}\, \lambda\, v^2$

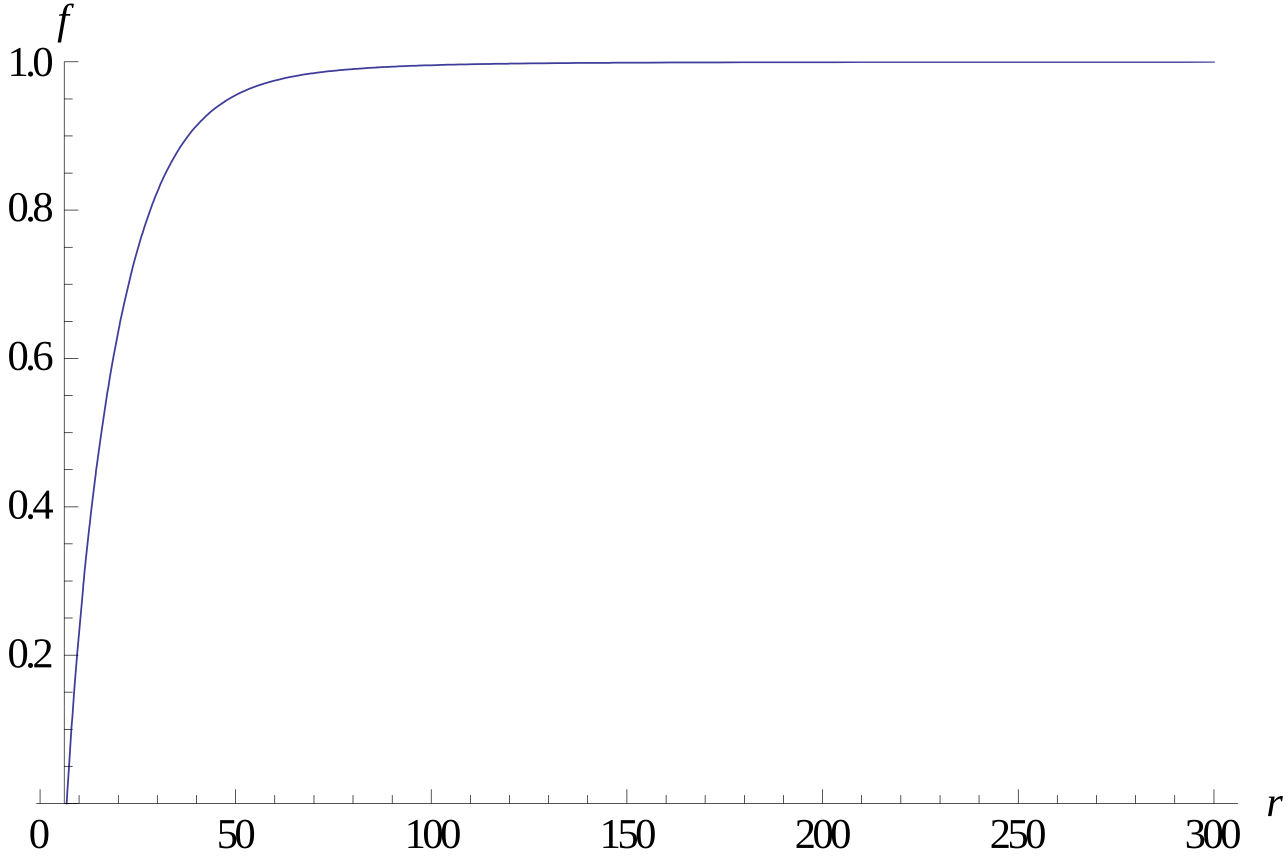
Composite Operator [E]

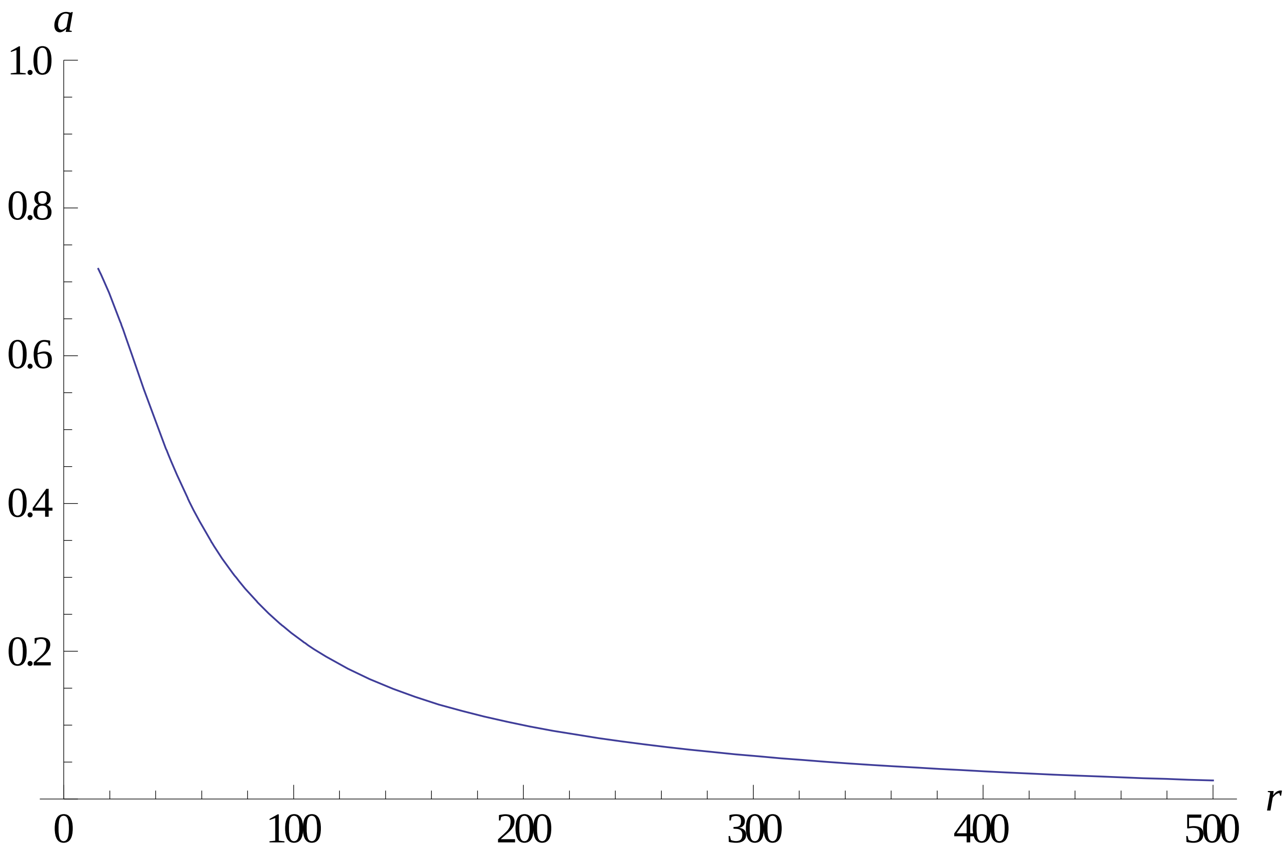
[E] appears in the scalar field equation of motion as a quantum correction.

[E] is the derivative of the one loop part of the effective potential.

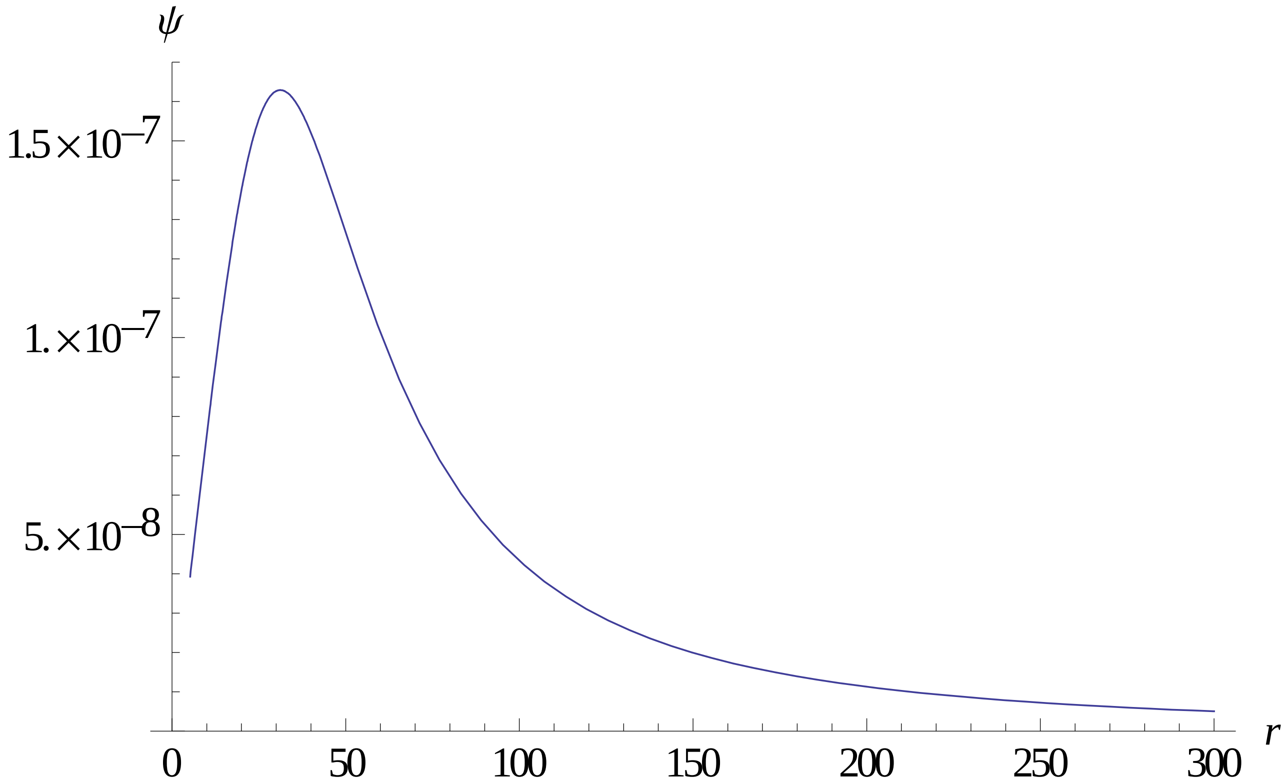
$$
\begin{aligned}
[E] = -\phi_c \frac{dV_{\rm loop}}{d\phi_c} = -\frac{1}{1152\pi^2}\Bigg[& 36\left(-\frac{R}{3}+\frac{\lambda\phi_c^2}{2}\right)\ln\left[\frac{2R-3\lambda\phi_c^2}{2R}\right]\lambda\,\phi_c^2 - 108\left(-\frac{R}{3}+\frac{\lambda\phi_c^2}{2}\right)^2\frac{\lambda\phi_c^2}{2R-3\lambda\phi_c^2} \\
& + 24\left(-\frac{R}{3}+\frac{\lambda\phi_c^2}{6}\right)\ln\left[\frac{2R-\lambda\phi_c^2}{2R}\right]\lambda\phi_c^2 - 72\left(-\frac{R}{3}+\frac{\lambda\phi_c^2}{6}\right)^2\frac{\lambda\phi_c^2}{2R-\lambda\phi_c^2} \\
& + 10R\,\lambda\,\phi_c^2 + \lambda^2\phi_c^4\left(18\,\ln\left[\frac{2R}{2R-3\lambda M^2}\right] + 4\ln\left[\frac{2R}{2R-\lambda M^2}\right]\right)\Bigg] \\
& - 4\Delta\,.
\end{aligned}
$$

Numerical solutions for the magnetic monopole: scalar field f, gauge field a and metric function ψ .





a
1.0
0.8
0.6
0.4
0.2
0
100
200
300
400
500
r


ψ
1.5×10−7
1.×10−7
5.×10−8
0
50
100
150
200
250
300
r

Future work

- Add gauge field fluctuations to the effective potential. Gauge fields would now run around loops.

$\Rightarrow$ λ will now be expressed in terms of the electromagnetic coupling constant *e.* The two parameters in the theory become *e* and the dimensionful VEV.

- Determine if de-Sitter (dS) space is a viable solution for the quantum-corrected effective potential.

END

Notation

We use the notation of Mukhanov & Winitzki, *Introduction to Quantum Effects in Gravity* (2007).

Metric signature is (+,-,-,-),

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \ldots \text{ and } R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$$

Commutator Curvature

$$[D_\mu, D_\nu]\,\phi^a = \mathscr{R}^a{}_{b\,\mu\,\nu}\,\phi^b \text{ with } \hat{\mathscr{R}}_{\mu\nu} \equiv \mathscr{R}^a{}_{b\,\mu\,\nu}$$

$$\begin{aligned} D_\mu D_\nu \phi^a &= \nabla_\mu(D_\nu \phi^a) + \varepsilon^a{}_{bc}\,A^b_\mu\,D_\nu\phi^c \\ &= \nabla_\mu(\nabla_\nu\phi^a + \varepsilon^a{}_{de}\,A^d_\nu\,\phi^e) + \varepsilon^a{}_{bc}\,A^b_\mu(\nabla_\nu\phi^c + \varepsilon^c{}_{fg}\,A^f_\nu\,\phi^g) \\ &= \nabla_\mu\nabla_\nu\phi^a + \varepsilon^a{}_{de}\,\nabla_\mu A^d_\nu\,\phi^e + \varepsilon^a{}_{de}\,A^d_\nu\nabla_\mu\phi^e + \varepsilon^a{}_{bc}\,A^b_\mu\nabla_\nu\phi^c + \varepsilon^a{}_{bc}\,\varepsilon^c{}_{fg}\,A^b_\mu\,A^f_\nu\,\phi^g\,. \end{aligned}$$

The commutator is then given by

$$\begin{aligned} [D_\mu, D_\nu]\phi^a &= \varepsilon^a{}_{de}\left(\nabla_\mu A^d_\nu - \nabla_\nu A^d_\mu\right)\phi^e + \varepsilon^a{}_{bc}\,\varepsilon^c{}_{fg}\left(A^b_\mu\,A^f_\nu - A^b_\nu\,A^f_\mu\right)\phi^g \\ &= \varepsilon^a{}_{de}\left(\nabla_\mu A^d_\nu - \nabla_\nu A^d_\mu + \varepsilon^d{}_{fg}\,A^f_\mu\,A^g_\nu\right)\phi^e \\ &= \varepsilon^a{}_{de}\,F^d_{\mu\nu}\,\phi^e \end{aligned}$$

$$\hat{\mathscr{R}}_{\mu\nu} \equiv \mathscr{R}^a{}_{e\,\mu\,\nu} = \varepsilon^a{}_{de}\,F^d_{\mu\nu}$$

$$P_{ij} = -\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}\lambda^2\left(\phi_a\phi^a\right)^2 = -4\,\lambda^2\left(\delta_{ij}\,\phi_a\phi^a + 2\,\phi_i\,\phi_j\,\right)$$

Delta in effective potential

$$\Delta = \lambda^2 \phi_c^4 \, \frac{(-1584 R^4 + 11904 M^2 R^3 \lambda - 20360 M^4 R^2 \lambda^2 + 12480 M^6 R \lambda^3 - 2475 M^8 \lambda^4)}{13824\, \pi^2 (-2R + M^2 \lambda)^2 (-2R + 3M^2 \lambda)^2} \, .$$

Quantum corrections

One loop divergent part of effective action

$$W_{div} = \frac{1}{n-4}\int d^4x\,\sqrt{g}\;\;\mathrm{tr}\,\hat{a}_2(x,x) \qquad (n\to 4)$$

Schwinger-Dewitt coefficient

$$\hat{a}_2(x,x) = \frac{1}{180}\Big(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} + \Box R\Big)\hat{1} + \frac{1}{12}\,\hat{\mathscr{R}}_{\mu\nu}\hat{\mathscr{R}}^{\mu\nu} + \frac{1}{2}\,\hat{P}^2 + \frac{1}{6}\,\Box\hat{P}\,.$$

Renormalized constants

$$S_{ren} = \int d^4x\,\sqrt{-g}\,\Big(\alpha_R\,R^2 + \beta_R\,R_{\mu\nu}R^{\mu\nu} - \frac{1}{4e_R^2}\,F^2 + (D\phi)^2 + \frac{1}{6}\,R\,\phi^2 - \lambda_R^2\,\phi^4\Big)$$

running coupling constants governed by an RG equation
-> length scale introduced

Trace anomaly involves composite operators

The SO(2,3) symmetry of AdS background yields

$$< T_{\mu\nu} > \equiv \frac{2}{\sqrt{-g}} \frac{\delta W_{ren}}{\delta g^{\mu\nu}} = \frac{1}{4} g_{\mu\nu} < T^{\mu}_{\mu} >$$

$$< T^{\mu}_{\mu} > = \frac{3\lambda^2}{16\,\pi^2}[\phi^4] - [E] + \frac{1}{16\pi^2}\Big[\frac{1}{60}\Big(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} + \Box R\Big) - \frac{1}{6}F^2 - \frac{10}{3}\lambda^2\Box[\phi^2]\Big]$$

$$[E] \equiv \frac{\phi^a}{\sqrt{-g}} \frac{\delta W_{ren}}{\delta \phi^a}.$$

[ ]=composite operators

need to calculate value of composite operators in vacuum with spontaneous symmetry breaking (non-zero VEV Φ_0)

use the effective potential formalism

Conformally invariant action for the magnetic monopole

Previous classical work (Paranjape et al. 2006)

$$S = \int d^4x \sqrt{-g} \left(C_{\mu\nu\sigma\tau} C^{\mu\nu\sigma\tau} - \frac{1}{4e^2} F^{\,a}_{\mu\nu} F^{\mu\nu}_{a} + D_\mu \phi^a D^\mu \phi_a + \frac{1}{6} R \phi^a \phi_a - \frac{\lambda}{4!} (\phi^a \phi_a)^2 \right)$$

where $F^{\,a}_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + \varepsilon^a{}_{bc} A^b_\mu A^c_\nu$ and $D_\mu \phi^a = \nabla_\mu \phi^a + \varepsilon^a{}_{bc} A^b_\mu \phi^c$.

Action invariant under the conformal transformation

$$g_{\mu\nu} \to \Omega^2(x)\, g_{\mu\nu} \text{ and } \phi^a \to \Omega^{-1}(x) \phi^a$$

Spontaneous symmetry breaking (SSB) via gravitation.

VEV: $\phi_0^2 = \dfrac{2R}{\lambda}$ AdS background, R=positive constant

This does not introduce a length scale. At the classical level, this must be done “by hand” by picking an R background.

Paranjape et al. obtained a magnetic monopole solution in AdS background.

Global scale invariance is not broken classically

SSB at the classical level does not introduce a scale (scale had to be introduced by "hand" by choosing a specific R background)

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \quad ; \quad \phi_0 \to \frac{\phi_0}{\Omega(x)} \quad ; \quad R \to \frac{R}{\Omega^2(x)} + \frac{6}{\Omega^3(x)} \Box\, \Omega(x)$$

Two vacuums are related by a conformal transformation if

$$\Box\, \Omega(x) \;\equiv\; g^{\mu\nu}\, \nabla_\mu \nabla_\nu\, \Omega(x) \;=\; 0.$$

For a global scale transformation, $\Omega(x)$=constant and the above automatically holds.

No natural length scale classically.

Quantum corrections:
Trace anomaly in AdS space

$$< T_\mu^\mu > = \frac{a_2(x)}{16\pi^2} - [E]$$
$$= \frac{1}{16\pi^2}\left\{\frac{1}{60}\left(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} + \Box R\right) - \frac{1}{6}F^2 + \frac{11}{3}\frac{\lambda^2}{4!}\phi^4\right\} - [E]$$

where composite operator $$[E] \equiv \frac{\phi^a}{\sqrt{-g}}\frac{\delta W_{ren}}{\delta\phi^a}$$

To evaluate [E] calculate one loop effective potential V_{eff} .

Yields VEV and [E] as derivative of V_{eff} .

Asymptotically, in AdS space, one has SO(2,3) symmetry and

$$< T_{\mu\nu} > \equiv \frac{2}{\sqrt{-g}}\frac{\delta W_{ren}}{\delta g^{\mu\nu}} = \frac{1}{4}g_{\mu\nu} < T_\mu^\mu >$$