The Pressurized Bouncing Ball

A simple model

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Context – Physics

- Introductory physics – Motion
  - Parabolic motion
  - Energy transformations, losses
  - Air resistance, other sources of errors
- Intermediate/advanced physics – Impact
  - Mechanics of impact
  - Impulse forces, deformation, etc.
  - **No simple model available!**

Context – Sports rules

- NBA – "The ball shall be an officially approved NBA ball between $7\frac{1}{2}$ and $8\frac{1}{2}$ pounds pressure [51.7 to 58.6 kPa]."
- FIBA – "[The ball shall] be inflated to an air pressure such that, when it is dropped onto the playing floor from a height of approximately 1,800 mm measured from the bottom of the ball, it will rebound to a height of between 1,200 mm and 1,400 mm, measured to the top of the ball."

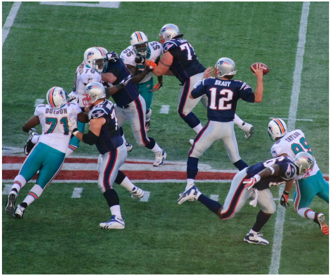


Image taken by Paul Keleher

Definition

The coefficient of restitution e of a ball impacting against an immovable body is

$$e = \left| \frac{v_{\mathsf{f}}}{v_{\mathsf{i}}} \right| \tag{1}$$

For balls, e ranges between 0 [no bounce] and 1 [perfectly bouncy].

The Question

Q: How does internal pressure affect the bouncing of a ball?
Q: What is $e\left(P\right)$?
A: No model exists!

- Polynomial?
- Exponantial?
- Something else?

Our Answer

- Pressure forces
- Wall forces
- Dissipative forces
- Final model

Pressure Forces 1

Geometry

$$A = \pi \left[R^2 - (R - x)^2 \right] \qquad (2)$$

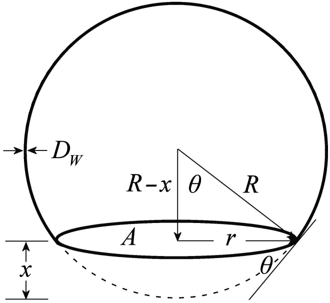
$$V = \frac{4}{3}\pi R^3 - \frac{1}{3}\pi x^2 (3R - x) \qquad (3)$$

Pressure force

$$F_{\mathrm{P}} = (P - P_0) A \qquad (4)$$

Isothermal compression

$$PV = P_{\mathrm{i}} V_{\mathrm{i}} \qquad (5)$$



Pressure Forces 2

Combining (2)–(5) together yields

$$F_{\mathsf{P}} = \left[\frac{4R^3}{4R^3 - x^2\,(3R - x)} P_{\mathsf{i}} - P_0 \right] \pi x\,(2R - x) \tag{6}$$

Taylor expansion in terms of the gauge pressure $P_{\mathsf{G}} = P_{\mathsf{i}} - P_0$

$$F_{\mathsf{P}} = 2\pi R P_{\mathsf{G}} x \left[1 - \frac{1}{2}\left(\frac{x}{R}\right) + \frac{3}{4}\left(1 + \frac{P_0}{P_{\mathsf{G}}}\right)\left(\frac{x}{R}\right)^2 + \ldots \right] \tag{7}$$

If $x \ll R$ and $P_{\mathsf{G}} \gg 0$

$$F_{\mathsf{P}} \approx 2\pi R P_{\mathsf{G}} x \tag{8}$$

F_{P} is <u>linear in x</u>, with a force constant of

$$k_{\mathsf{P}} = 2\pi R P_{\mathsf{G}} \tag{9}$$

Wall Forces 1

Wall forces (shear forces)

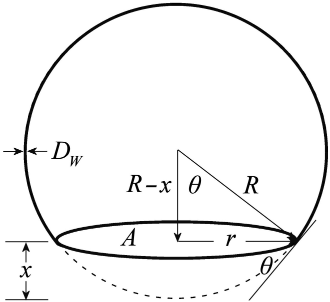
$$F_{\mathsf{W}} = A_{\mathsf{p}} G \theta \tag{10}$$

Cross-sectional area of perimeter

$$A_{\mathsf{p}} = 2\pi D_{\mathsf{W}} \sqrt{R^2 - (R-x)^2} \tag{11}$$

Angle of contact

$$\theta = \arccos\left(\frac{R-x}{R}\right) \tag{12}$$



Wall Forces 2

Combining (10)–(12) yield

$$F_{\mathrm{W}} = 2\pi G D_{\mathrm{W}} \sqrt{R^2 - (R-x)^2} \arccos\left(\frac{R-x}{R}\right) \tag{13}$$

Taylor expansion

$$F_{\mathrm{W}} = 2\pi G D_{\mathrm{W}} x \left[2 - \frac{1}{3}\left(\frac{x}{R}\right) - \frac{1}{15}\left(\frac{x}{R}\right)^2 + \ldots\right] \tag{14}$$

If $x \ll R$

$$F_{\mathrm{W}} \approx 4\pi G D_{\mathrm{W}} x \tag{15}$$

F_{W} is linear in x, with a force constant of

$$k_{\mathrm{W}} = 4\pi G D_{\mathrm{W}} \tag{16}$$

Total Restoring Force

According to our model, the combined restoring effect of wall strength and pressure is

$$\begin{aligned} F_{\mathsf{R}} &= F_{\mathsf{P}} + F_{\mathsf{W}} \\ &\approx \left(2\pi R P_{\mathsf{G}} + 4\pi G D_{\mathsf{W}}\right) x \end{aligned} \tag{17}$$

and the ball will effectively have a spring constant of

$$k = 2\pi R P_{\mathsf{G}} + 4\pi G D_{\mathsf{W}} \tag{18}$$

Dissipative Forces 1

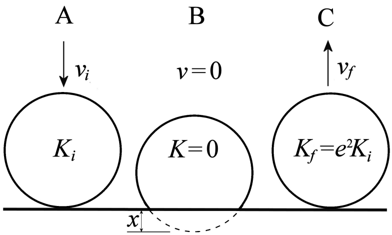
Let us consider a ball of spring-like restoring force F_{R}, with spring constant k, subject to a uniform dissipative force F_{D}.

In compression (A $\rightarrow$ B)

$$K_{\mathsf{i}} = \frac{1}{2}kx_0^2 - mgx_0 + F_{\mathsf{D}}x_0 \tag{19}$$

In decompression (B $\rightarrow$ C)

$$K_{\mathsf{f}} = e^2 K_{\mathsf{i}} = \frac{1}{2}kx_0^2 - mgx_0 - F_{\mathsf{D}}x_0 \tag{20}$$



Dissipative Forces 2

Combining (20) and (21), we obtain

$$\frac{\left(1+e^2\right)}{\left(1-e^2\right)^2} = \frac{kK_\mathsf{i}}{4F_\mathsf{D}^2} - \frac{mgK_\mathsf{i}}{2F_\mathsf{D}^2 x_0^2} \tag{21}$$

If $\frac{1}{2}kx_0^2 \gg mgx_0$, we can ignore the last term, and

$$\frac{\left(1+e^2\right)}{\left(1-e^2\right)^2} \approx \frac{kK_\mathsf{i}}{4F_\mathsf{D}^2} \tag{22}$$

Final Model

Incorporating (19) into (23), we obtain

$$\frac{\left(1+e^2\right)}{\left(1-e^2\right)^2} = \frac{\left(2\pi R P_\mathsf{G} + 4\pi G D_\mathsf{W}\right) K_\mathsf{i}}{4F_\mathsf{D}^2} \tag{23}$$

Or

$$\frac{\left(1+e^2\right)}{\left(1-e^2\right)^2} = A P_\mathsf{G} + B \tag{24}$$

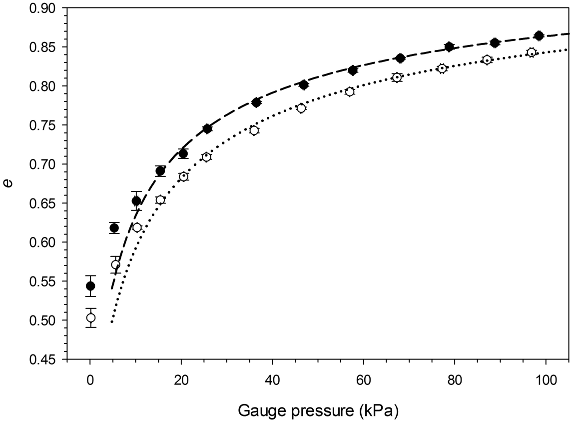
where

$$A = \frac{\pi R K_\mathsf{i}}{2F_\mathsf{D}^2} \tag{25}$$

$$B = \frac{\pi G D_\mathsf{W} K_\mathsf{i}}{F_\mathsf{D}^2} \tag{26}$$

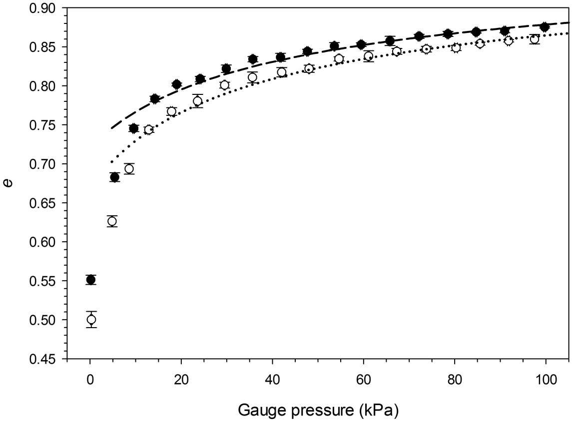
$$\frac{B}{A} = \frac{2G D_\mathsf{W}}{R} \tag{27}$$

Reality check – Basketballl



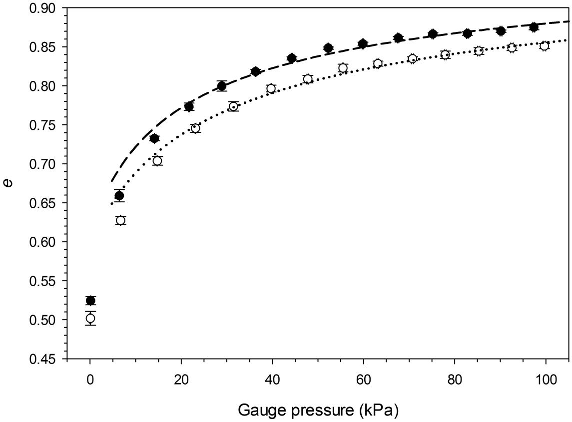
Height (m)	A $(10^{-4}\ \text{Pa}^{-1})$	B	G $(10^{5}\ \text{Pa})$	F_D (N)
● 0.75	2.576 ± 0.052	1.35 ± 0.28	1.01 ± 0.27	55.92 ± 0.92
○ 1.50	1.916 ± 0.036	1.29 ± 0.20	1.29 ± 0.27	91.7 ± 1.5

Reality check – Soccerball



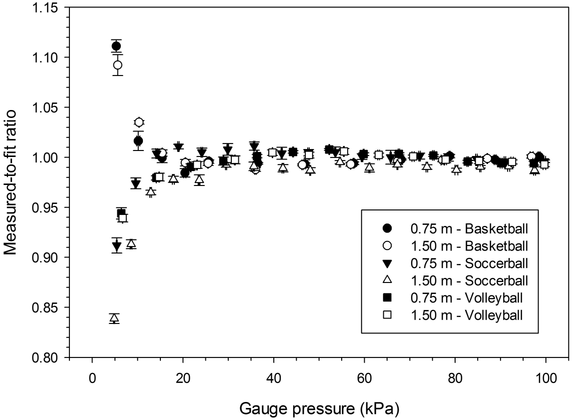
Height (m)	A $(10^{-4}\ \mathrm{Pa}^{-1})$	B	G $(10^{5}\ \mathrm{Pa})$	F_{D} (N)
• 0.75	2.76 ± 0.11	6.60 ± 0.65	2.90 ± 0.49	43.8 ± 1.2
○ 1.50	2.277 ± 0.088	3.24 ± 0.46	2.52 ± 0.44	55.7 ± 1.4

Reality check – Volleyball



Height (m)	A (10^{-4} Pa^{-1})	B	G (10^5 Pa)	F_D (N)
● 0.75	3.14 ± 0.14	3.50 ± 0.80	1.18 ± 0.39	32.16 ± 0.93
○ 1.50	2.091 ± 0.074	4.74 ± 0.50	1.62 ± 0.39	68.1 ± 1.8

Fit vs Data



If $P_{\mathrm{G}} > 25$ kPa, spread <2.5%, individual points <1.5%!

Conclusions 1

- Model is very accurate at $P_{\mathsf{G}} > 25$ kPa.
- F_{D} increases by a factor of 1.3 to 2.1 when height is doubled.
- G is constant within error when height is doubled.
- G has correct order of magnitude.
  - $G_{\mathsf{exp}} \approx 10^5$ Pa vs $G_{\mathsf{rubber}} = 3 \times 10^5$ Pa.

Conclusions 2

- Could include higher-order correction terms in the analysis if greater accuracy is desired at $P_{\mathsf{G}} < 25$ kPa.
- Ultimately, could go back to the specific forms of $A(x)$, $V(x)$, $A_{\mathsf{p}}(x)$ and $\theta(x)$ for more accurate $F_{\mathsf{P}}(x)$ and $F_{\mathsf{W}}(x)$.
- Non-uniform dissipative forces?
- Non-isothermal compressions?

Acknowledgments

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A. Georgallas, G. Landry. "The Coefficient of Restitution of Pressurized Balls: A Mechanistic Model." Submitted to *Canadian Journal of Physics* on 8 June 2015.

Experimental Method 1

Several methods exist to probe e

$$e = \left| \frac{v_\mathsf{f}}{v_\mathsf{i}} \right| \tag{28}$$

v_f and v_i are related to several other quantities, like the height of bounces, times of flight, etc. In terms of typical accuracy

Time methods > Height methods > Velocity methods

Experimental Method 2

Since e depend on K_i, we need to control for K_i. Easiest way is to control for H_i, and study first impact.

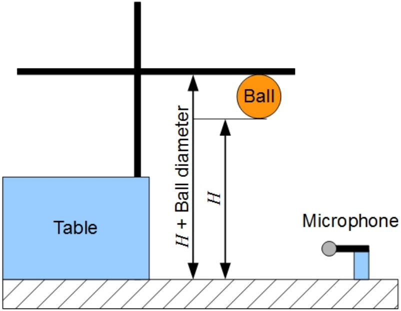
Assuming no air resistance

$$v_\mathsf{i} = \sqrt{2gH} \tag{29}$$

$$v_\mathsf{f} = \frac{1}{2}gt_\mathsf{f} \tag{30}$$

Therefore

$$e = \sqrt{\frac{gt_\mathsf{f}}{8H}} \tag{31}$$



Experimental Method 3

- Manual release ($\pm < 1$ cm) at 0.75 m and 1.50 m
- Sound-based time-of-flight measurement ($\pm < 1$ ms)
- Inflated with bike pump, but accurate sensor ($\pm < 0.5$ kPa)
- At least 5 trials per pressure, per ball, per height

Ball & Model	R (cm)	m (g)	D_{W} (mm)
(Bask.) Wilson WTB0935	11.75 ± 0.15	592.9 ± 0.1	3.10 ± 0.09
(Socc.) Nike SC2400-471	10.80 ± 0.15	422.2 ± 0.1	4.51 ± 0.08
(Voll.) Wilson WTH3501	10.35 ± 0.15	271.2 ± 0.1	5.02 ± 0.28