Some discrete-flavoured approaches to Dyson-Schwinger equations

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Dyson-Schwinger equations

Consider a Dyson-Schwinger equation like

$$G(x, L) = 1 - \frac{x}{q^2} \int d^4k \frac{k \cdot q}{k^2 G(x, \log k^2/\mu^2)(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$.

This is a little piece of Yukawa theory. (See Broadhurst and Kreimer arXiv:hep-th/0012146, but blame me for the funny mathematician's normalization.)

Today's level of generality

We can deal with more than just this example. For today we need

- single scale, i.e. only propagator insertions into propagator functions
- only one insertion place (can fix this, see arXiv:0810.2249)

But we can have

- any number of primitive diagrams
- a wide variety of theories

Diagrammatics

Consider Dyson-Schwinger equations diagramatically.
Eg:

B_+ for graphs

A Feynman graph is primitive if it has no subdivergences. Write B_+^γ for insertion into the primitive Feynman graph γ.
Eg:

By weighting the insertions by an appropriate combinatorial coefficient we can avoid double counting overlapping subdivergences.
Eg:

The coefficient for avoiding double counting is hairy

Combinatorial Dyson-Schwinger equations

Using B_+ we can rewrite the diagrammatic Dyson-Schwinger equations as follows
Eg:

The coupling has become a counting variable.

Rooted trees

Let's get some experience with the simpler example of rooted trees instead of Feynman graphs. This is the same as the situation with no overlapping subdivergences by thinking of a Feynman graph as the tree of its subdivergences.

Let $B_+(F)$ be the tree constructed by adding a new root above each tree from the forest F.
Eg:

Expanding tree equations – I

Eg:

$$T = \mathbb{I} + xB_+(T)$$

Expanding tree equations – II

Eg:

$$T = \mathbb{I} - xB_+\left(\frac{1}{T}\right)$$

This is a kind of combinatorial specification language using only sums, products, and sequences.

Trees, leading logs, and tree factorial

Staying with trees for the moment, the Feynman rules would give a map from trees to some target algebra. For just the leading log part we would have

$$t \mapsto c_t L^{|t|}$$

A nice such map comes from the tree factorial. For a vertex v of t, let t_v be the subtree rooted at v. Then

$$t! =$$

Eg:

The tree factorial Feynman rules are

$$t \mapsto \frac{L^{|t|}}{t!}$$

Hook weight formulas

The combinatorics community has studied this in a different language (see Jones, Y., arXiv:1412.6053 regarding unifying the communities). Given a series $B_1, B_2, \ldots$ define the hook weight of a tree

$$w_B(t) = \prod_{v \in t} B_v$$

Then for a class $\mathcal{T}$ of trees we can form the weighted generating function

$$F_{B,\mathcal{T}}(x) = \sum_{t \in \mathcal{T}} w_B(t) x^{|t|}$$

The question is, what choices of **tree class** and **hook weights** give nice **weighted series** $F_{B,\mathcal{T}}$ where nice either means a closed formula or a combinatorial interpretation.

In our language this is, what choices of **combinatorial Dyson-Schwinger equation** and (leading log) **Feynman rules** give nice **Green function**.

Example - I

(Postnikov's formula)
Combinatorial DSE:

$$T(x) = \mathbb{I} + xB_+(T(x)^2)$$

Hook weight:

$$B_k = 1 + \frac{1}{k}$$

Green function:

$$G(x, L) = \frac{-W(-2xL)}{2xL}$$

where W is the Lambert W-function

Example - II

(Jones and Kreimer independently)
Combinatorial DSE:

$$T(x) = \mathbb{I} - xB_+\left(\frac{1}{T(x)^2}\right)$$

Tree factorial Feynman rules.
Green function:

$$G(x, L) = (1 - 3xL)^{1/3}$$

Back to something closer to the physics

Recall

$$G(x, L) = 1 - \frac{x}{q^2} \int d^4k \frac{k \cdot q}{k^2 G(x, \log k^2/\mu^2)(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

Manipulate using standard tricks

- plug in $G(x, L) = 1 - \sum \gamma_k(x) L^k$
- use $\partial_\rho^k x^{-\rho}|_{\rho=0} = (-1)^k \log^k(x)$
- switch the order of $\int$ and ∂

to obtain

$$G(x, L) = 1 - xG(x, \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\Big|_{\rho=0}$$

Where $F(\rho)$ is the integral for the primitive regularized by a parameter ρ which marks the insertion place.

The question

Given

$$G(x,L) = 1 - \sum_{k\geq 1} x^k G(x, \partial_{-\rho})^{1-sk}(e^{-L\rho} - 1)F_k(\rho)|_{\rho=0}$$

and

$$F_k(\rho) = f_{k,0}\rho^{-1} + f_{k,1} + f_{k,2}\rho + \cdots$$

How to solve for $G(x, L)$?

Rooted connected chord diagrams

We can solve this by a chord diagram expansion (with N. Marie for $k = 1$, $s = 2$, arXiv:1210.5457, general case with M. Hihn).

A chord diagram is *rooted* if it has a distinguished vertex.
A chord diagram is *connected* if no set of chords can be separated from the others by a line.
Eg:

These are really just irreducible matchings of points along a line.

Recursive chord order

Let C be a connected rooted chord diagram. Order the chords recursively:

- c_1 is the root chord
- Order the connected components of $C \smallsetminus c_1$ as they first appear running counterclockwise, $D_1, D_2, \ldots$. Recursively order the chords of D_1, then of D_2, and so on.

Eg:

Terminal chords

A chord is terminal if it only crosses chords which come before it in the recursive chord order. Let

$$t_1 < t_2 < \cdots < t_\ell$$

be the terminal chords of C. Then

- $b(C) = t_1$ and
- $f_C = f_{t_\ell - t_{\ell-1}} \cdots f_{t_3 - t_2} f_{t_2 - t_1} f_0^{|C|-\ell}$

Eg:

Result

Theorem 1

$$G(x,L) = 1 - \sum_{i\geq 1} \frac{(-L)^i}{i!} \sum_{\substack{C \\ b(C)\geq i}} x^{|C|} f_C f_{b(C)-i}$$

solves

$$G(x,L) = 1 - xG(x,\partial_{-\rho})^{-1}(e^{-L\rho}-1)F(\rho)\Big|_{\rho=0}$$

where

$$F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \cdots$$

A weighted result holds for the general case.

Conclusion

- The recursive shape of Dyson-Schwinger equations gives us a lot of information.
- The combinatorics community has a family of toys with nice solutions under the name of hook weight formulas.
- We can understand the series solution to a general family of Dyson-Schwinger equations using a chord diagram expansion.

Bonus – The renormalization group equation

The **renormalization group equation** is very important physically. For us it says

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} + \gamma(x)\right) G(x, L) = 0$$

What happens if we apply it to the chord diagram expansion?

Chord diagram decomposition

We can insert a rooted connected chord diagram C_1 into another C_2, by

- choosing an interval of C_2 other than the one before the root
- putting the root of C_1 just before the root of C_2 and
- putting the rest of C_2 in the chosen interval

Eg:

Since the diagrams are connected C_1 and C_2 can be recovered.

A classical recurrence

This decomposition is classical. Nijenhuis and Wilf (1978) use it to prove the recurrence (originally due to Stein (1978) and rephrased by Riordan)

$$s_n = \sum_{k=1}^{n-1} (2k-1) s_k s_{n-k} \qquad \text{for } n \geq 2$$

where s_n is the number of connected rooted chord diagrams with n chords.

The recurrence translated

This recurrence can be extended to keep track of the terminal chords. Let

$$g_{k,i} = \sum_{\substack{C \\ |C|=i \\ b(C)\geq i}} f_C f_{b(C)-i}$$

where C runs over rooted connected chord diagrams. Then

$$g_{k,i} = \sum_{\ell=1}^{i-1}(2\ell - 1)g_{1,i-\ell}g_{k-1,\ell} \quad \text{for } 2 \leq k \leq i$$

This is exactly the renormalization group equation on chord diagrams.

This gives a combinatorial view of the renormalization group equation.