Yang-Mills Flow in the Abelian Higgs Model

Paul Mikula

University of Manitoba

mikulap@myumanitoba.ca

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The Abelian Higgs Model

Starting with the action:

$$S[A,\phi] = \int d^4x \, \sqrt{-g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \, - (D^\mu \phi)^\dagger (D_\mu \phi) \, - V(\phi^\dagger \phi) \right]$$

Where μ,ν are the usual space-time indices and g is the metric determinant.
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
$D_\mu \phi = \partial_\mu \phi + \imath q A_\mu \phi$
$A_\mu(x)$ is the Electromagnetic vector potential.
$V = \frac{\lambda}{4} \left(\phi^\dagger \phi - \frac{v^2}{2} \right)^2$

Yang-Mills Flow

We generalize the Yang-Mills Flow as the gradient flow of the action:

$$\begin{aligned}
\frac{\partial A_\nu}{\partial \tau} &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\gamma}F_{\gamma\nu}) - f^2q^2\left(A_\nu + \frac{\partial_\nu\omega}{qv}\right) - \partial_\nu\bar{\chi}(x) \\
\frac{\partial \omega}{\partial \tau} &= \frac{1}{\sqrt{-g}}\partial_\gamma\left(\frac{\sqrt{-g}g^{\mu\gamma}qf^2}{v}\left[A_\mu + \frac{\partial_\mu\omega}{qv}\right]\right) - qv\bar{\chi}(x) \\
\frac{\partial f}{\partial \tau} &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\gamma}\partial_\gamma f) - g^{\mu\gamma}fq^2\left(A_\gamma + \frac{\partial_\gamma\omega}{qv}\right)\left(A_\mu + \frac{\partial_\mu\omega}{qv}\right) \\
&\quad -\frac{\lambda}{4}\left(f^3 - v^2 f\right)
\end{aligned}$$

for an arbitrary flow parameter τ.
We have added the $\bar{\chi}$ terms in order to ensure the flow remains gauge-invariant.
For Riemannian metrics the Yang-Mills Flow equations are coupled parabolic PDEs similar to the Heat equation.

Why Yang-Mills Flow?

The Yang-Mills action is used to describe several different physical theories such as electromagnetism and chromodynamics.

Due to the AdS/CFT correspondence, we have an equivalence between a gravity theory and a Yang-Mills theory in one fewer dimension.

The Yang-Mills flow can be studied in the context of particle physics, as well as in gravity, where we can compare it to better understood flows such as the Ricci flow.

Ginzburg-Landau Theory

Ginzburg-Landau Theory describes superconductors near a critical temperature where the material undergoes a phase transition between superconducting and non-superconducting by expanding around a complex parameter $|\phi|^2$

The parameter $|\phi|^2$ is zero when the material has lost superconductivity and a constant value when it is superconducting.

Neglecting terms of order $|\phi|^6$ and greater this is equivalent to the Abelian Higgs model.

We will consider the case of with cylindrical symmetry where we have some magnetic flux through the material at the origin.

Vortex Solutions

$$A_t = A_\rho = 0, A_\theta = A_\theta(\rho)\,,\; A_z = A_z(\rho)\,,\; \phi = \frac{f(\rho)}{v\sqrt{2}} e^{\frac{\imath n\theta}{v}}$$

We rescale the radial coordinate and our flow time:

$$\tau \to t = (vq)^2\tau\,; \quad \rho \to r = vq\rho$$

Now we can write the flow equations in terms of the gauge invariant quantities

$$B = (qA_\theta + n)\,,\; Z = q^2 v A_z \text{ and } f$$

$$\begin{aligned} \frac{\partial B}{\partial t} &= B'' - \frac{B'}{r} - f^2 B \\ \frac{\partial Z}{\partial t} &= Z'' + \frac{Z'}{r} - f^2 Z \\ \frac{\partial f}{\partial t} &= f'' + \frac{f'}{r} - f\left[\frac{B^2}{r^2} + Z^2\right] - \frac{\lambda}{4q^2}\left(f^3 - f\right) \end{aligned}$$

Numerical Calculations

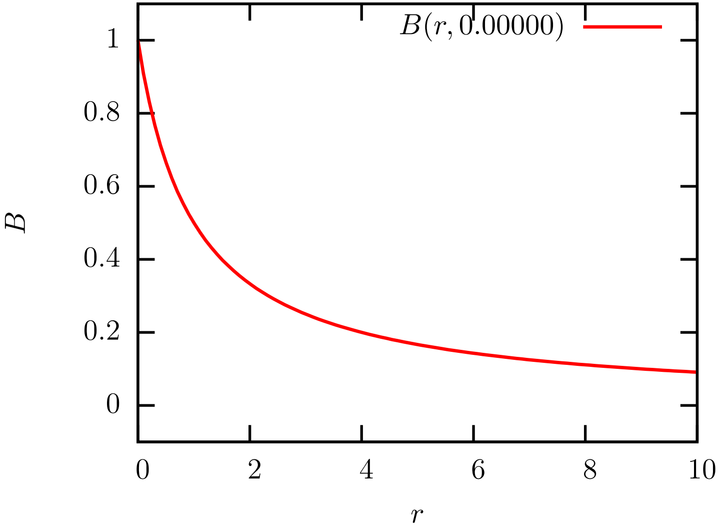
We can compute the flow numerically for any given initial data using a simple finite difference scheme.

The appropriate boundary conditions of the flow equations for the vortex solutions are

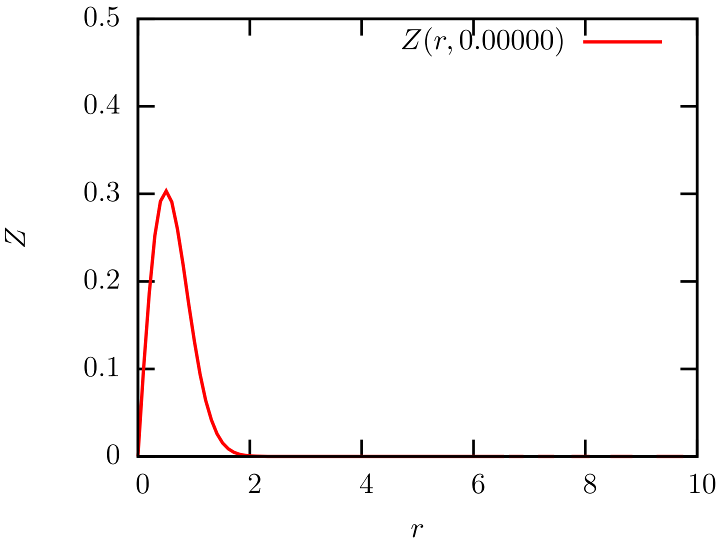
$$f(0,\tau) = 0 \quad f(\infty,\tau) = 1 \tag{1}$$
$$B(0,\tau) = n \quad B(\infty,\tau) = 0 \tag{2}$$
$$Z(0,\tau) = 0 \quad Z(\infty,\tau) = 0 \tag{3}$$

The stationary points of the flow correspond to solutions of the equations of motion.

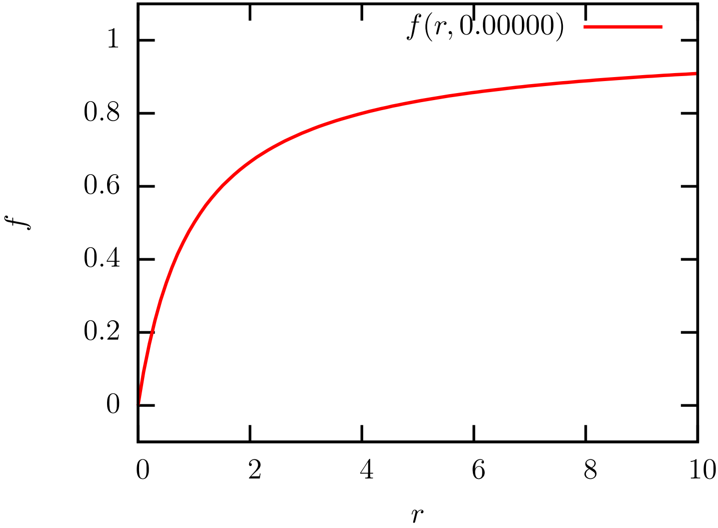
B(t,r)



Z(t,r)



f(t,r)



Stability and Constant Solutions

We find that the flow moves B and f towards the solution for any values of n or the parameter $\frac{\lambda}{4q^2}$, and Z will always decrease to zero.

For $n = 0$ we have two constant solutions:

- $B = 0, f = 1$: Stable
- $B = 0, f = 0$: Unstable

For $n \neq 0$ we have the solution $B = n, f = 0$ which is unstable.

AdS/CFT and Black holes

We can also consider a scalar field coupled with an electromagnetic potential in curved spacetimes.

An AdS black hole can allow for symmetry breaking of the scalar field, which produces the properties of a superconductor. Here we can consider the flow in the AdS theory and look for the equivalent flow in a CFT on the boundary.

In this model we could also consider flow that acts on the components of the metric as well.

End

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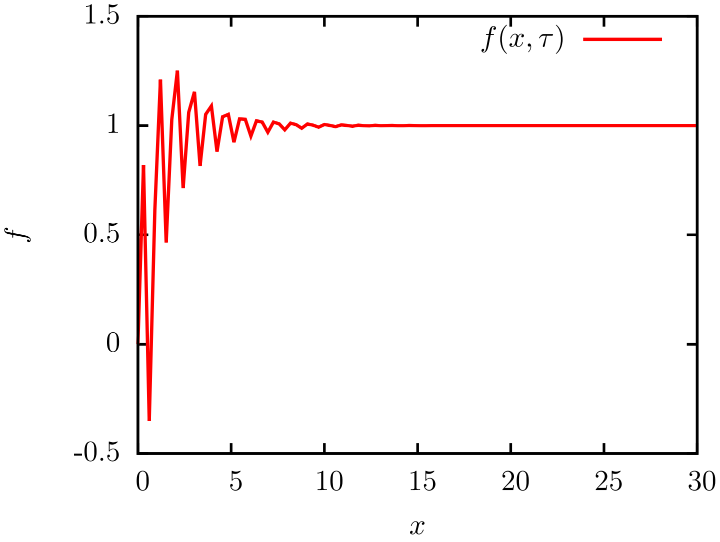
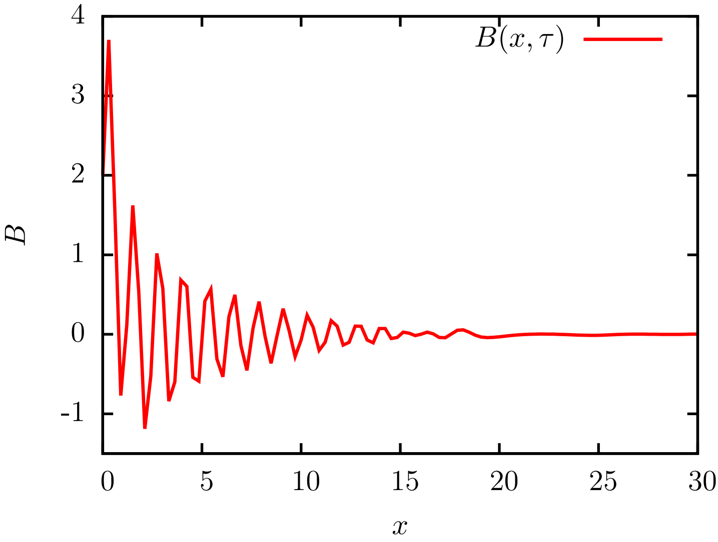
Questions?

$f(r, \tau)$



$B(r, \tau)$



$Z(r, \tau)$