# The Physics of X-ray Tomography: Not as simple as it looks

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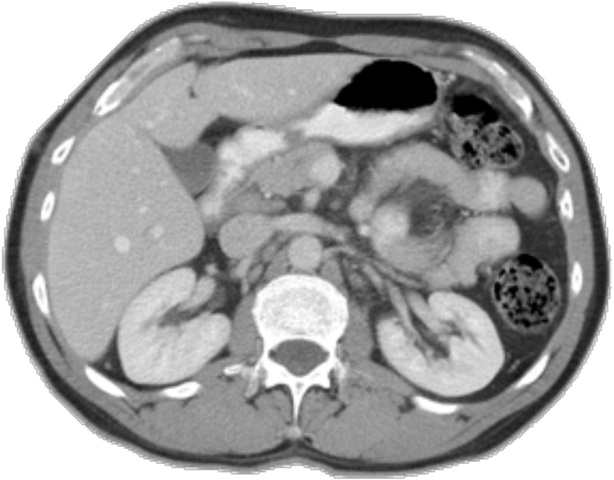
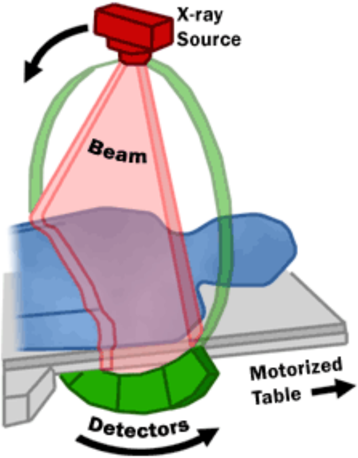
# X-Ray Tomography

- Greek:
  - “tomos”for “slice” or “section”.
  - “graphe” means “drawing”.
- Tomograph: a cross-sectional image or a “slice”.



# CT Scanner

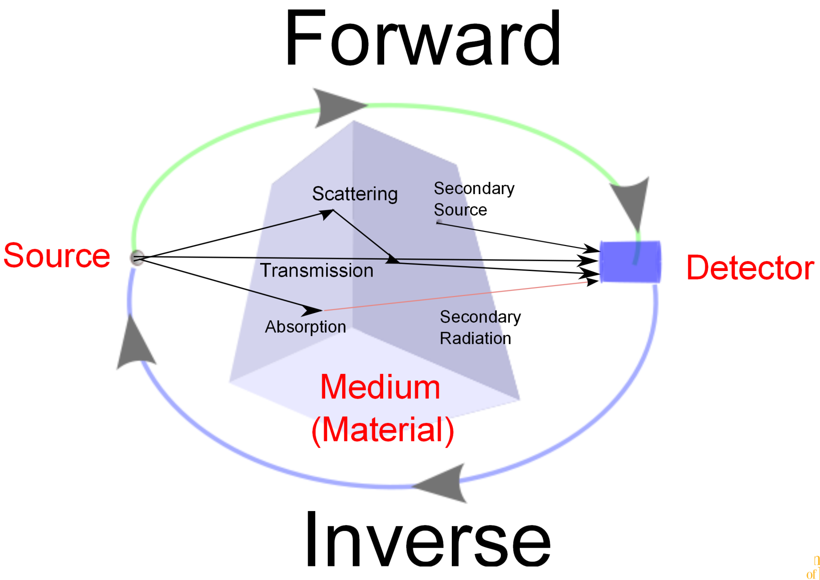


# Computed Tomography (CT)

- Image can be only obtained by computation, solving an **inverse problem**:

$$\text{Measurements} \xrightarrow{\text{Map}^{-1}} \text{Parameters}$$

- Measurements: intensity of transmitted radiation.
- Parameters: attenuation coefficient of incident radiation in each pixel.


Forward
Scattering
Secondary
Source
Source
Transmission
Detector
Absorption
Secondary
Radiation
Medium
(Material)
Inverse

## Forward Model

Best if based on Particle (Boltzmann) Transport Equation:

$$
\begin{aligned}
&\frac{1}{v}\frac{\partial}{\partial t}\phi(\vec{r},E,\vec{\Omega},t) \text{ [Volumetric Rate of Change] } = \\
&Q(\vec{r},E,\vec{\Omega},t) \text{ [Independent Source]} \\
&\qquad -\vec{\Omega}\cdot\nabla\phi(\vec{r},E,\vec{\Omega},t) \text{ [Streaming/Divergence]} \\
&-\Sigma_t(\vec{r},E,t)\ \phi(\vec{r},E,\vec{\Omega},t) \text{ [Removal (Absorption + Scattering)]} \\
&+\int \Sigma_s(\vec{r},E'\rightarrow E;\vec{\Omega}'\rightarrow\vec{\Omega},t)\ \phi(\vec{r},E',\vec{\Omega}',t)\ dE'\ d\vec{\Omega} \text{ [Scattering in]} \\
&+\int \nu\Sigma_g(\vec{r},E'\rightarrow E;\vec{\Omega}'\rightarrow\vec{\Omega},t)\ \phi(\vec{r},E',\vec{\Omega}',t)\ dE'\ d\vec{\Omega} \text{ [Generation]}
\end{aligned}
$$

Solution is difficult:

- Many variables (3 position, 1 Energy, 2 Direction, 1 time).
- Direction has no point of origin.
- Integro-differential equation.
- Not self-adjoint: $\Sigma(E' \rightarrow E) \neq \Sigma(E \rightarrow E')$.

Solvable by sophisticated methods (Spherical Harmonics, Discrete Ordinates, Monte Carlo): not directly invertible.

# One Common Simplification

## Exponential Law of Attenuation

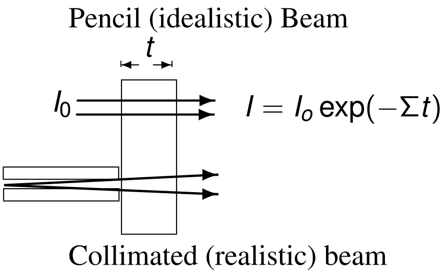
$$\phi(\vec{r}, E, \vec{\Omega}) = \phi(\vec{r}_0, E, \vec{\Omega}) \ \exp\left[-\int_{\vec{r}_0}^{\vec{r}} \Sigma_t(E, \vec{r}')\, dr'\right]$$

Obtained upon the solution of the Transport equation under the following conditions:

1. steady-state,
2. away from any sources of radiation,
3. at a particular direction,
4. when the radiation energy does not change.

Suitable for modeling the transmission of (i) narrow (pencil) radiation beams, while (ii) not accounting for radiation spread with distance.

# Simplification: Pencil-Beam Attenuation




- There is no such a thing as a pencil beam.
- Need to measure away from source.
- For a multi-energetic source, e.g. x-rays, unless the energy spectrum is measured:
  - Recoded intensity will involve all energies.
  - Cannot discern the attenuation coefficient at each energy.
  - Only an effective attenuation coefficient is obtainable.
  - Beam hardening problem.

# Another Common Simplification

## Inverse-Square Law

$$\frac{\phi(\vec{r}_1 - \vec{r}_0, E)}{\phi(\vec{r}_2 - \vec{r}_0, E)} = \frac{|\vec{r}_2 - \vec{r}_0|^2}{|\vec{r}_1 - \vec{r}_0|^2}$$

Obtained upon the solution of the Transport equation under the following conditions:

- steady state,
- for a point source,
- for an isotropic source,
- in vacuum (i.e. in the absence of any material).
- radiation energy will not change.

# Inverse Problem

Solves for:

- Source energy spectrum, $Q(E)$, when characterizing a radiating source,
- Internal source spatial distribution, $Q(\vec{r})$, in emission imaging (SPECT or PET),
- External source spatial and angular distribution, $Q(\vec{r}, \vec{\Omega})$, in radiotherapy planning, or
- Material distribution, $\Sigma_t(\vec{r})$, in imaging (CT).

Simplified models are typically used for ease of inversion, but attempts are made at full utilization of the Boltzmann transport equation.

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## Inverse Transport

**Adjoint Transport:** calculations initiated from detectors to determine the most likely location of a concealed source[1].

**Iterative Matching:** Nonlinear optimization to find source terms and medium that minimize the difference between calculations and measurements[2].

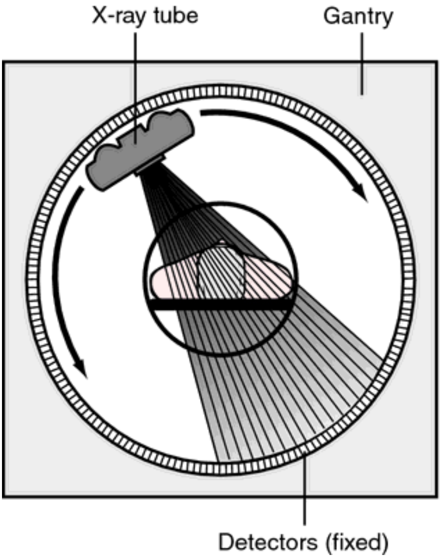
**Response Matrix:** for detector unfolding[3] and for SPECT[4].

**Perturbation Method (Inverse Method):** Inverse problem is viewed as a perturbation of a nominal reference configuration, and Monte Carlo simulations are used to estimate detector responses with factors that contain the unknown parameters[5,6].

[1] Jarman, K.D. et al., April 2010. ANS Topical Meeting, Las Vegas, NV.
[2] Mattingly, J., Mitchell, D.J., 2010. IEEE Trans. Nucl. Sci. 57, 3734-3743.
[3] Search (`http://rsicc.ornl.gov/Catalog.aspx?c=PSR`).
[4] Floyd Jr., C.E. at el.1985. IEEE Trans. Nucl. Sci. NS-32, 779-785.
[5] Yacout, A.M., Dunn, W.L., 1987. Adv. X-ray Anal., 30, 113-120.
[6] Dunn, W.L., 2006. Trans. Amer. Nucl. Soc. 5, 532-533.

# Computed Tomography

## Pencil Beams

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## CT: Simple-Model Inversion

### Exponential Attenuation Law

$$p(\vec{r}, E, \vec{\Omega}) = -\ln \frac{\phi(\vec{r}, E, \vec{\Omega})}{\phi(\vec{r}_0, E, \vec{\Omega})} = -\int_{\vec{r}_0}^{\vec{r}} \Sigma_t(E, \vec{r}')\, dr'$$

- RHS = integral along line $\equiv$ Radon Transform.
- Radon Transform is closely related to Fourier transform (in the frequency domain).
- Fourier transform is readily amenable to numerical manipulation, via FFT.
- Fourier inversion is not commonly used in image reconstruction, because of its sensitivity to error.
- Inverse Fourier transform $\equiv$ backprojection of transmission projections, with the magnitude of frequency as a filter.
- Fourier filter backprojection is the most widely used method in transmission imaging.

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## Pencil-Beam Inversion

To obtain meaningful results from Radon transforms, one must:

- Collimate detector field-of-view (FoV).
- Eliminate scattering.
- Remove low-energy radiation (scattering & beam hardening)

## Well-posedness

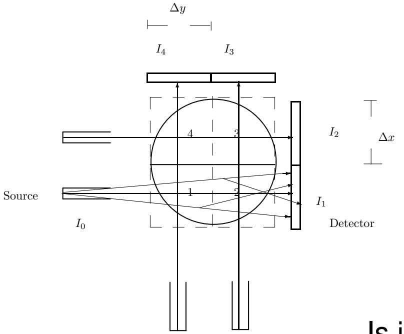
The inverse problem should be well-posed, i.e.:

1. There exists a solution for a given set of measurements,
2. The solution is unique, and
3. The problem is continuous.

- Numerical solution requires discretization, which violates the third condition.
- Practical inverse problems are ill-posed:
  - A small change in measurements $\Longrightarrow$ large change in solution values.
  - Solution is sensitive to modeling error and measurement uncertainties.
- Solution is regularized with the aid of *a priori* information, constraints on solution, smoothing, etc.

# Transmission Imaging

## Example: 2 × 2 X-ray Tomography in a Circular Section




Is it really Simple?

## Challenges to Radon Transform: integration over a line:

**Discretization:** square pixels at edges, partial filling.

**Section Depth:** averaging of content.

**Pencil Beam:** there is no such a thing.

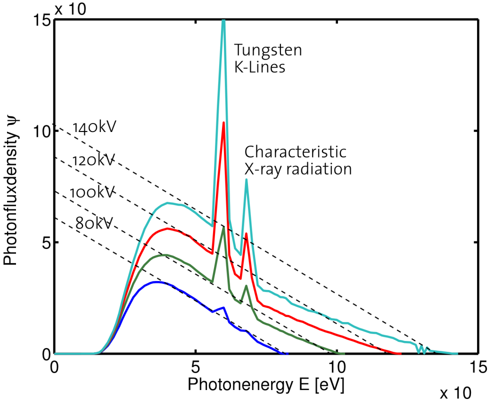
**Source Collimation:** wider beam, an uneven coverage, divergence effect.

**Detector Collimation:** two conical intersecting FoV's.

**Uncollimated Detector:** unequal travel distance, scattering.

# $2 \times 2$ X-ray Tomography in a Circular Section

## Source/Detection Energy Challenges




**Source Energy:** x-ray source, $0 < E < eV_p$.
**Attenuation Coefficient:** $\Sigma(E)$ varies with energy.
**Detection Energy:** Spectrum or total, detection efficiency $f(E)$, beam hardening.

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# $2 \times 2$ X-ray Tomography in a Circular Section
## Complex Forward Model

- Integrate over source surface, $\int dS$.
- Integrate over Source Cone: $\int d\Omega$.
- Detect all energies, integrate over energy: $\int dE$.
- Incorporate Detector Efficiency, $\eta(E)$.
- Include effect of scattering, $B(\Sigma_{r,\theta,\phi}, R_{\theta,\phi})$.
- Include divergence.

$$\phi = \oint \int_0^{E_p} \int_0^{\Omega_s} \phi_0(E, \vec{\Omega})\, B(\Sigma_{r,\theta,\phi}, R_{\theta,\phi})\, \frac{e^{-\int_0^{R_{\theta,\phi}} \Sigma_{r,\theta,\phi,E}\,\mathrm{d}r}}{4\pi R_{\theta,\phi}^2}\, \mathrm{d}\Omega\, \eta(E)\, \mathrm{d}E\, \mathrm{d}S$$

Bye Bye Inverse Radon/Fourier, Filter backprojection, or any other straightforward inversion.

# $2 \times 2$ X-ray Tomography in a Circular Section

## Simplifying Forward Model

1. Replace cone beam with an equivalent pencil beam:
   - Burdened with inherent assumptions of attenuation law.
   - Forcing an even, but unrealistic FoV.
2. Normalize measured intensity (flux) to that measured in air:
   - Reduce divergence effect.
   - Evaluated $\Sigma$ is with respect to that of air.
3. Reduce scattering:
   - Place detector as far as possible from object.
   - Use high-energy source.
4. Ignore or measure attenuation in air.
5. Assume an equivalent monoenergetic source energy: invert for an effective attenuation coefficient, $\Sigma^e$.
6. Filter source to remove low-energy component: reduce beam hardening.
7. Ensure source-detector alignment.

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# $2 \times 2$ X-ray Tomography in a Circular Section

## Forward Model

Logarithmic transformation for linearization + discretiziation:

$$p_i = -\ln \frac{\phi_i}{\phi_{0_i}} = -\sum_{j}^{N_i} \Sigma_j^e \, \Delta r_{ij}$$

Discretiziation has a homogenizing effect.

Matrix form:

$$\left\{ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \right\} = \begin{bmatrix} \Delta x & \Delta x & 0 & 0 \\ 0 & 0 & \Delta x & \Delta x \\ 0 & \Delta y & \Delta y & 0 \\ \Delta y & 0 & 0 & \Delta y \end{bmatrix} \left\{ \begin{array}{c} \Sigma_1^e \\ \Sigma_2^e \\ \Sigma_3^e \\ \Sigma_4^e \end{array} \right\}$$

$$\mathbf{p} = \mathbf{H}\boldsymbol{\Sigma}^{e}$$

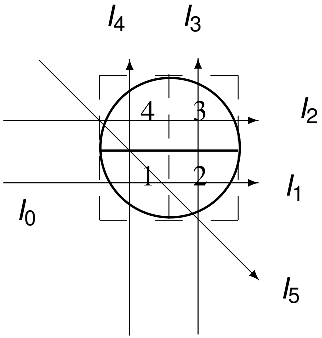
## $2 \times 2$ X-ray Tomography in a Circular Section
### Inverse Problem

- Matrix **H** is singular: one equation is obtainable from the linear combination of the other three.
- Setup was flawed!
- Add one more projection at an angle $\neq \frac{\pi}{2}$.
- Non-square matrix, minimize:

$$\chi^2 = \left[\mathbf{H}\boldsymbol{\Sigma}^e - \mathbf{p}\right]^2$$

Leading to:

$$\boldsymbol{\Sigma}^e = \left[\mathbf{H}^T\mathbf{H}\right]^{-1}\mathbf{H}^T\mathbf{p}$$

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# Dealing with Ill-posedness

## Regularization

Minimize:

$$\chi^2 = \mathbf{W}\left[\mathbf{H}\boldsymbol{\Sigma}^e - \mathbf{p}\right]^2 + \alpha^2\left[\mathbf{G}(\boldsymbol{\Sigma}^e - \boldsymbol{\Sigma}^\star)\right]^2$$

**W**: Weight matrix to favor more accurate measurements.
**G**: a regularization matrix aiming at smoothing solution.
$\alpha^2$: a regularization parameter, controls the degree of smoothing.
$\boldsymbol{\Sigma}^\star$: a credible estimate of the solution, if any.

Leading to:

$$\boldsymbol{\Sigma}^e = \left[\mathbf{H}^T\mathbf{W}\mathbf{H} + \alpha^2\mathbf{G}^2\right]^{-1}\left[\mathbf{H}^T\mathbf{W}\mathbf{p} + \alpha^2\mathbf{G}^2\boldsymbol{\Sigma}^\star\right]$$

A comprehensive list of regularization methods is given in:
Hussein, E.M.A., 2011. Computed Radiation Imaging, Elsevier Insights, Elsevier Burlington, MA.

# Emission Imaging: SPECT & PET

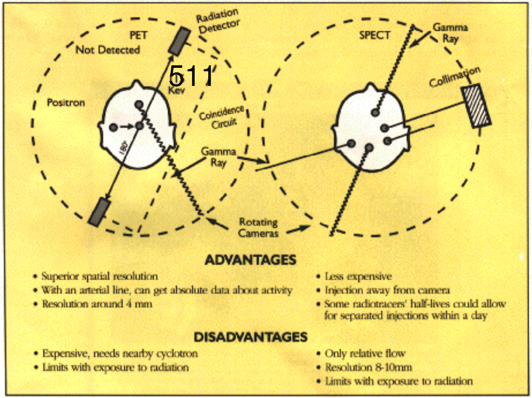



**Figure 1** - This is a diagram of the imaging technique behind SPECT (right of image) and PET (left of image).

# Emission Imaging: Simple Model

- When attenuation is ignored $\longrightarrow$ Radon Transform = integration of intensity over lines of detection.
- Can compensate for attenuation by:
  - Associated CT image: SPECT-CT Systems.
  - Some independent transmission measurements.
  - Estimated average value.
- Incorporating attenuation into the Radon transform results in an exponential Radon transform $\longrightarrow$ shifting frequency of the unattenuated Fourier coefficient and altering the corresponding amplitude.

# Scatter Imaging

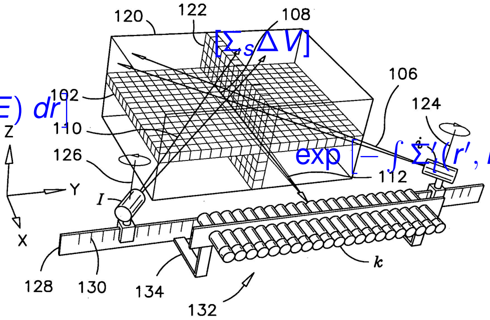
## Arsenault & Hussein: US Patent No. 7,203,276

Radon/Fourier transforms are not applicable.

$\exp\left[-\int \Sigma_t(r,E)\, dr\right]$

$[\Sigma_s \Delta V]$

$\exp\left[-\int \Sigma_t'(r',E')\, dr'\right]$



# Conclusions

Computed Tomography is an Inverse Problem.
On the Forward problem:

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.
Albert Einstein

On the Inverse Problem:

[S]ometimes we tend to resort to inversion techniques too blindly, without using our judgment or “feel” about handling a given problem, which may lead to “antiaesthetic” excesses.
Diran Deirmendjian[7]

[7] In: Deepak, A., Ed. (1977). Inversion methods in atmospheric remote sounding, p. 138, Academic Press, New York.

**http://www.elsevierdirect.com/ISBN/9780123877772/Computed-Radiation-Imaging**