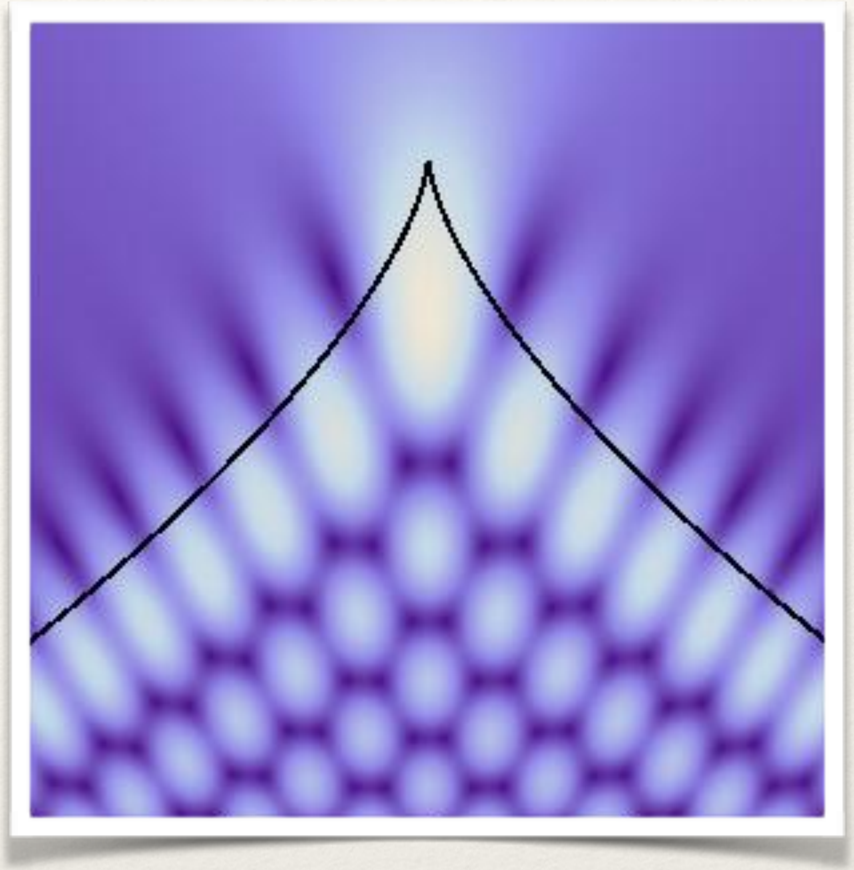
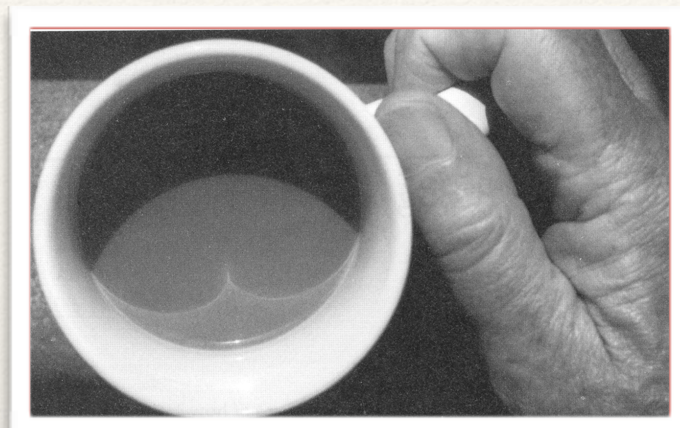


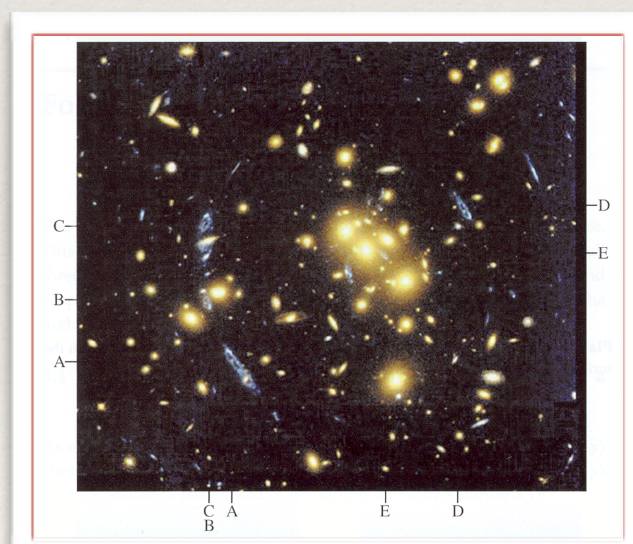
CAP congress, 16 June 2015

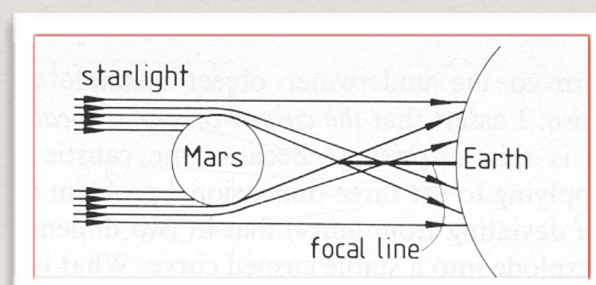
Universal features of quantum dynamics: Quantum Catastrophes

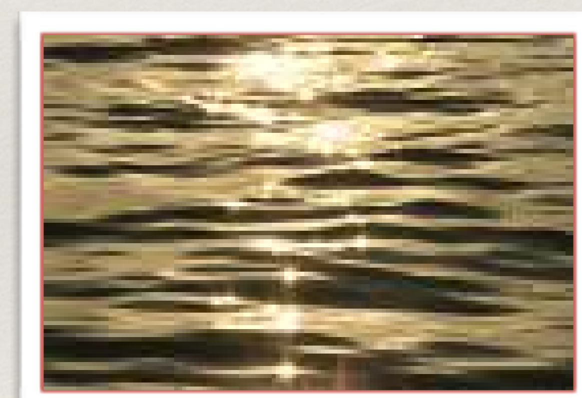
Duncan O'Dell,
McMaster University

Natural focusing: caustics





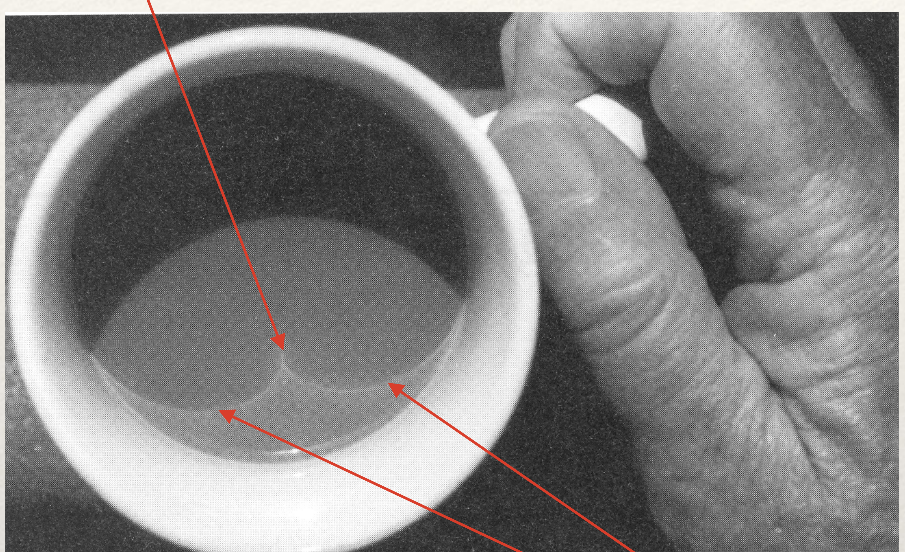




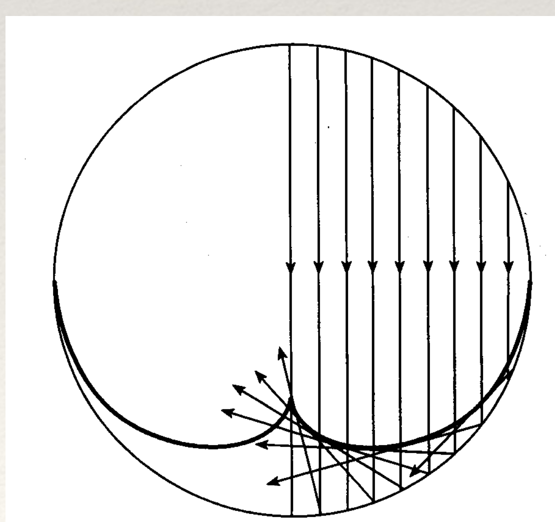
Pictures from: *Natural Focusing and the Fine Structure of Light* by J.F. Nye

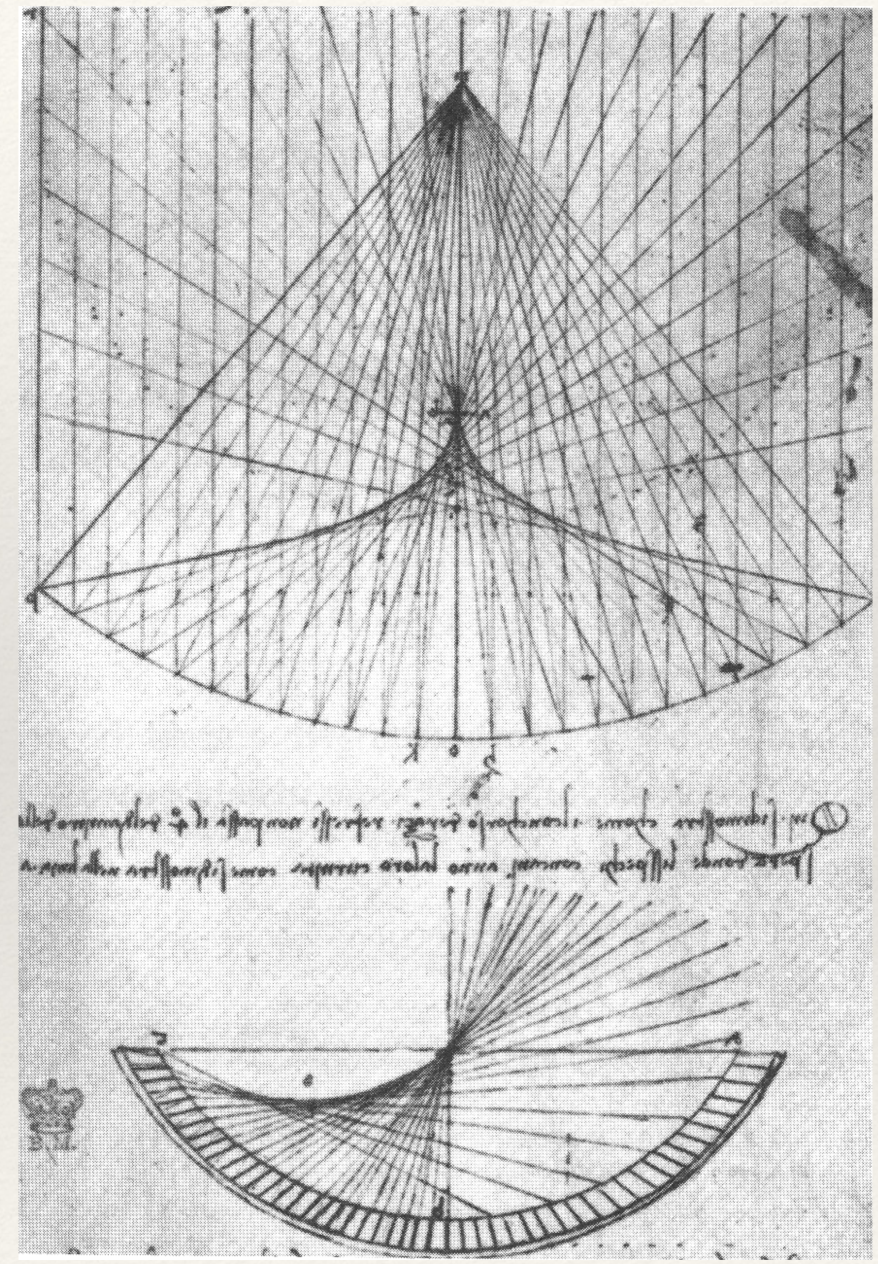
Cusp caustic

cusp point



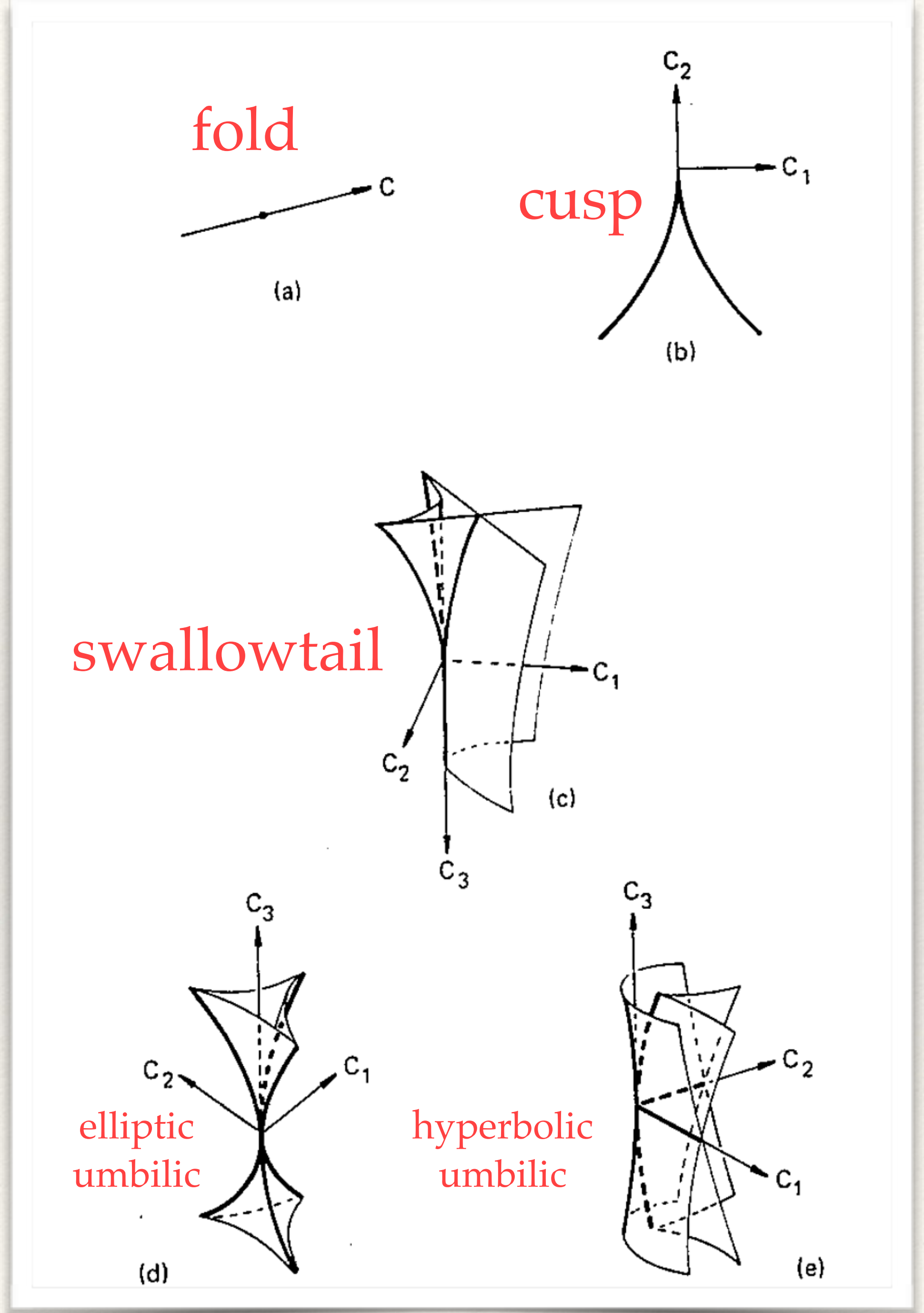
fold lines





Leonardo de Vinci c. 1508

Structurally stable catastrophes with K≤3



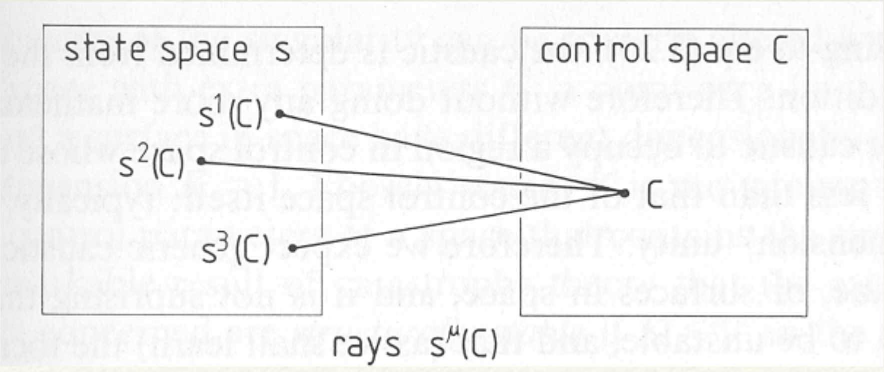
Berry & Upstill, Prog. Opt. **18**, 257 (1980)

Structurally stable caustics and their generating functions with K≤4

name	**codimension K**	**$\phi(s;C)$** [generating function]
Fold	1	$s^3/3+Cs$
Cusp	2	$s^4/4+C_2s^2/2+C_1s$
Swallowtail	3	$s^5/5+C_3s^3/3+C_2s^2/2+C_1s$
Elliptic umbilic	3	$s_1^3-3s_1s_2^2-C_3(s_1^2+s_2^2)-C_2s_2-C_1s_1$
Hyperbolic umbilic	3	$s_1^3+s_2^3-C_3s_1s_2-C_2s_2-C_1s_1$
Butterfly	4	$s^6/6+C_4s^4/4+C_3s^3/3+C_2s^2/2+C_1s$
Parabolic umbilic	4	$s_1^4+s_1s_2^2+C_4s_2^2+C_3s_1^2+C_2s_2+C_1s_1$

R. Thom *Structural Stability and Morphogenesis* (Benjamin, 1975); **V.I. Arnol'd***,* Russ. Math. Survs. *30 (5) (1975) p.1*

Mathematically, catastrophe theory describes the **singularities of gradient map** $\frac{\partial \phi}{\partial s_i} = 0$



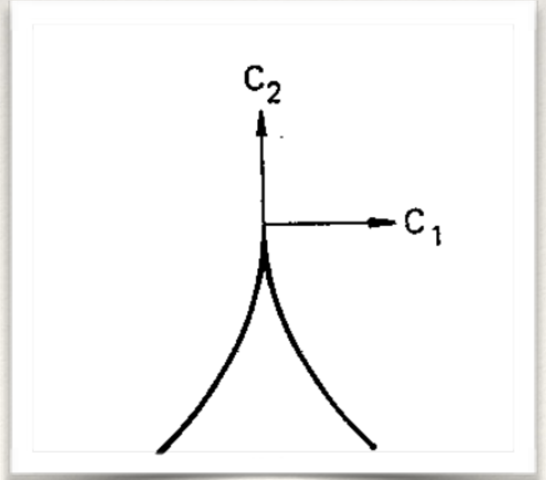
How it works

Example of the cusp: $\phi(s;C) = s^4/4 + C_2 s^2/2 + C_1 s$

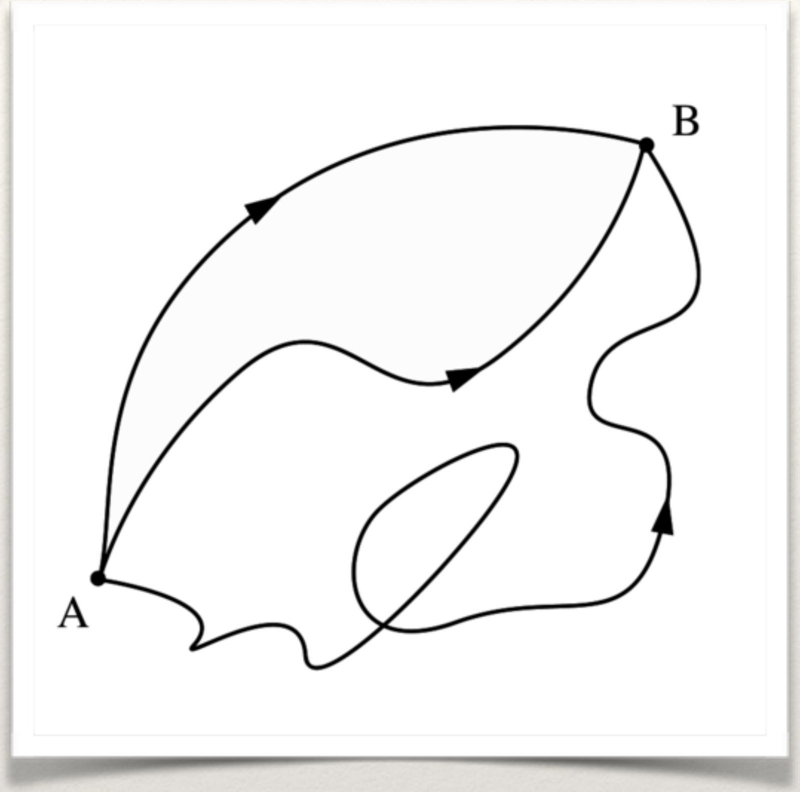
Ray equation (Fermat's principle) : $\dfrac{\partial \phi}{\partial s} = s^3 + C_2 s + C_1 = 0$

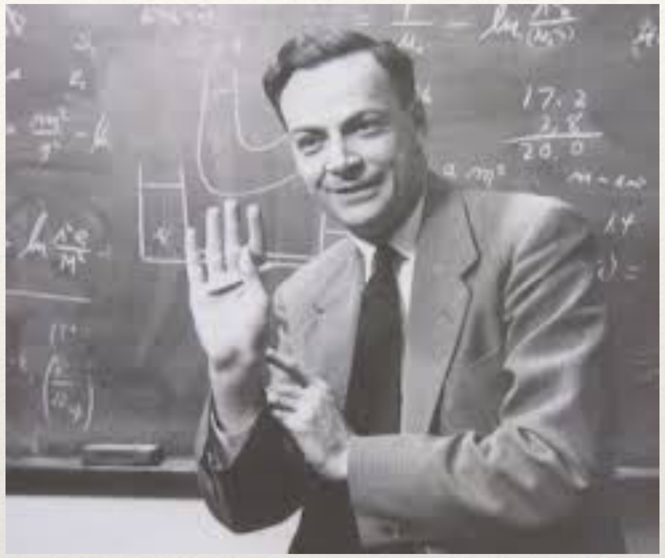
Caustic equation : $\dfrac{\partial^2 \phi}{\partial s^2} = 3s^2 + C_2 = 0$

Eliminate s : $C_1 = \pm\sqrt{\dfrac{8}{27}}(-C_2)^{3/2}$



Wave theory: Feynman path integral





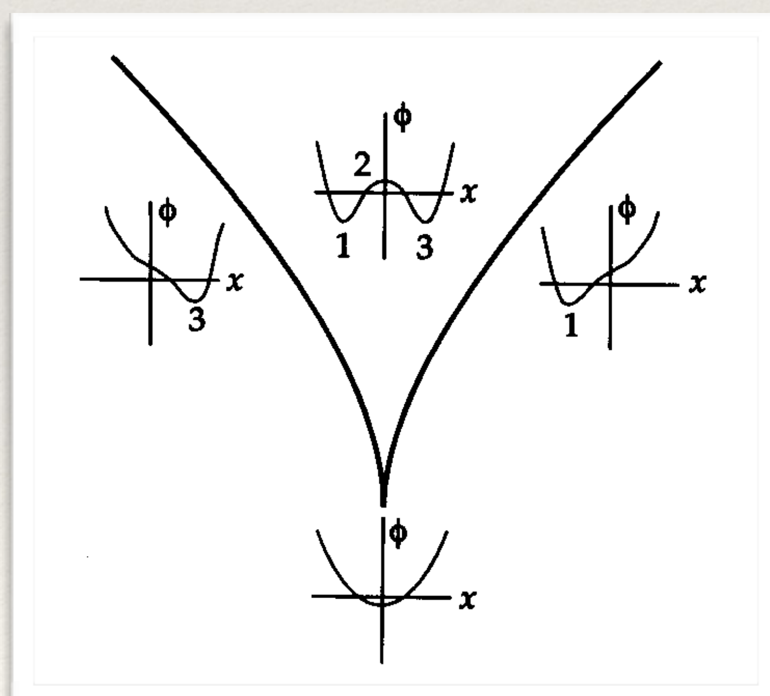
Richard Feynman

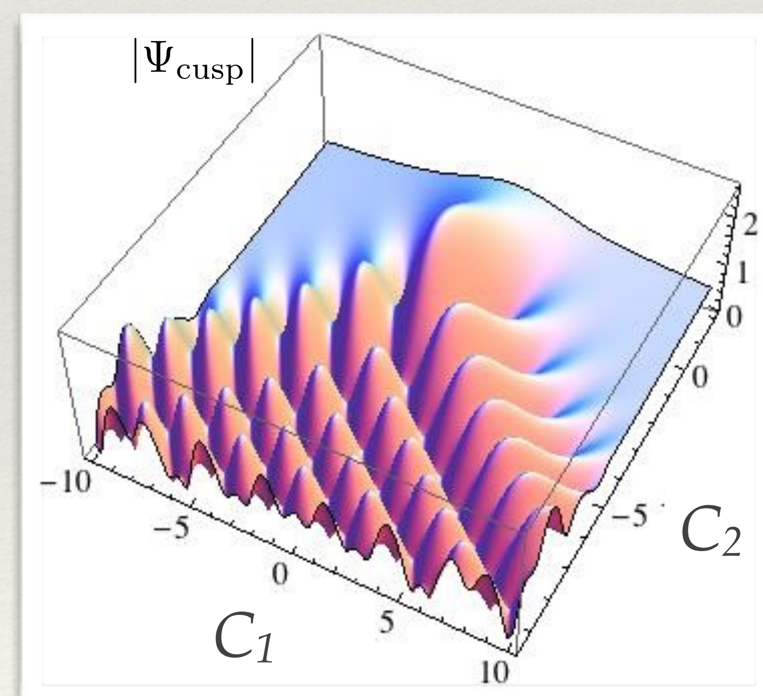
$$\Psi(B) = \mathcal{N} \sum_{\text{paths } j} e^{iS_j/\hbar}$$

The Pearcey function

$$\Psi_{\text{cusp}}(C_1, C_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s^4/4 + C_2 s^2/2 + C_1 s)} ds$$

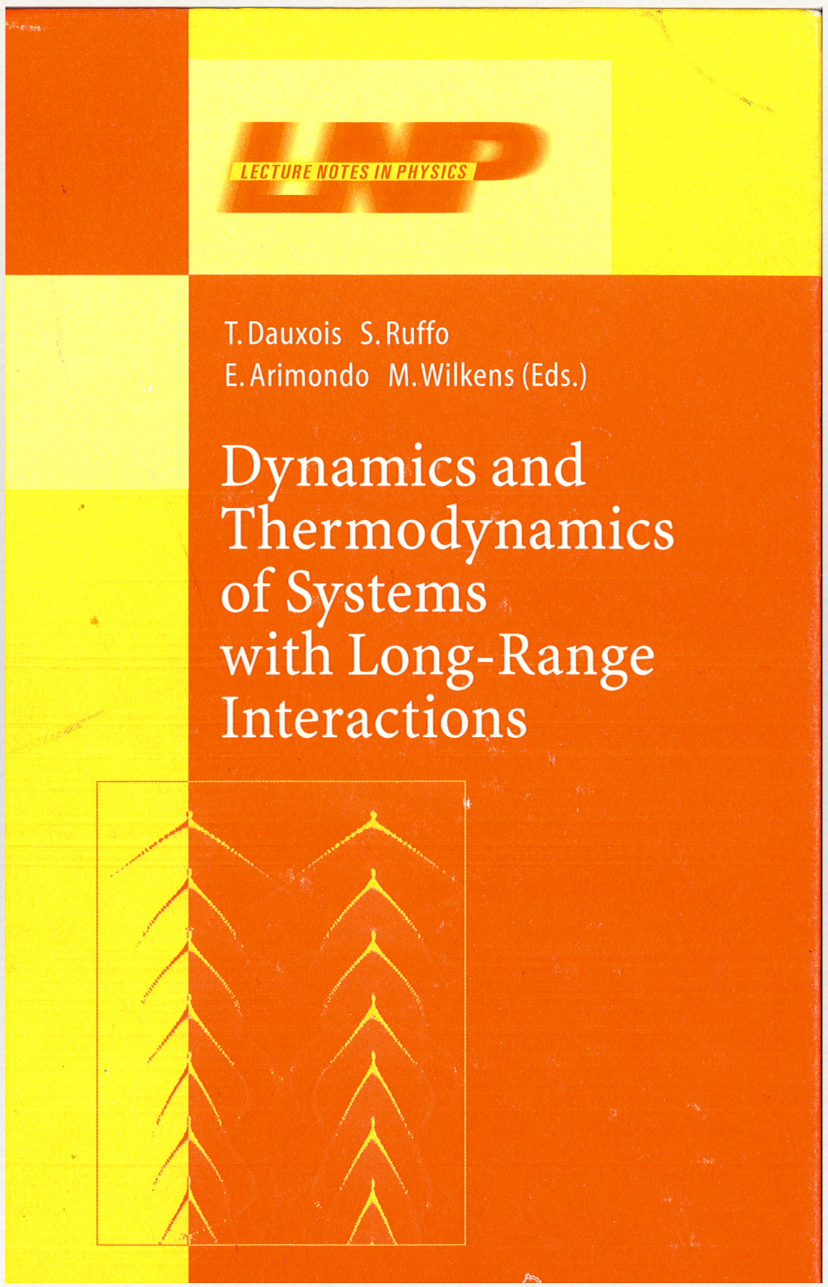
T. Pearcey, Phil. Mag. **37**, 311 (1946)



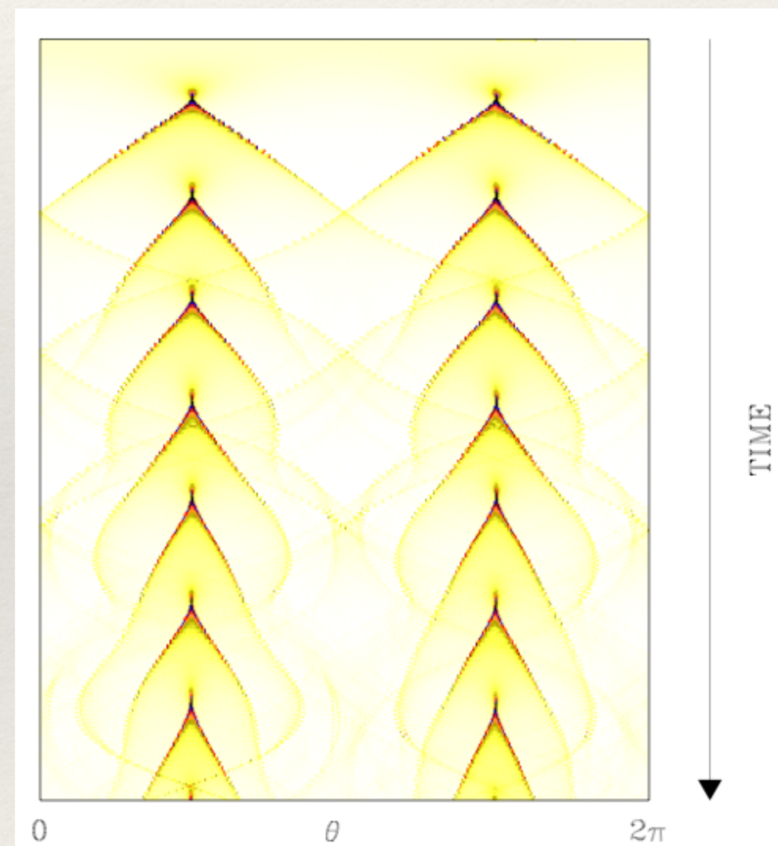


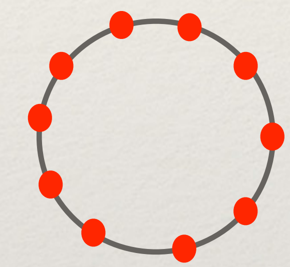
There are three rays inside the cusp and one outside

Dynamics of N particles on a ring



$$\sum_i \frac{p_i^2}{2} + \frac{\epsilon}{2N} \sum_{i,j} [1 - \cos(\theta_i - \theta_j)]$$

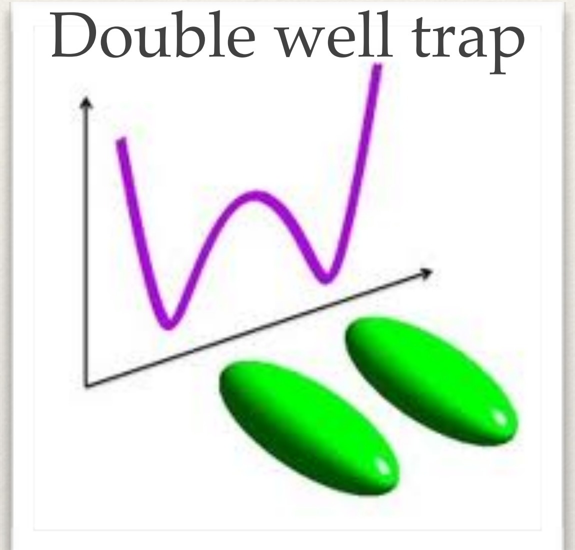


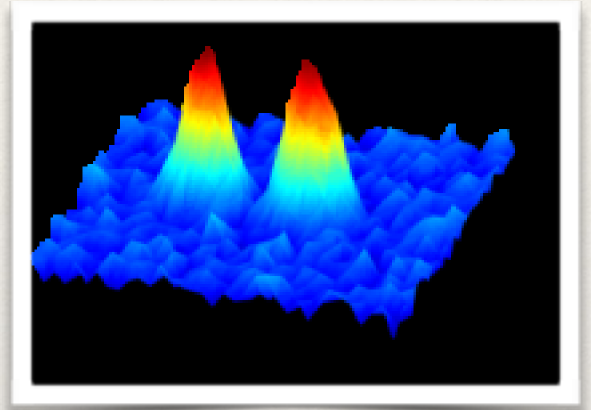


Particle density as a function of time. Initial density on ring at *t=0* is uniform. Interaction is *repulsive.*

Catastrophes in superfluids

1. Bosonic Josephson junction (two tunnel-coupled Einstein condensates)

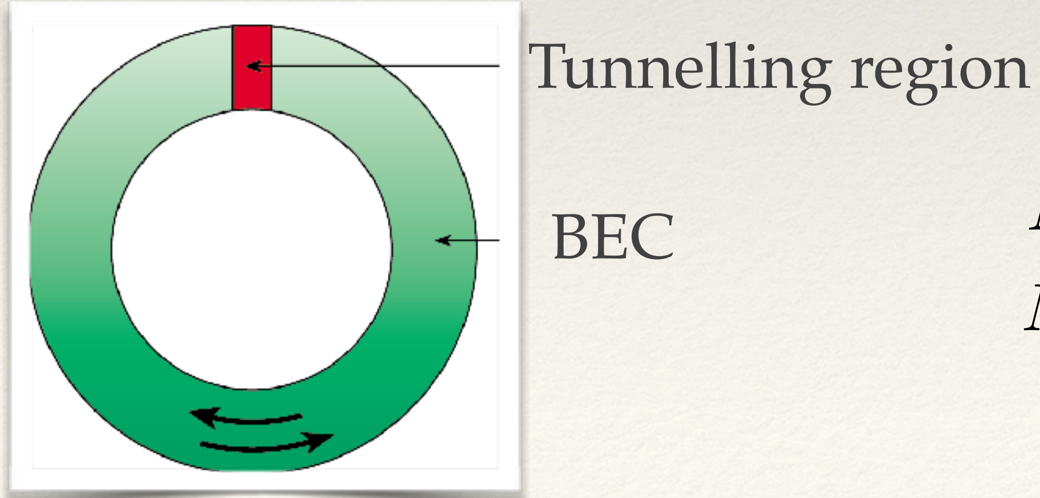




$N_L = \#$ of atoms in left well

$N_R = \#$ of atoms in right well

2. Rotation of a Bose-Einstein condensate around a ring



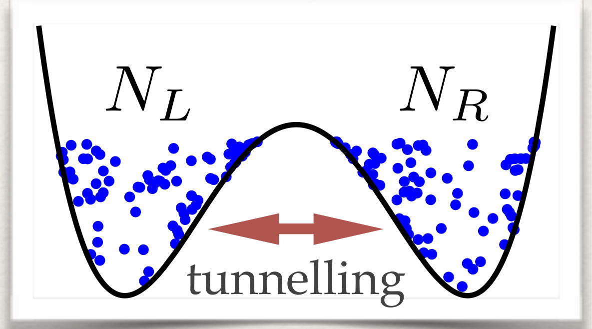
$N_c = \#$ of clockwise rotating atoms

$N_a = \#$ of anticlockwise rotating atoms

Quantum field theory description

Bose-Hubbard model: $$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i$$

Reduce to two sites:



$$\hat{H} = -J(\hat{a}_l^\dagger \hat{a}_r + \hat{a}_r^\dagger \hat{a}_l) + \frac{U}{4}(\hat{a}_l^\dagger \hat{a}_l - \hat{a}_r^\dagger \hat{a}_r)^2 + \text{constant terms}$$

In Josephson junction language:

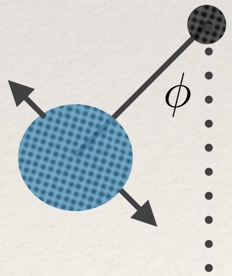
$$\hat{H} = -\frac{E_J}{N}(\hat{a}_l^\dagger \hat{a}_r + \hat{a}_r^\dagger \hat{a}_l) + \frac{E_c}{2}(\hat{a}_l^\dagger \hat{a}_l - \hat{a}_r^\dagger \hat{a}_r)^2$$

Classical field theory (mean-field theory)

$$\hat{H} = -\frac{E_J}{N}(\hat{a}_l^\dagger \hat{a}_r + \hat{a}_r^\dagger \hat{a}_l) + \frac{E_c}{2}(\hat{a}_l^\dagger \hat{a}_l - \hat{a}_r^\dagger \hat{a}_r)^2$$

$$\begin{cases} \hat{a}_l \to \sqrt{n_l} e^{i\phi_l} \\ \hat{a}_l^\dagger \to \sqrt{n_l} e^{-i\phi_l} \end{cases} \qquad \begin{cases} \hat{a}_r \to \sqrt{n_r} e^{i\phi_r} \\ \hat{a}_r^\dagger \to \sqrt{n_r} e^{-i\phi_r} \end{cases}$$

$$H = \to -E_J\sqrt{1 - 4\frac{n^2}{N^2}}\cos\phi + \frac{E_c}{2}n^2 \approx -E_J\cos\phi + \frac{E_c}{2}n^2$$



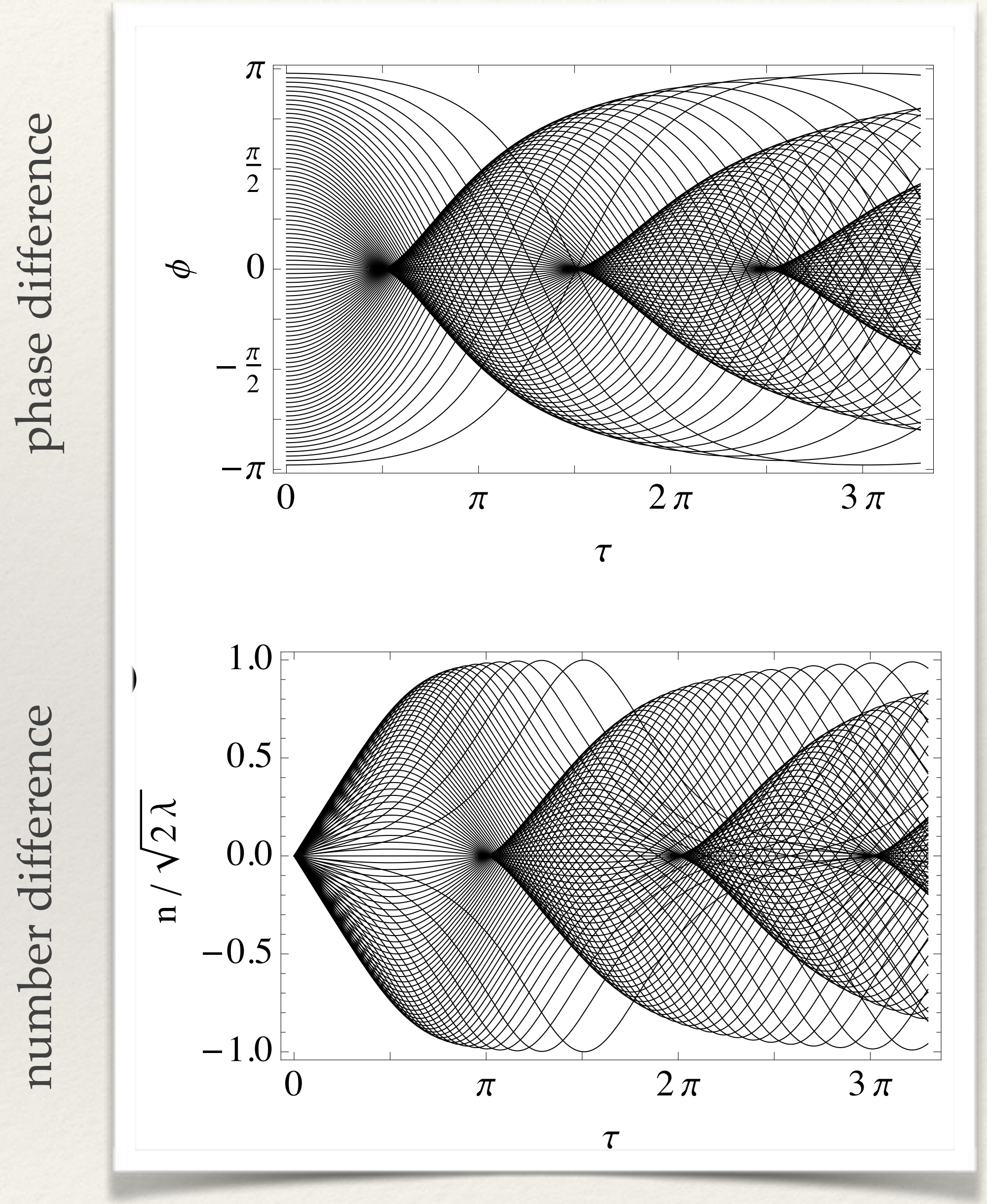
pendulum

where: $$n \equiv \frac{1}{2}(n_l - n_r) \quad , \quad \phi \equiv \phi_l - \phi_r$$

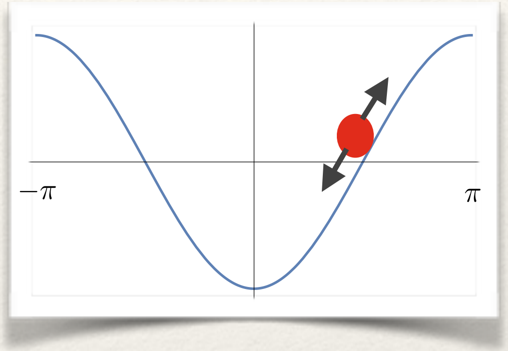
population difference phase difference

Mean-field theory is equivalent to Maxwell's theory for light…

Classical-field cusps in the dynamics of a bosonic Josephson junction



$$H = \frac{E_c}{2} n^2 - E_J \cos\phi$$



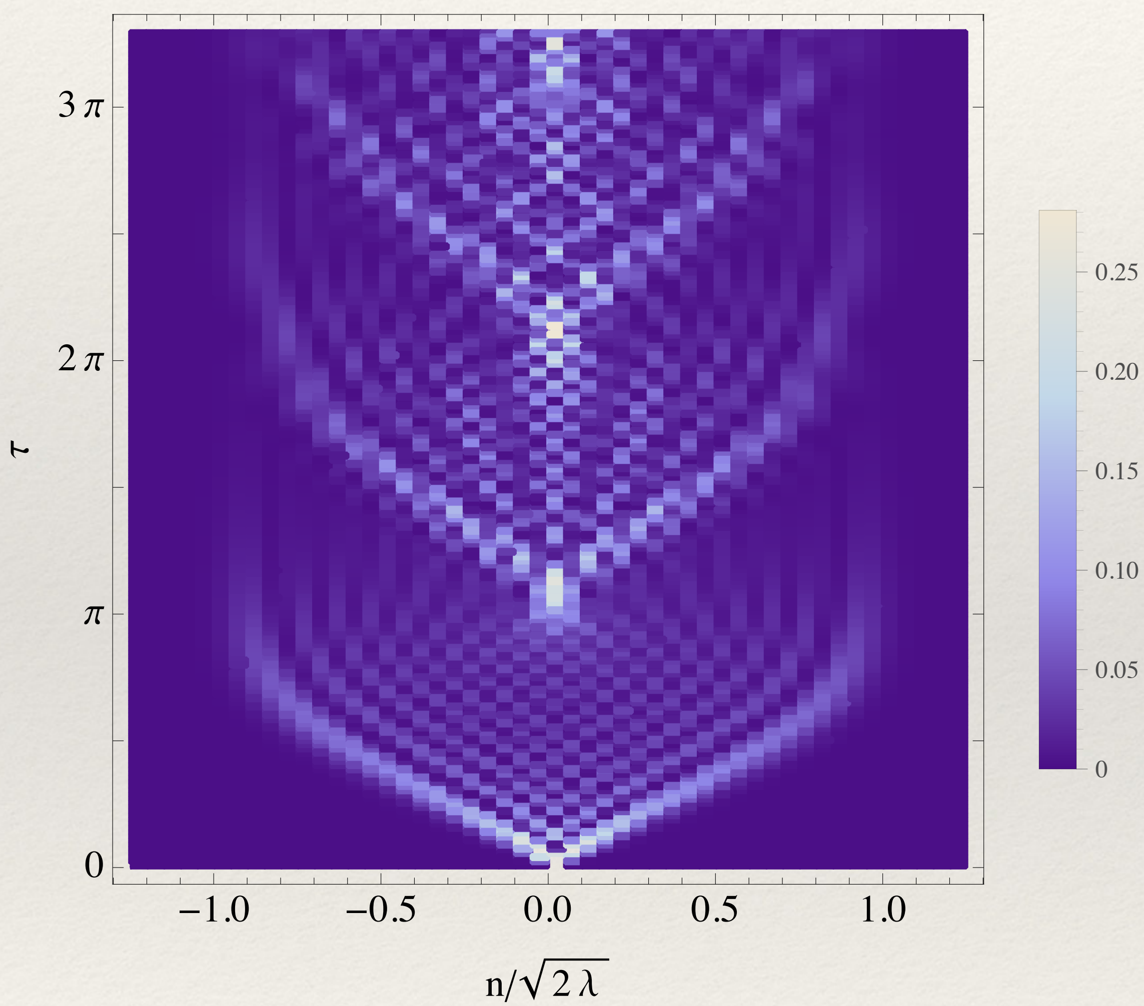
Josephson's equations [mean-field theory] :

$$\dot{\phi} = \frac{E_c}{\hbar} n$$

$$\dot{n} = -\frac{E_J}{\hbar} \sin\phi$$

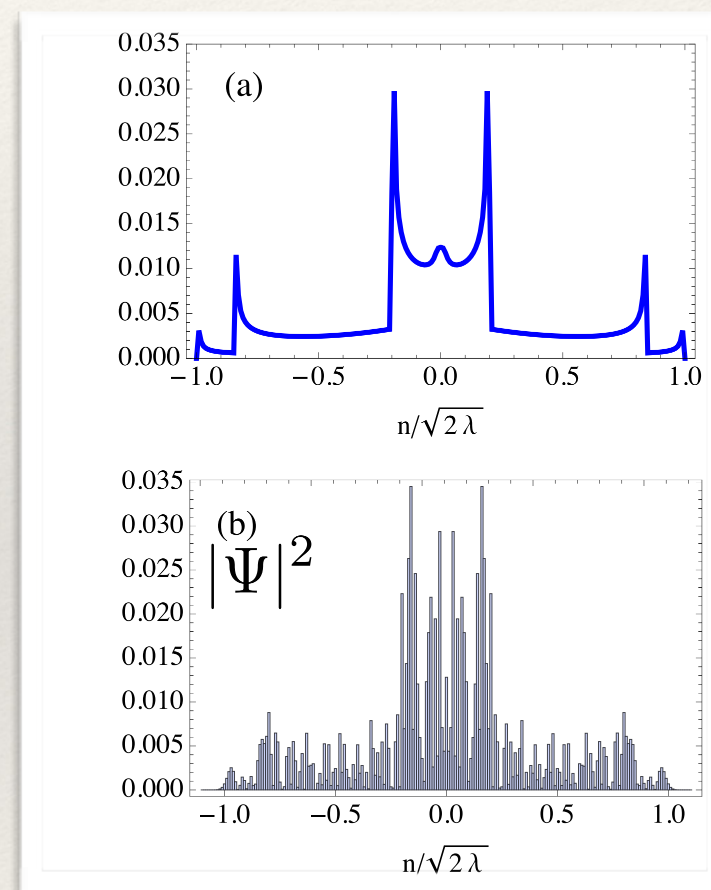
Note that in quantum mechanics: $[\hat{\phi}, \hat{n}] \approx i \longrightarrow \Delta\phi \Delta n \geq \frac{1}{2}$

Quantum field dynamics



$$\lambda \equiv \frac{2E_J}{E_c} = 200$$

$$\tau = 3.3\pi$$



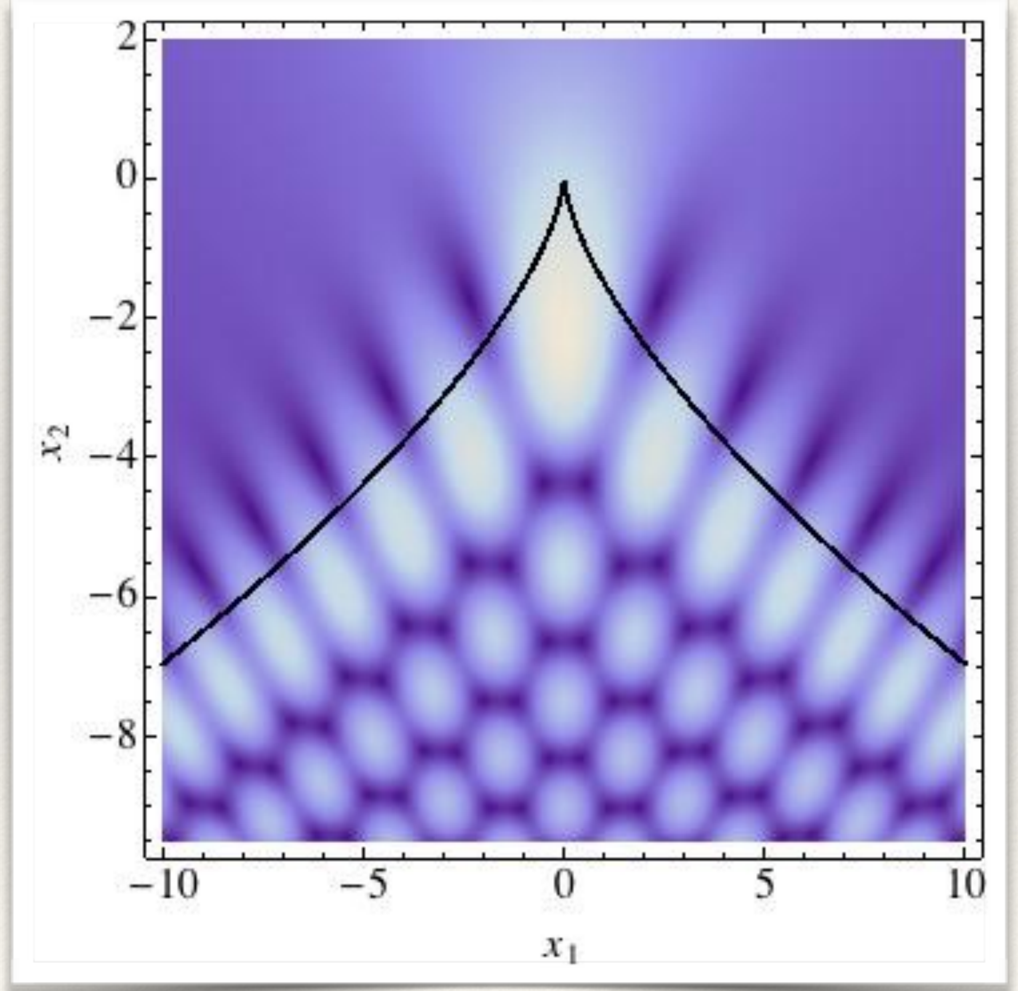
Classical field (Gross-Pitaevskii theory)

Quantum field (Bose-Hubbard theory)

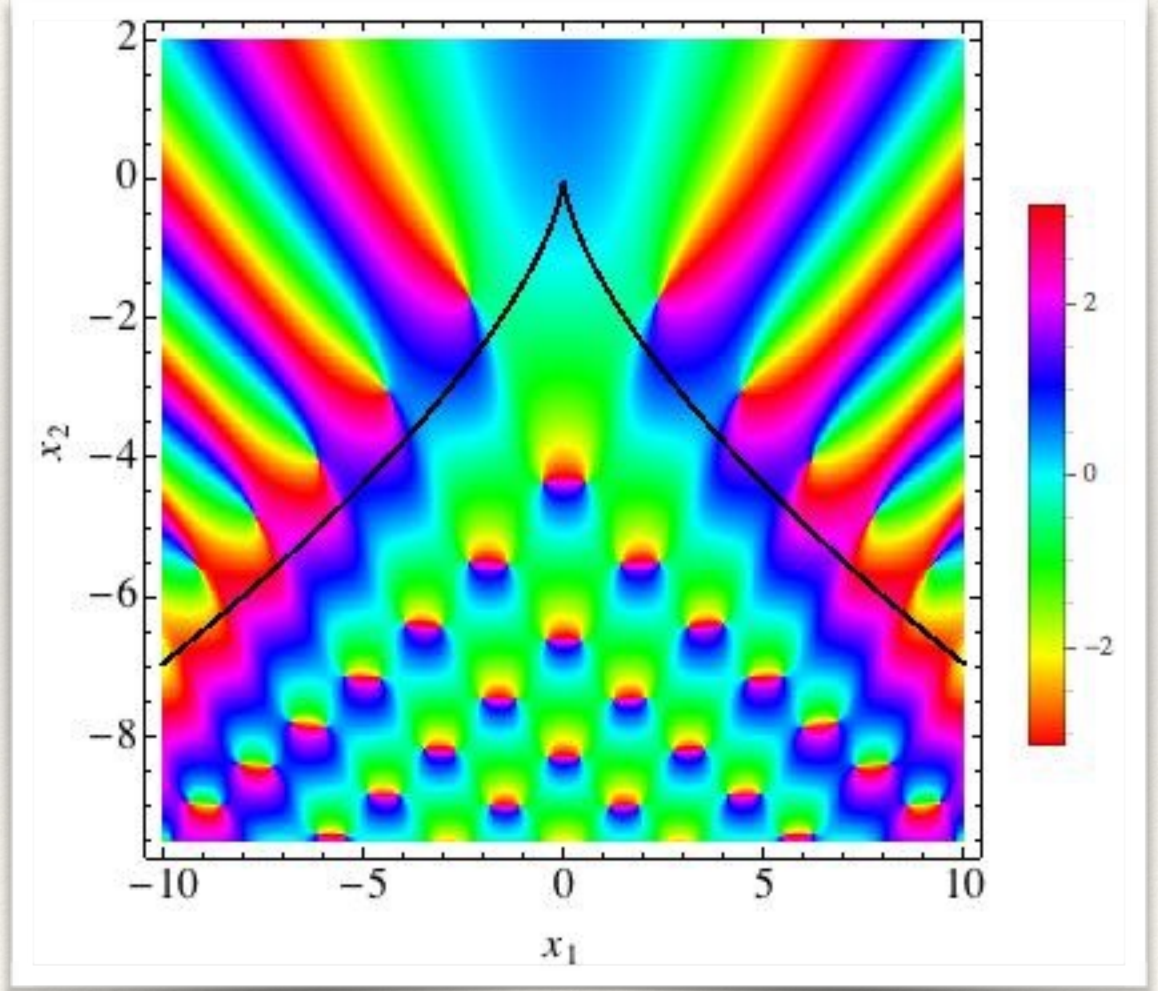
$$\lambda \equiv \frac{2E_J}{E_c} = 5000$$

Fine structure: vortex-antivortex pairs

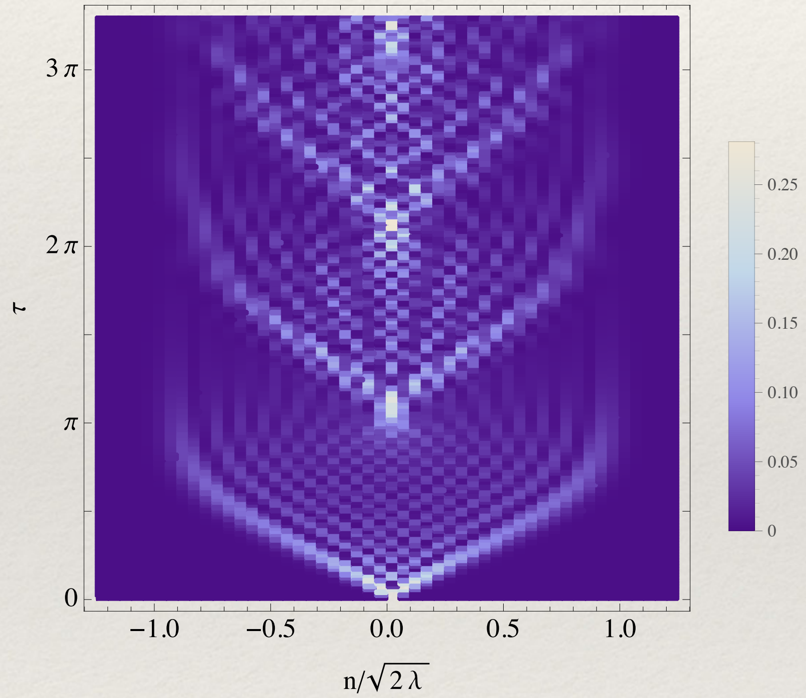
amplitude



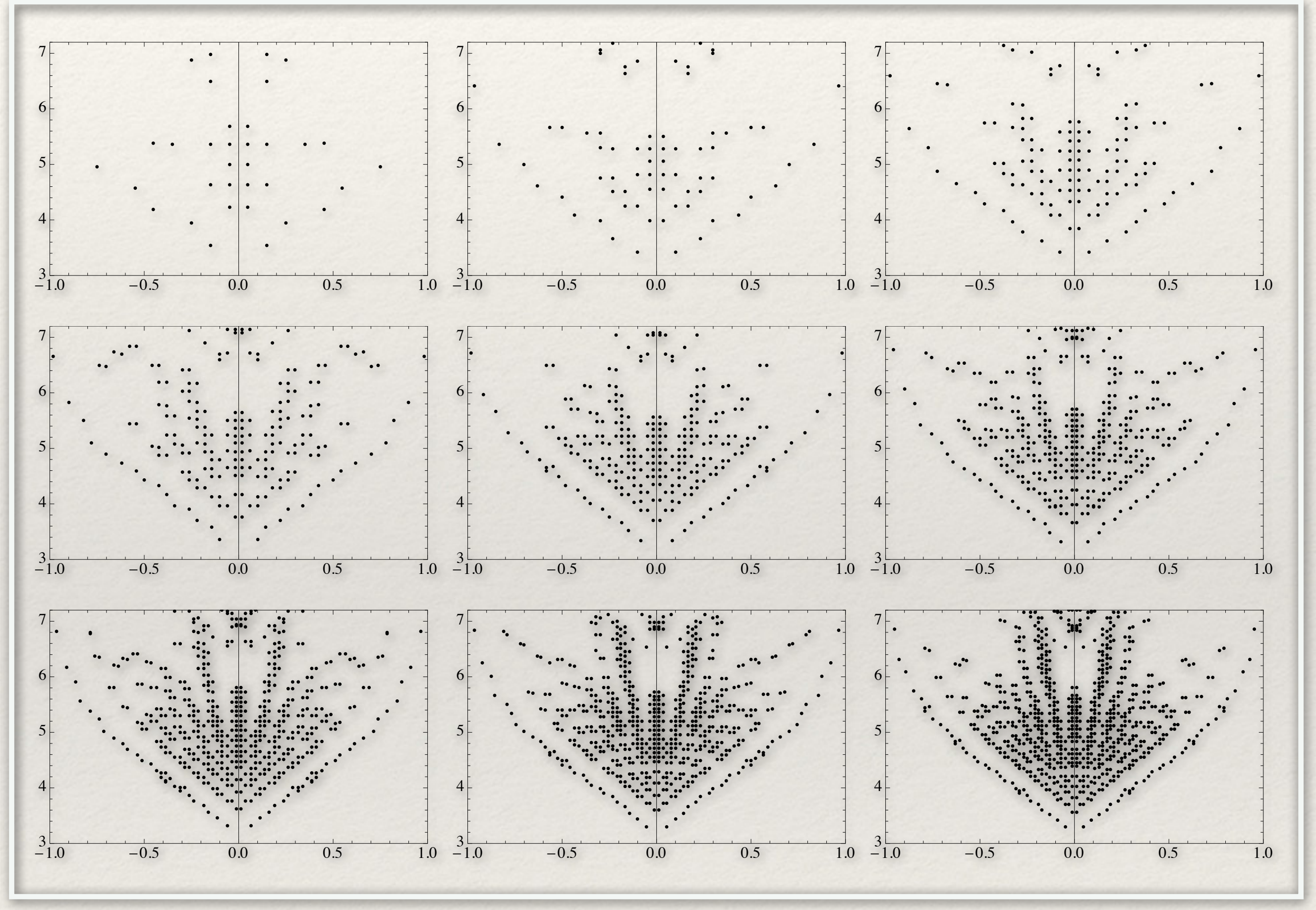
phase



Fine structure in the quantum cusp: vortices in Fock space



λ=200



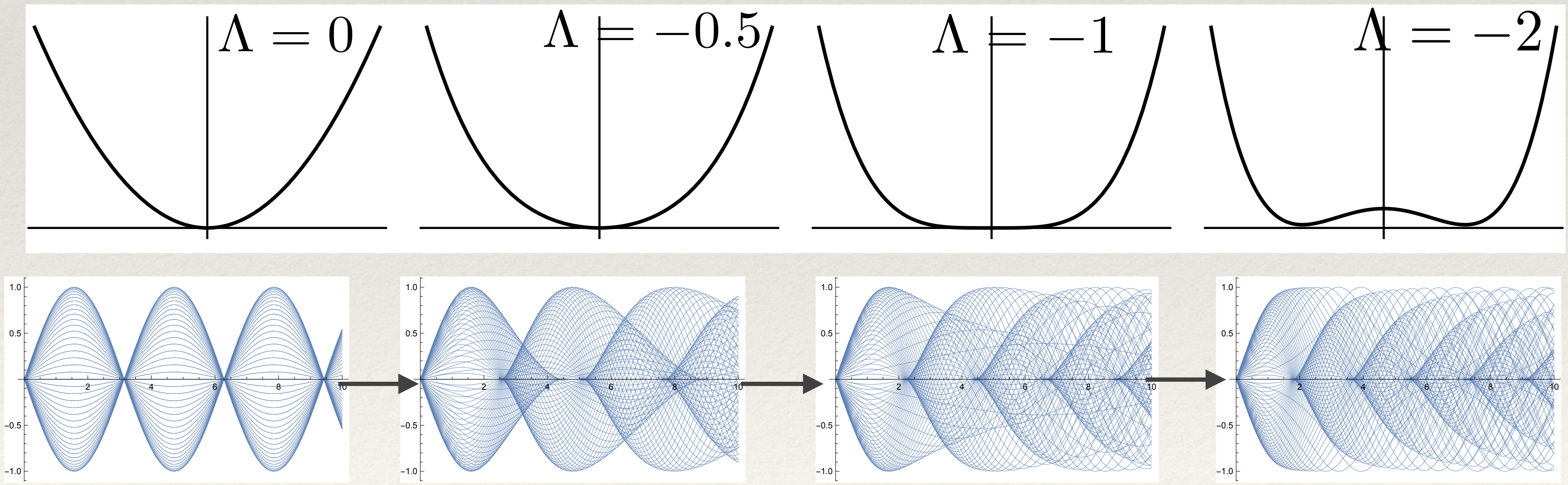
λ=50,112.5, 200, 312.5, 450, 612.5, 800, 1012.5 and 1200

Dynamics near a quantum phase transition

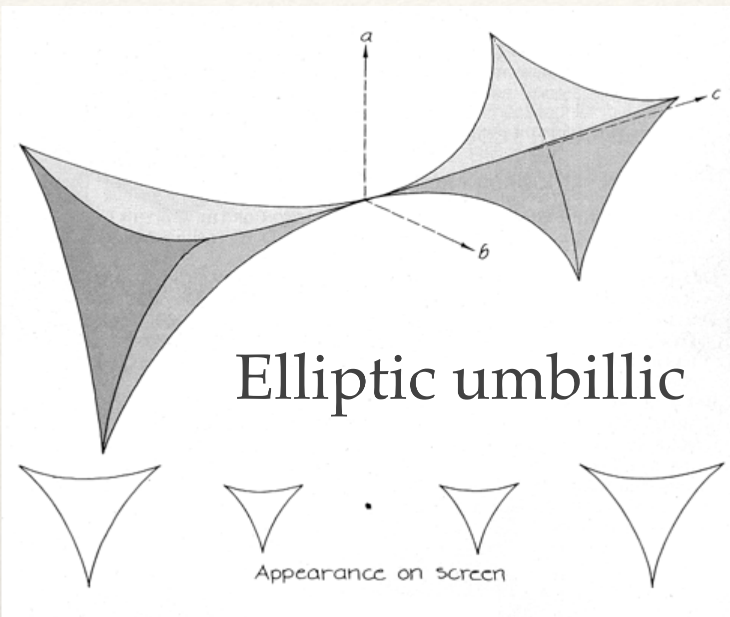
$$H = \frac{E_c}{2} n^2 - E_J \sqrt{1 - 4n^2/N^2} \cos\phi$$

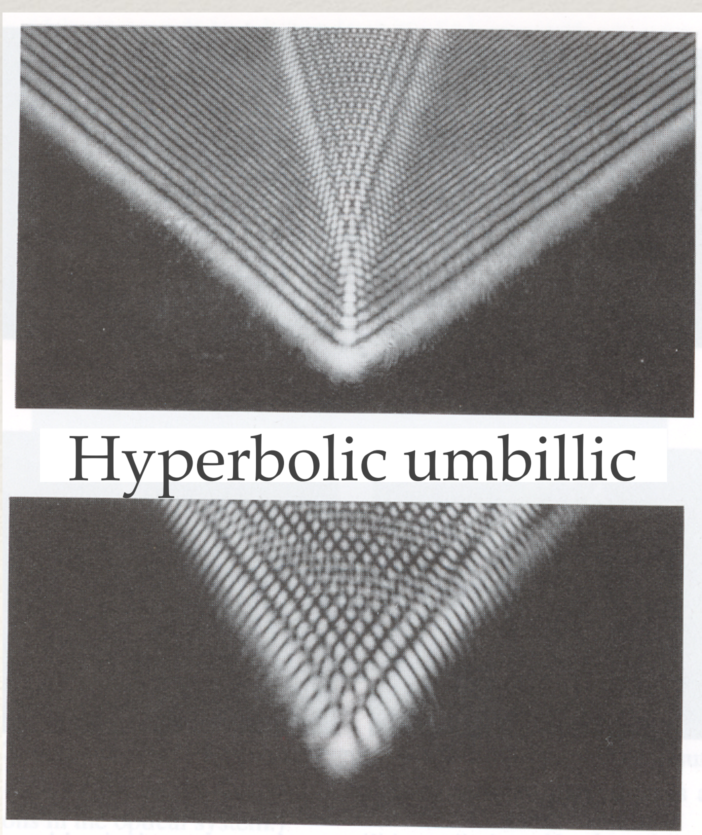
$$\dot{n}^2 + n^2(1 - \Lambda H_0 + \Lambda^2 n^2/4) = 1 - H_0^2$$

$$\boxed{\Lambda_c = -1}$$

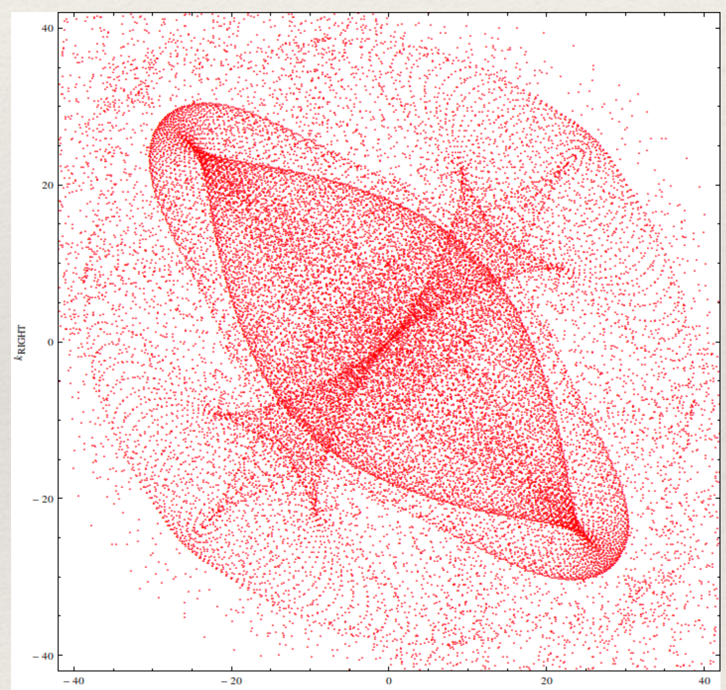


Triple well

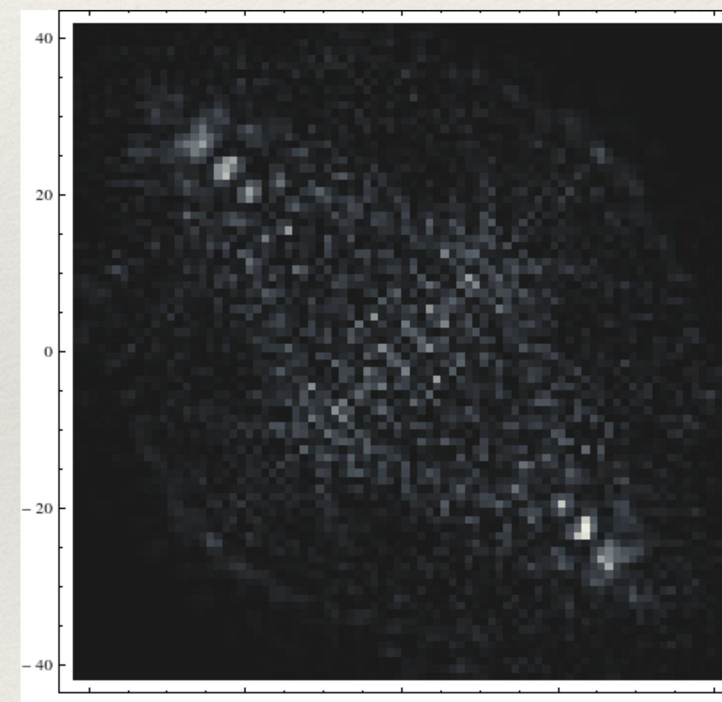




Triple well simulations



classical



quantum

$\lambda = 750 \qquad N_{\text{total}} = 42$

Summary

- Catastrophes are universal objects in classical and quantum in the dynamics.
- They fall into equivalence classes.
- The wave function and it scaling properties in the immediate vicinity of a catastrophe are given by one of the Thom-Arnold generating functions.
- Catastrophes have three levels of structure (geometric, interference fringes,vortices)
- Quantum catastrophes live in Fock space and are naturally discretized; they also contain discretized vortices.
- Dynamics near phase transitions can generate catastrophes.

Acknowledgements:

Donald Sprung, Yohan Yee, Eric Turner

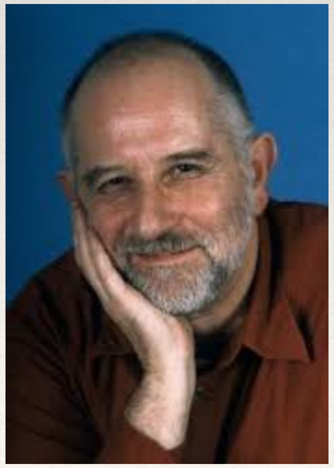
Outline

1. The big question
2. Gallery of catastrophes in nature
3. Catastrophes in quantum fluids
4. Fine structure of a quantum catastrophe

The big question

When do we need to second-quantize waves in order to avoid singularities?

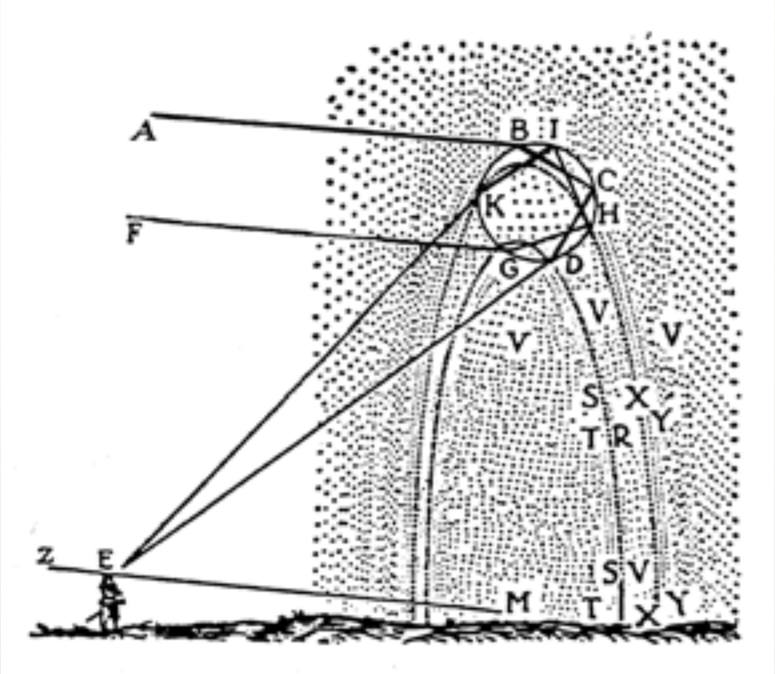
M.V. Berry, Nonlinearity **21**, T19 (2008),
"**Three quantum obsessions**"



When do we need to 1st quantize?

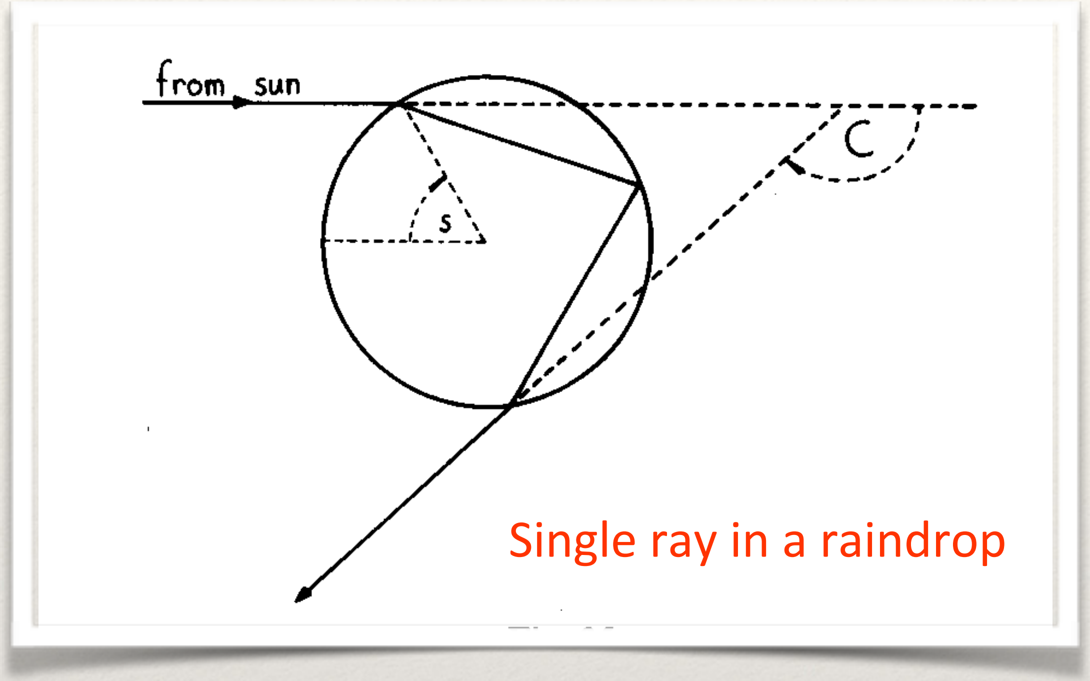
Ray ABCDE gives the primary bow

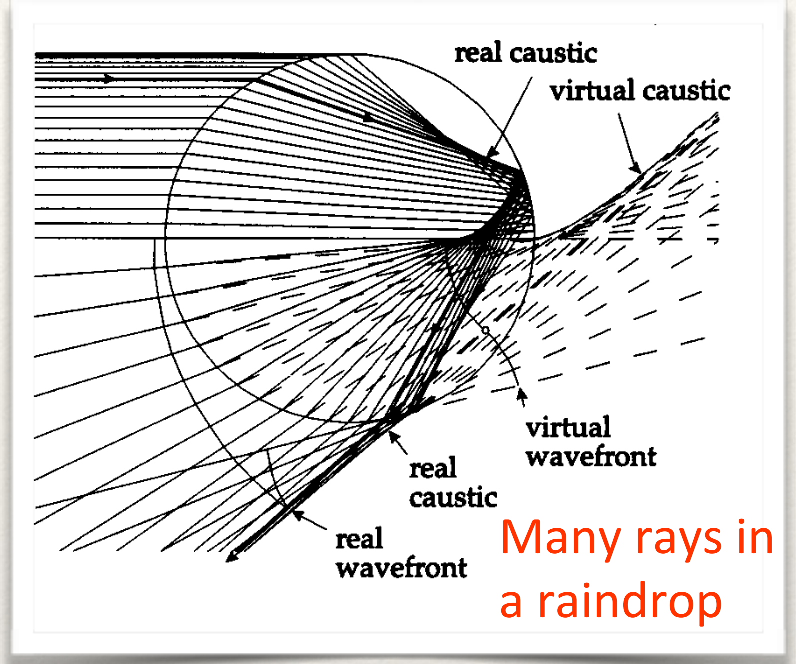
Ray FGHIKE gives the secondary bow



René Descartes' geometrical ray theory of the rainbow, *Discourse on Method* (1637)

The rainbow as a caustic

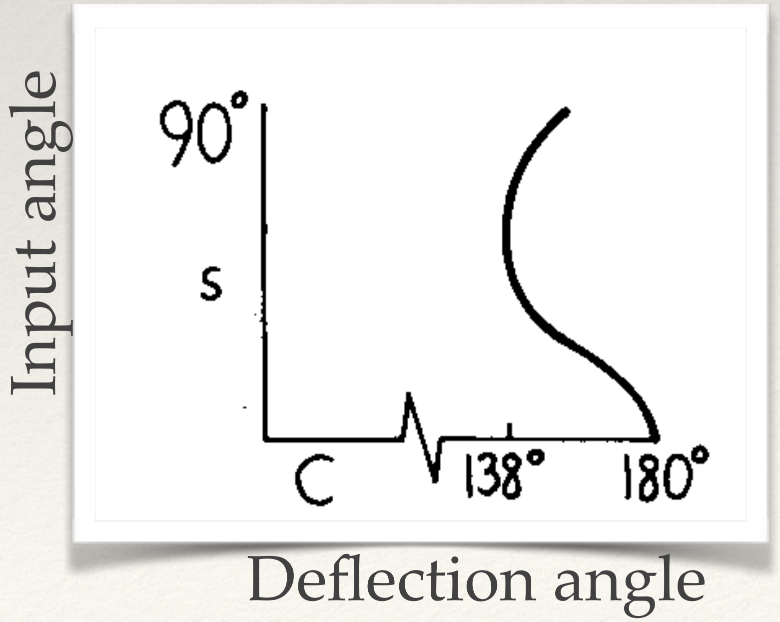




Caustic = envelope of a family of rays

In ray theory the light intensity **diverges** on a caustic: “a lot goes into a little”

Caustics are the singularities of ray theory, i.e. places where it fails

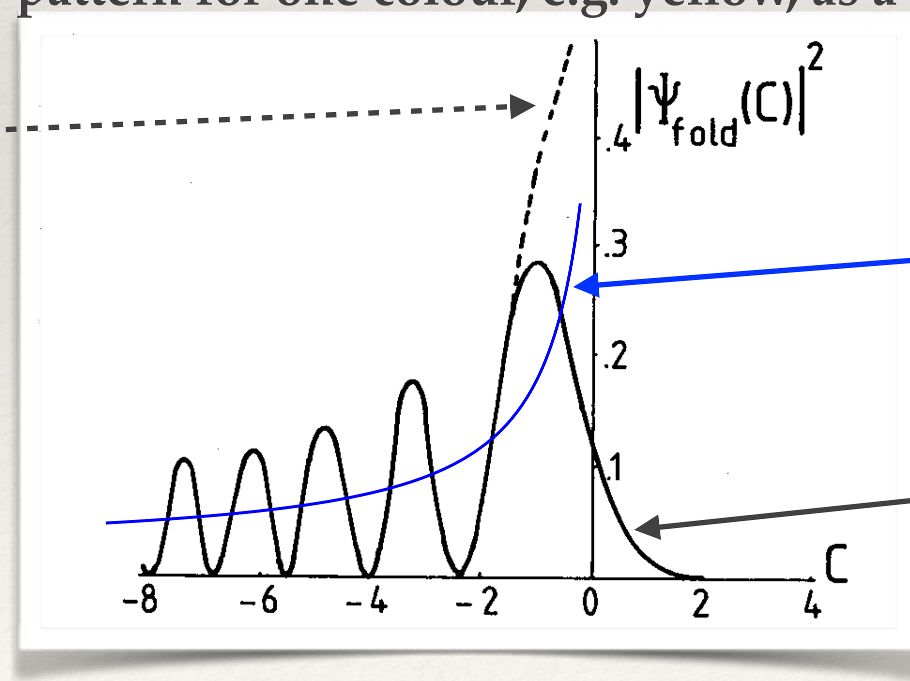


Taming the singularity: wave theory (1st quantization)

Supernumerary arcs = Airy fringes made by white light

Intensity pattern for one colour, e.g. yellow, as a function of angle

WKB theory
(Thomas Young, 1803)

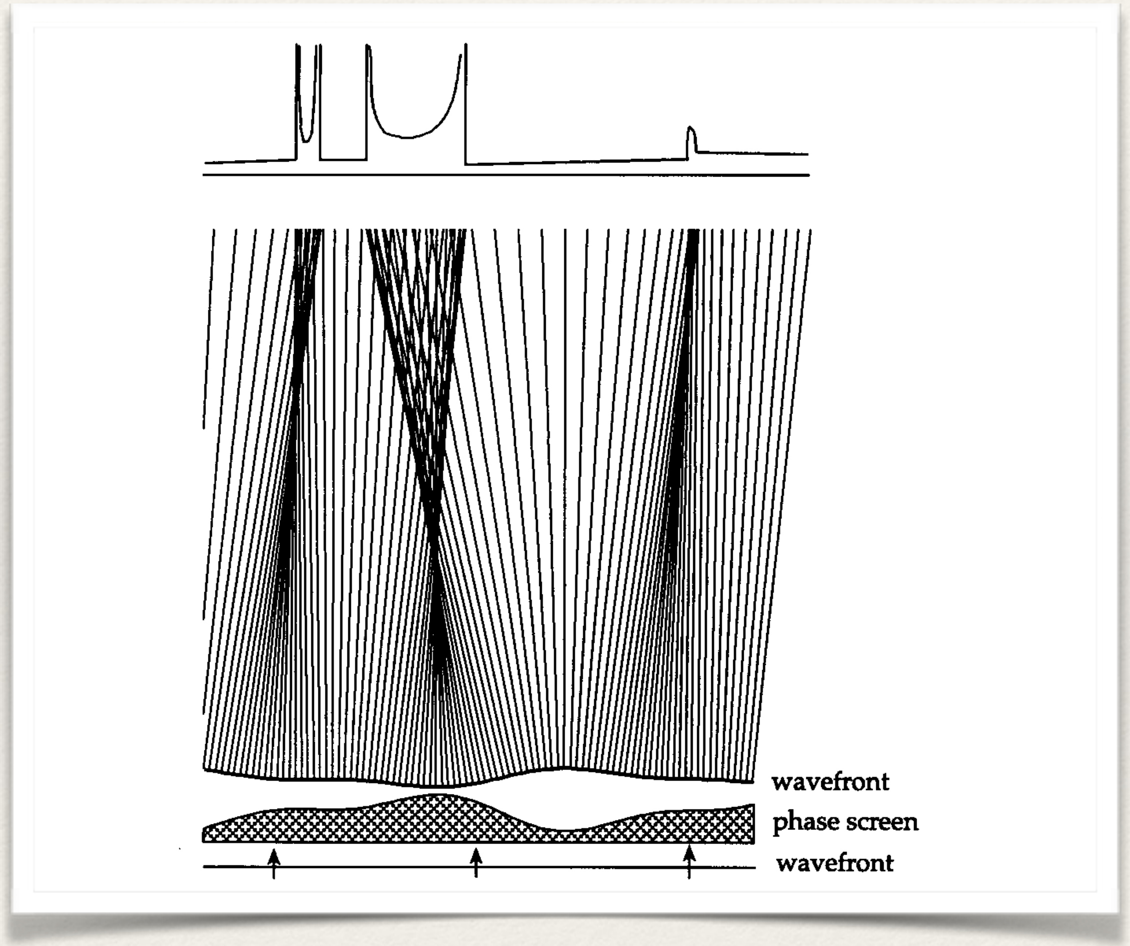


Geometric optics

Airy function

G.B. Airy, *On the intensity of light in the neighbourhood of a caustic*, Trans. Camb. Phil. Soc. **6**, 379 (1838)

Twinkling of starlight



Quantum catastrophe: Hawking radiation

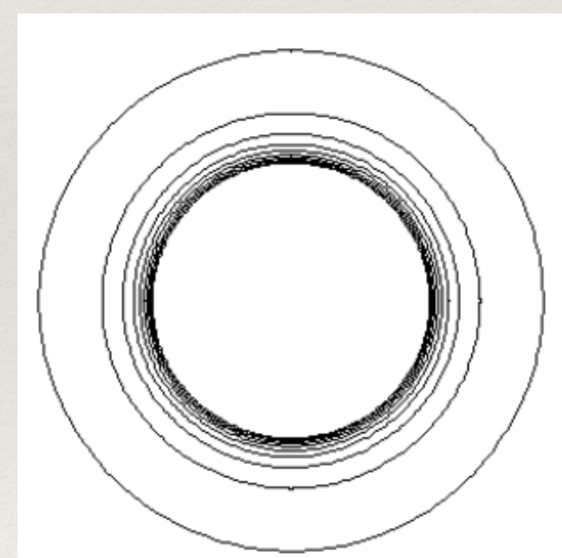
U. Leonhardt, Nature **415**, 406 (2002)

A laboratory analogue of the event horizon using slow light in an atomic medium

Ulf Leonhardt

School of Physics and Astronomy, University of St Andrews, North Haugh, St Andrews, Fife KY16 9SS, UK

Singularities underlie many optical phenomena[1]. The rainbow, for example, involves a particular type of singularity—a ray catastrophe—in which light rays become infinitely intense. In practice, the wave nature of light resolves these infinities, producing interference patterns. At the event horizon of a black hole[2], time stands still and waves oscillate with infinitely small wavelengths. However, the quantum nature of light results in evasion of the catastrophe and the emission of Hawking radiation[3]. Here I report a theoretical laboratory analogue of an event horizon: a parabolic profile of the group velocity[7] of light brought to a standstill in an atomic medium[4–6] can cause a wave singularity similar to that associated with black holes. In turn, the quantum vacuum is forced to create photon pairs with a characteristic spectrum, a phenomenon related to Hawking radiation[3]. The idea may initiate a theory of 'quantum' catastrophes, extending classical catastrophe theory[8,9].



Waves approaching an event horizon suffer a logarithmic phase singularity: $\lambda \sim (r-r_{eh})$

Rogue waves

PRL 104, 093901 (2010) PHYSICAL REVIEW LETTERS week ending 5 MARCH 2010

Freak Waves in the Linear Regime: A Microwave Study

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[2]*Department of Physics, Tulane University, New Orleans, Louisiana 70118, USA*
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(Received 4 September 2009; revised manuscript received 2 December 2009; published 1 March 2010)

Microwave transport experiments have been performed in a quasi-two-dimensional resonator with randomly distributed conical scatterers. At high frequencies, the flow shows branching structures similar to those observed in stationary imaging of electron flow. Semiclassical simulations confirm that caustics in the ray dynamics are responsible for these structures. At lower frequencies, large deviations from Rayleigh's law for the wave height distribution are observed, which can only partially be described by existing multiple-scattering theories. In particular, there are "hot spots" with intensities far beyond those expected in a random wave field. The results are analogous to flow patterns observed in the ocean in the presence of spatially varying currents or depth variations in the sea floor, where branches and hot spots lead to an enhanced frequency of freak or rogue wave formation.

FIG. 1 (color online). Photograph of one of the two scattering arrangements used. The platform has width 260 mm and length 360 mm. Each cone has diameter 25 mm and height 15 mm. The probe antenna is fixed in a horizontally movable top plate located 20 mm above the bottom (not shown).

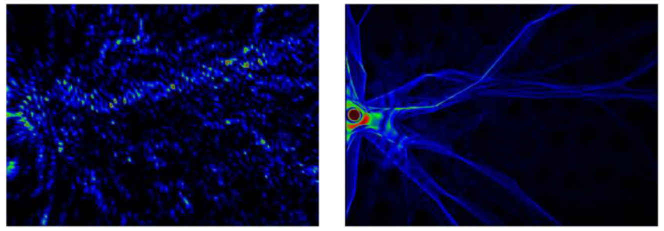


FIG. 2 (color online). Comparison of an experimental wave pattern with a classical ray simulation. Left: A wave function at frequency $f = 30.95$ GHz. Right: The corresponding semiclassical simulation, with modes 1 through 4 added together.

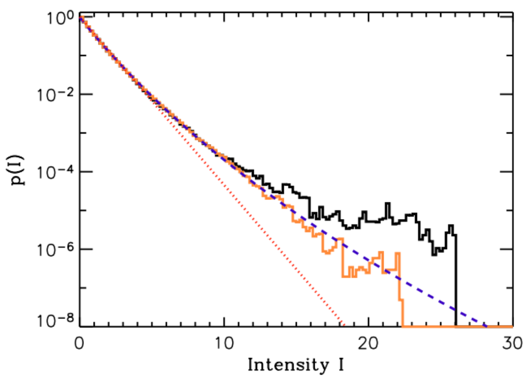


FIG. 3 (color online). Probability distribution of intensities. The dark (black) histogram includes all data, while the light (yellow) histogram excludes frequencies associated with the hot spots. The dotted line is the Rayleigh distribution, while the dashed (blue) line is a best fit using the theoretical distribution given by Eq. (2) ($\gamma \approx 23.5$).

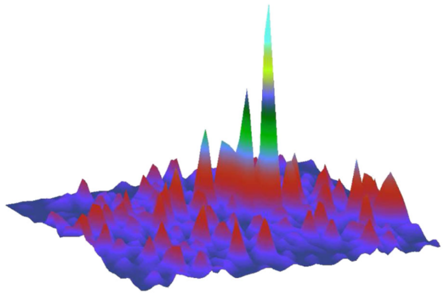
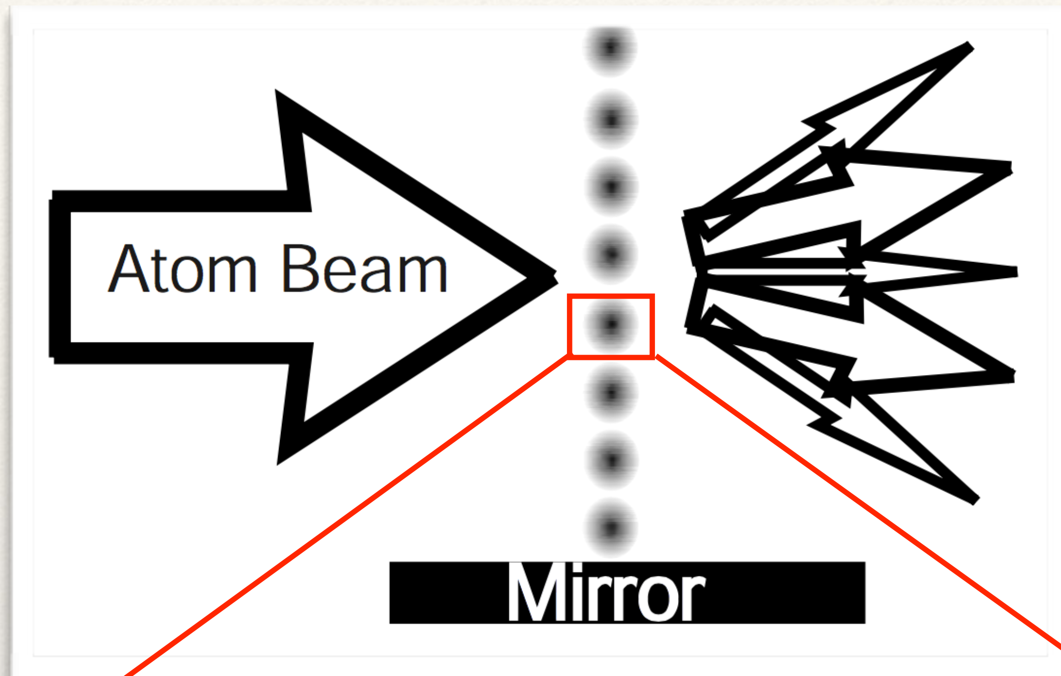
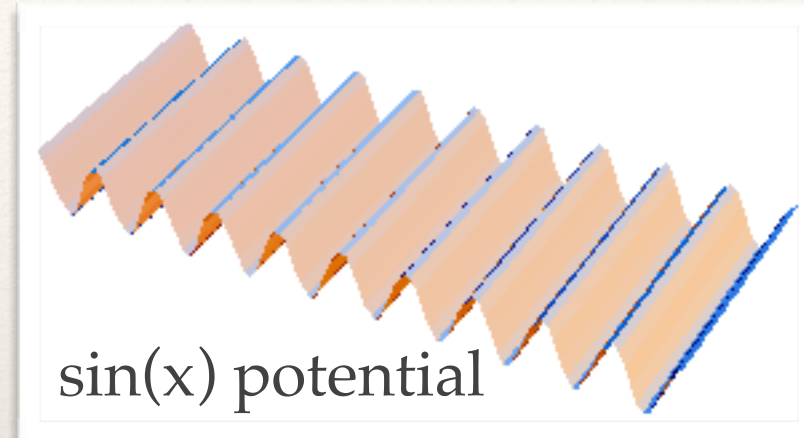


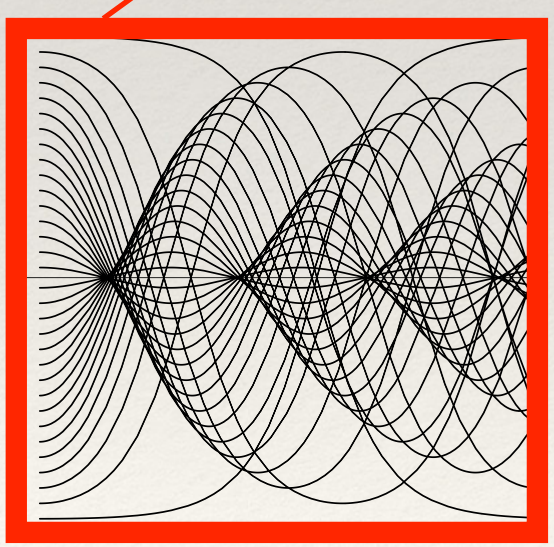
FIG. 4 (color online). A "hot spot," observed at a frequency of 8.85 GHz. The experimental probability density for observing such a hot spot is 1 to 2 orders of magnitude larger than that expected from multiple-scattering theory.

Caustics in atom diffraction

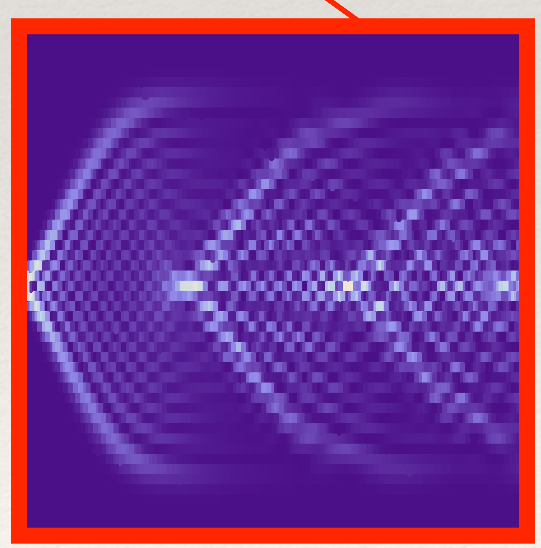


standing wave laser



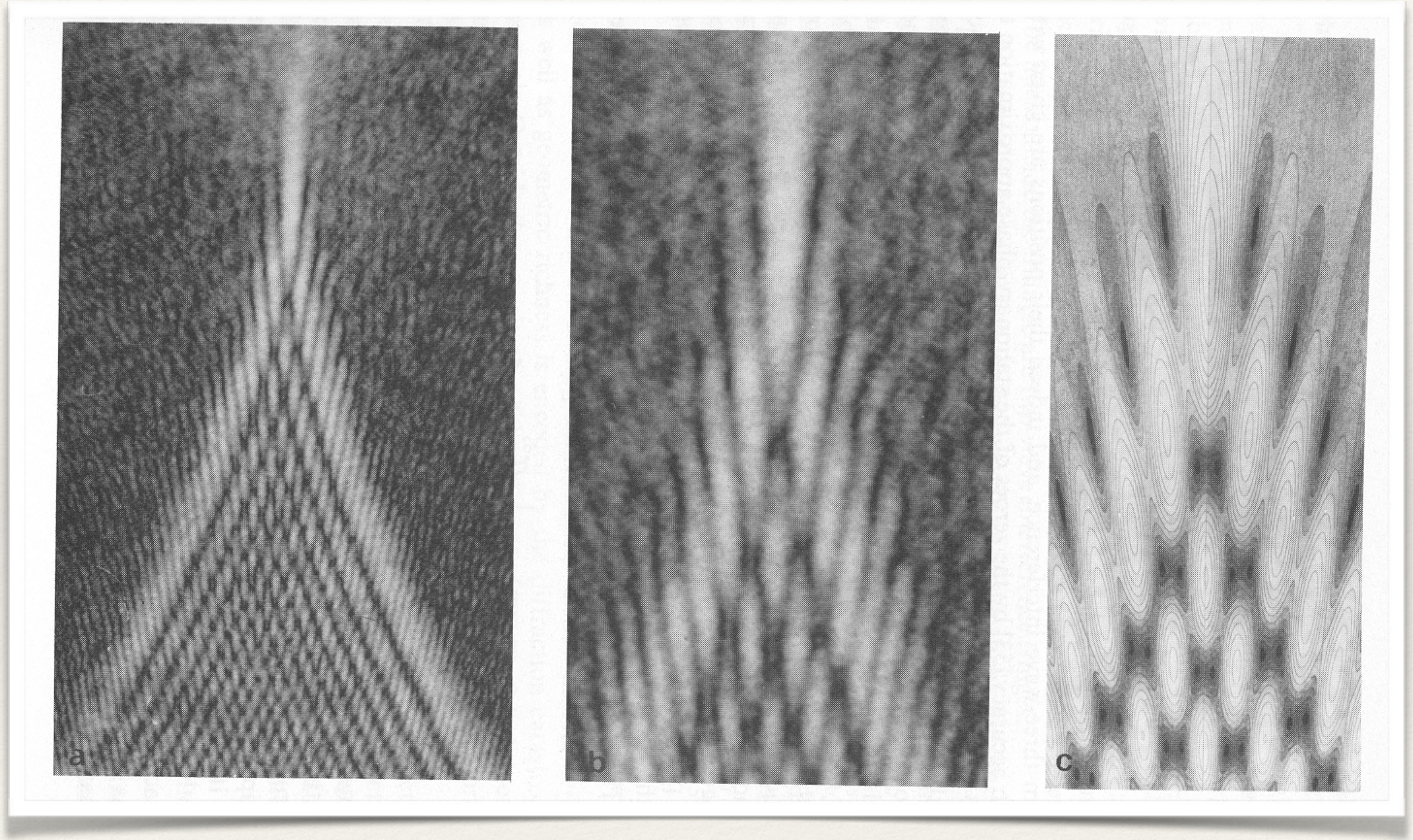


channeling (classical mechanics)



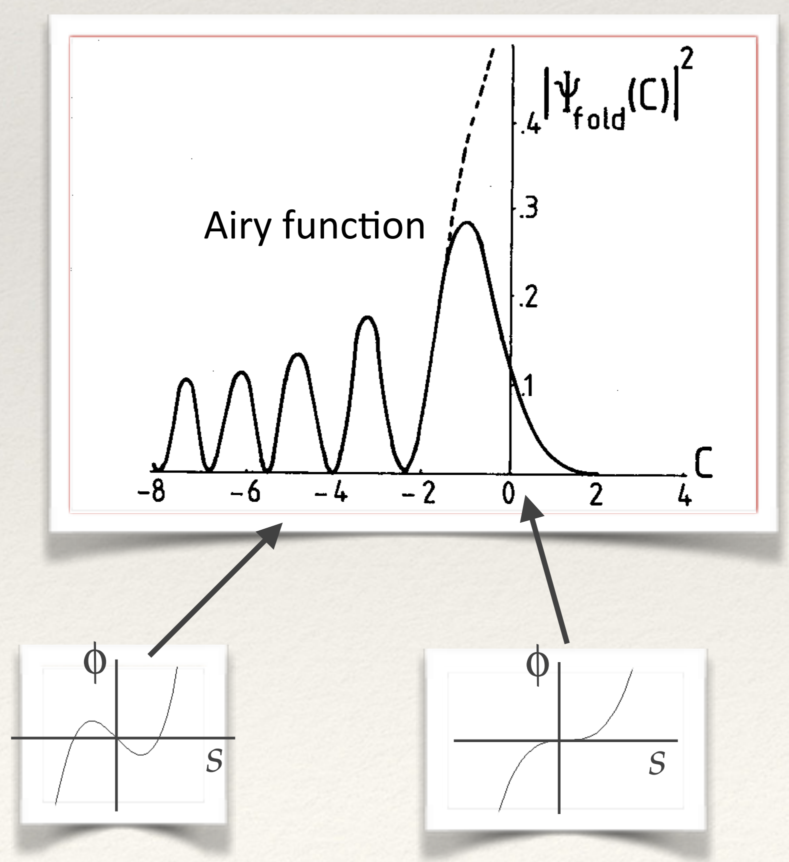
dynamical diffraction (matter waves)

Wave theory removes geometric singularity



The Airy function as a path integral

$$\Psi_{\text{fold}}(C) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s^3/3 + Cs)}\ ds$$
$$= \sqrt{2\pi}\,\text{Ai}[C]$$



gradient map:

$$\frac{\partial \phi}{\partial s} = s^2 + C = 0$$

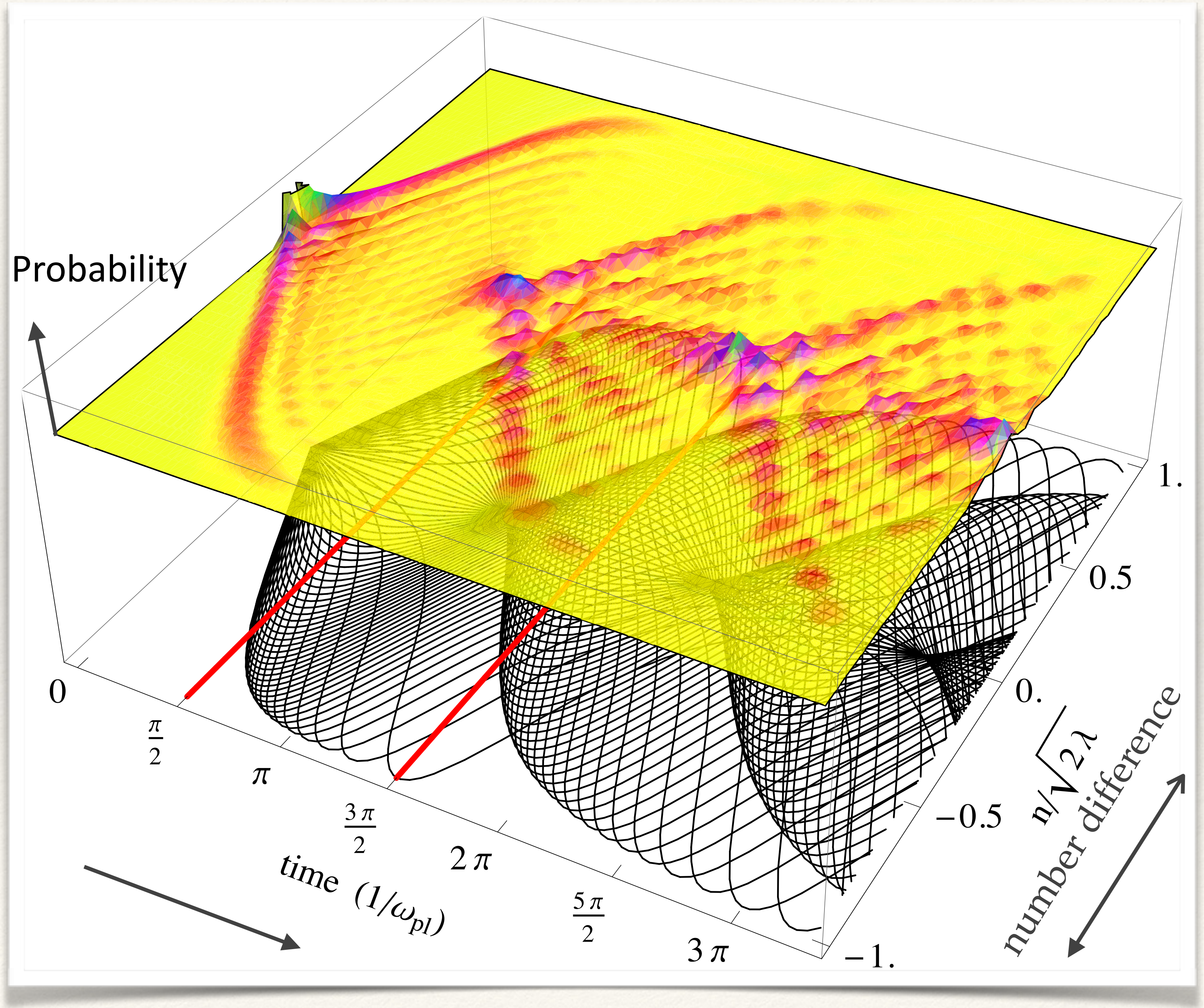
Two interfering rays when *C<0:*

$$s^{(\pm)} = \pm\sqrt{-C}$$

Rays coalesce on caustic at *C=0*.

Universal quantum dynamics! Catastrophes in Fock space following the sudden coupling of two independent BECs

D. O'D., Phys. Rev. Lett. **109**, 150406 (2012)



$$\lambda \equiv \frac{2E_J}{E_c}$$

$$= 200$$

Poisson resummation of wave function

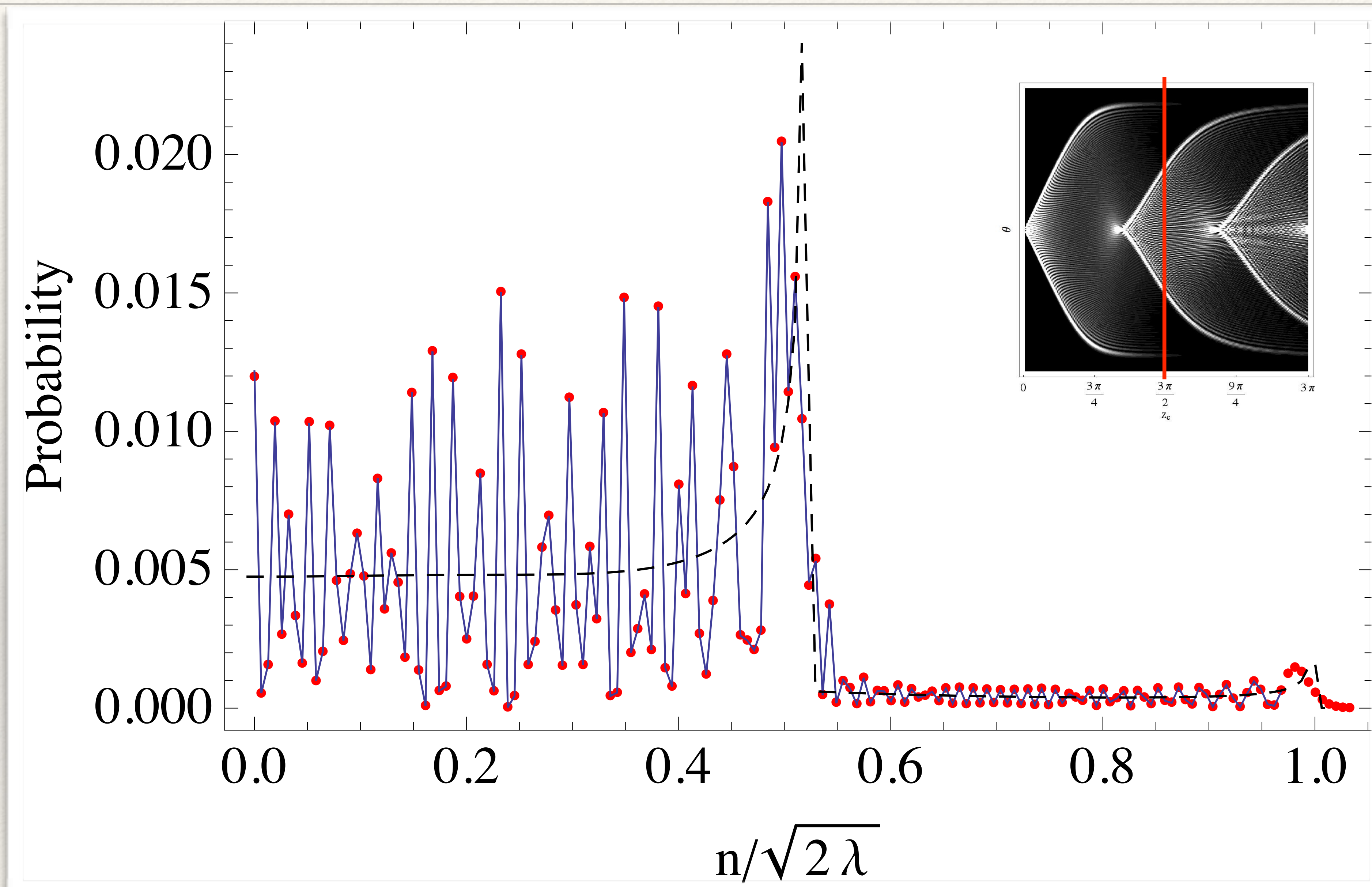
1) $$\Psi(t) = \sum_n A_n \, \psi_n(x) e^{-iE_n t/\hbar}$$ Energy eigenfunction superposition

2) $$\sum_{j=0}^{\infty} f(j) = \sum_{m=-\infty}^{\infty} \int_0^{\infty} f(j) \mathrm{e}^{2\pi \mathrm{i} m j} \, dj$$ Poisson resummation

3) $$\begin{aligned}\Psi_{\text{rainbow}} = \frac{\mathrm{e}^{\mathrm{i}\sqrt{\Lambda}\bar{\mathcal{V}}(y,z_c)}}{\sqrt{2}} \Bigg[& \left(\frac{1}{(1-(y^2-\beta_1)^2)^{1/4}(1-\beta_1^2)^{1/4}\sqrt{\mathcal{V}''(\beta_1)}} \right. \\ & \left. + \frac{1}{(1-(y^2-\beta_2)^2)^{1/4}(1-\beta_2^2)^{1/4}\sqrt{-\mathcal{V}''(\beta_2)}} \right) \left(\frac{3\Delta\mathcal{V}}{4\Lambda} \right)^{1/6} \mathrm{Ai}\left(-\left(\frac{3\sqrt{\Lambda}\Delta\mathcal{V}}{4} \right)^{2/3} \right) \\ & - \mathrm{i} \left(\frac{1}{(1-(y^2-\beta_1)^2)^{1/4}(1-\beta_1^2)^{1/4}\sqrt{\mathcal{V}''(\beta_1)}} \right. \\ & \left. - \frac{1}{(1-(y^2-\beta_2)^2)^{1/4}(1-\beta_2^2)^{1/4}\sqrt{-\mathcal{V}''(\beta_2)}} \right) \left(\frac{4}{3\Lambda^2\Delta\mathcal{V}} \right)^{1/6} \mathrm{Ai}'\left(-\left(\frac{3\sqrt{\Lambda}\Delta\mathcal{V}}{4} \right)^{2/3} \right) \Bigg].\end{aligned}$$

Already predicted by catastrophe theory!

Caustics emerge as N⟶∞



Relative to background, rainbow peak diverges as $N^{1/6}$, and cusp peak as $N^{1/4}$

$n \ll n_c$ (bright)	$n = n_c$	$n \gg n_c$ (dark)
$\mathcal{O}(\lambda^{-\frac{1}{2}}) \times \cos[\mathcal{O}(\lambda^{\frac{1}{2}})]$	$\mathcal{O}(\lambda^{-\frac{1}{3}})$	$\mathcal{O}(\lambda^{-\frac{1}{2}})\, \exp[-\mathcal{O}(\lambda^{\frac{1}{2}})]$
$\mathcal{O}(N^{-1/2}) \times \cos[\mathcal{O}(N^{\frac{1}{2}})]$	$\mathcal{O}(N^{-\frac{1}{3}})$	$\mathcal{O}(N^{-\frac{1}{2}})\, \exp[-\mathcal{O}(N^{\frac{1}{2}})]$

Scaling exponents

Catastrophe	Arnold Index β	Berry Indices σ_j	Berry Index γ
Fold	1/6	*2/3*	2/3
Cusp	1/4	*3/4, 1/2*	5/4
Swallowtail	3/10	*4/5, 3/5, 2/5*	9/5
Elliptic umbilic	1/3	*2/3, 2/3, 1/3*	5/3
Hyperbolic umbilic	1/3	*2/3, 2/3, 1/3*	5/3
Butterfly	1/3	*5/6, 2/3, 1/2, 1/3*	7/3
Parabolic umbilic	3/8	*5/8, 3/4, 1/2, 1/4*	17/8

$$\psi(C_j; k) = \left(\frac{k}{k_0}\right)^{\beta} \psi\left[(k/k_0)^{\sigma_j} C_j; k_0\right]$$

Airy function in critical Anderson model

PRL **105,** 090601 (2010) PHYSICAL REVIEW LETTERS week ending 27 AUGUST 2010

Critical State of the Anderson Transition: Between a Metal and an Insulator

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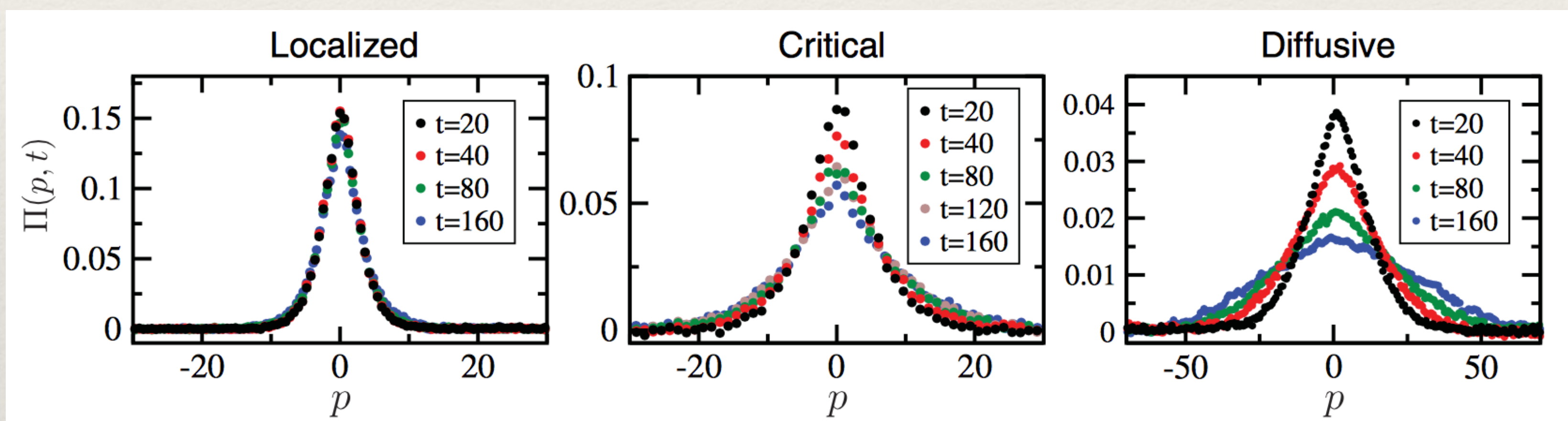

Using this critical behavior, we can compute the AIGF for the quasiperiodic kicked rotor [24]. The details of the calculation will be published elsewhere; we obtain:

$$\Pi(p,t) = \tfrac{3}{2}(3\rho^{3/2}t)^{-1/3}\mathrm{Ai}[(3\rho^{3/2}t)^{-1/3}|p|], \qquad (6)$$

where ρ is a parameter directly related to the critical quantity $\Lambda_c = \lim_{t\to\infty}\langle p^2\rangle/t^{2/3}$ (see [2,5]) via $\rho = \Gamma(2/3)\Lambda_c/3$, where Γ is the Gamma function and $\mathrm{Ai}(x)$ is the Airy function. The asymptotic form Eq. (3) comes simply from the limiting behavior of the Airy function for large x and is found perfectly intermediate between the exponential (localized) and the Gaussian (diffusive) shapes.