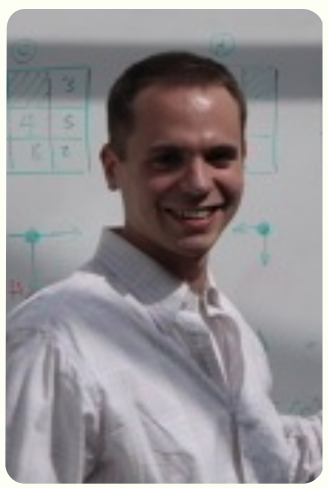
ENTANGLEMENT ENTROPY IN QUANTUM FLUIDS & GASES

Measuring quantum correlations in the spatial continuum



Chris Herdman
UVM

Stephen Inglis
U Waterloo /
LMU

P.N. Roy
U Waterloo

Roger Melko
U Waterloo

Phys. Rev. B, 89, 140501 (2014)
arXiv:1404.7104

2014 CAP Congress

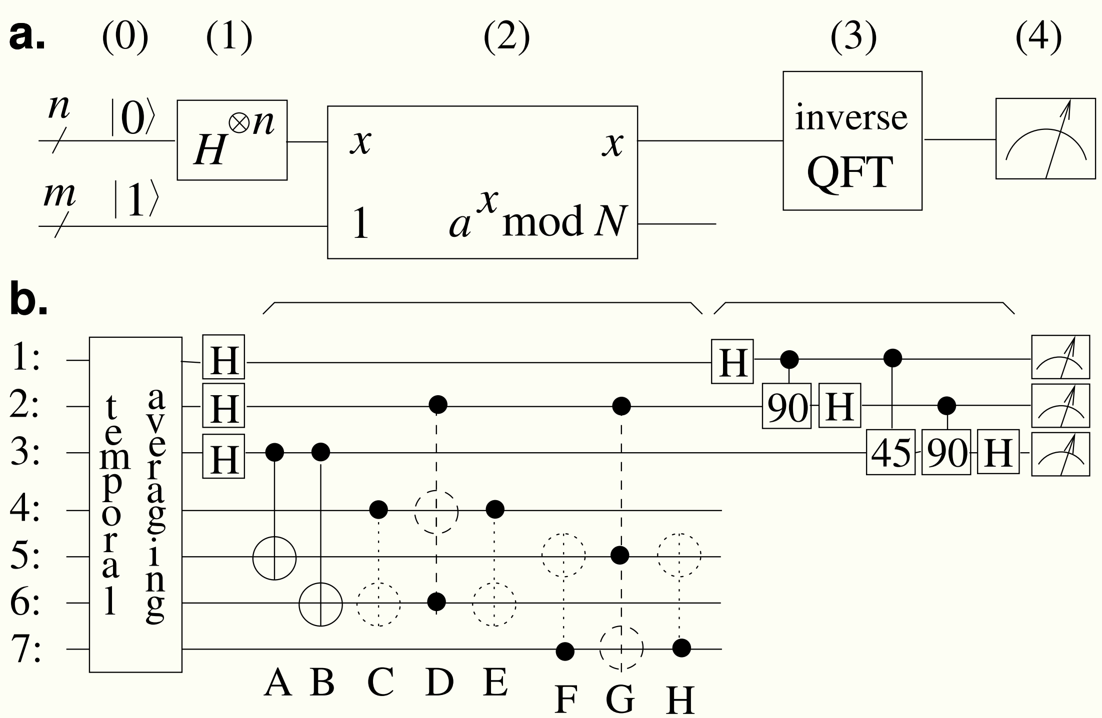
Adrian Del Maestro
University of Vermont

Entanglement is a resource for quantum information processing

necessary to provide an exponential speed-up over classical computation

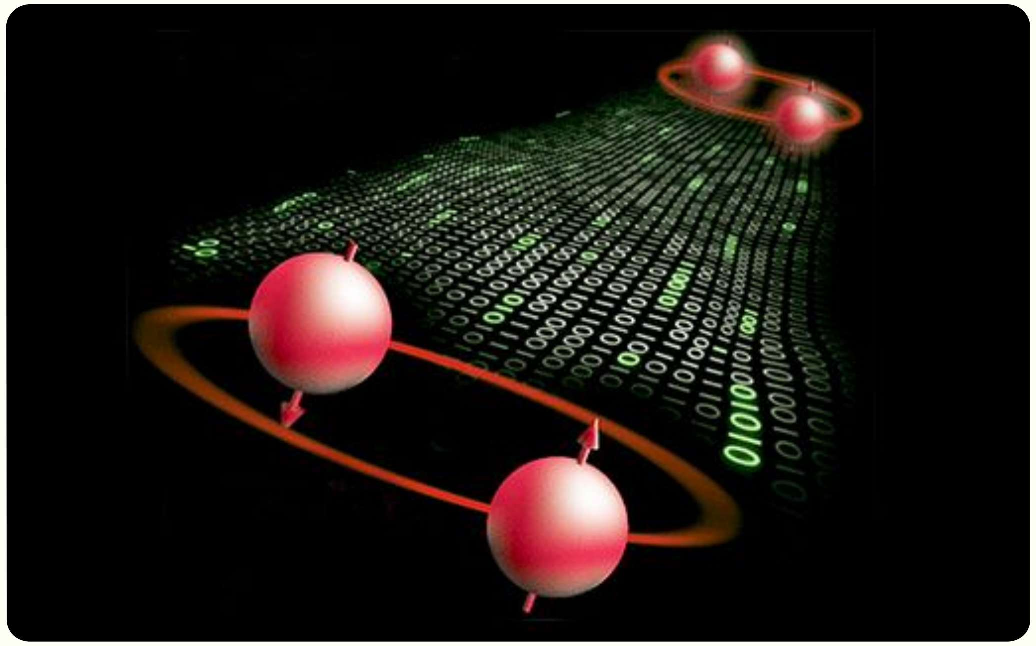
R. Jozsa and N. Linden, Proc. Roy. Soc. A: Math, Phys. and Eng. 459, 2011 (2003)

$$\mathrm{O}\left(\mathrm{e}^{1.9(\log N)^{1/3}}(\log\log N)^{2/3}\right) \to \mathrm{O}\left((\log N)^3\right)$$

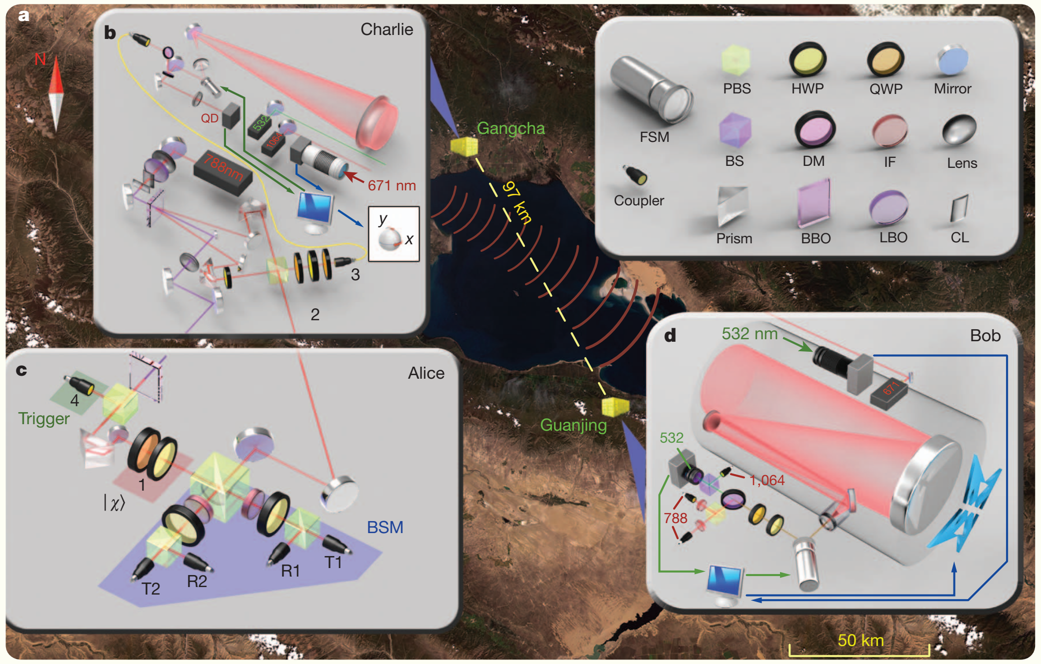


L. M. K. Vandersypen, *et. al.*, Nature 414, 883 (2001)

teleportation

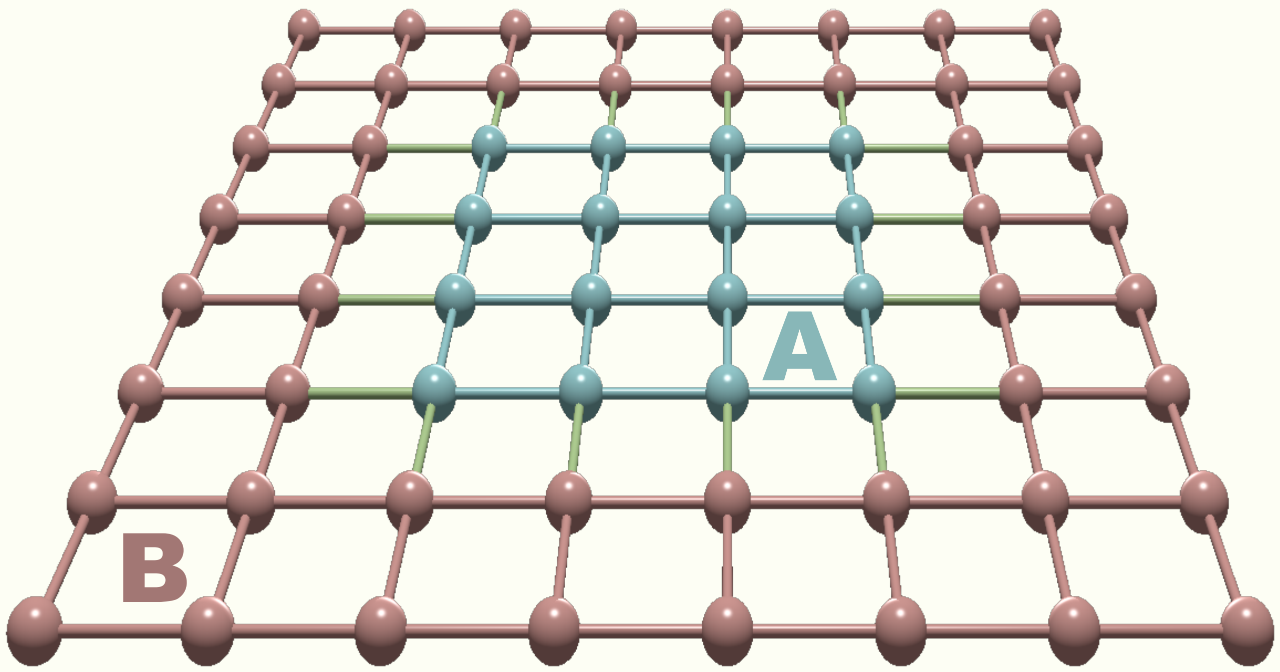


C.H.Bennett, et al. Phys. Rev. Lett. 70, 1895 (1993)



J. Yin et al., Nature 488, 185 (2012)

Detection and classification of quantum states of matter



area law

entanglement scales with the boundary size

$$S(A) \sim \ell^{d-1}$$

L. Amico, A. Osterloh, and V. Vedral, RMP 80, 517 (2008)
J. Eisert, M. Cramer, and M. B. Plenio, RMP 82, 277 (2010)

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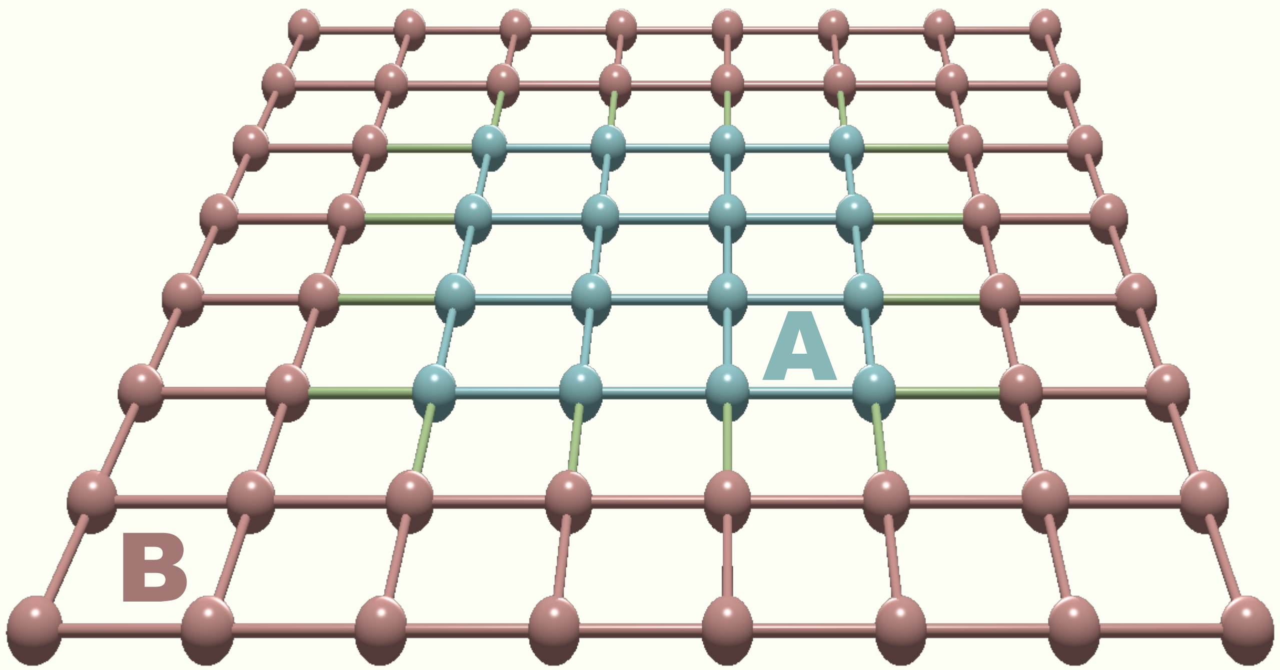

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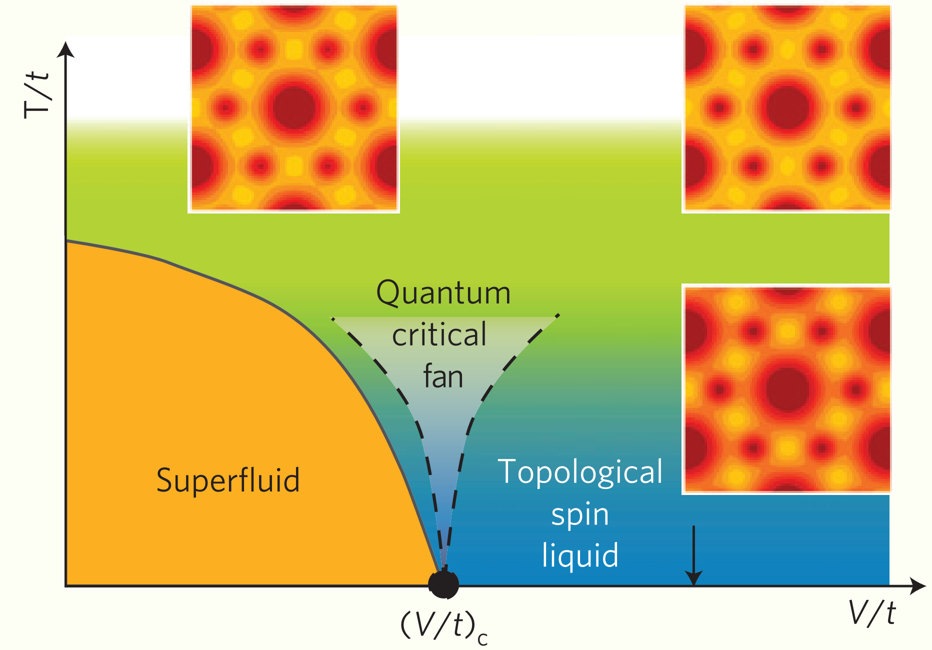


2d topological spin liquid

$$S(A) = \ell - \gamma$$

(1+1) conformal field theory

$$S = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c_1$$

L. Amico, A. Osterloh, and V. Vedral, RMP 80, 517 (2008)
J. Eisert, M. Cramer, and M. B. Plenio, RMP 82, 277 (2010)

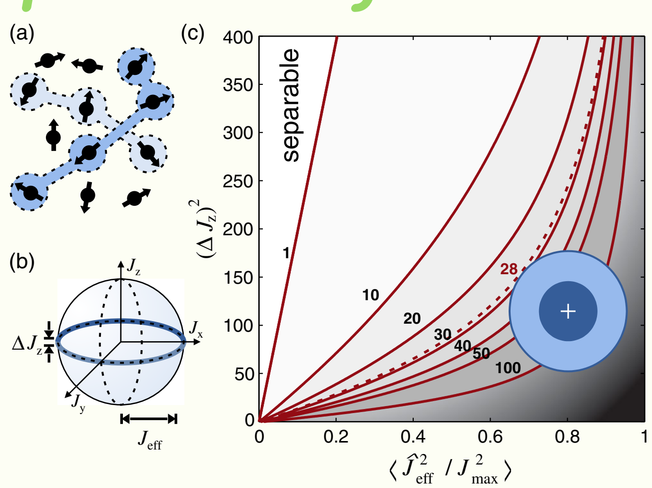
S. V. Isakov, *et al.*, Nat Phys 7, 772 (2011)
A. Kitaev and J. Preskill, PRL 96, 110404 (2006)
M. Levin and X.-G. Wen, PRL 96, 110405 (2006)
M. M. Wolf *et al.* PRL 100, 070502 (2008).

Entanglement in quantum fluids and gases

Theoretical work has focused on systems with discrete Hilbert spaces: qubits, insulating lattice models, ...

Experiments employ the quantum states of ultra-cold atomic gasses and BECs

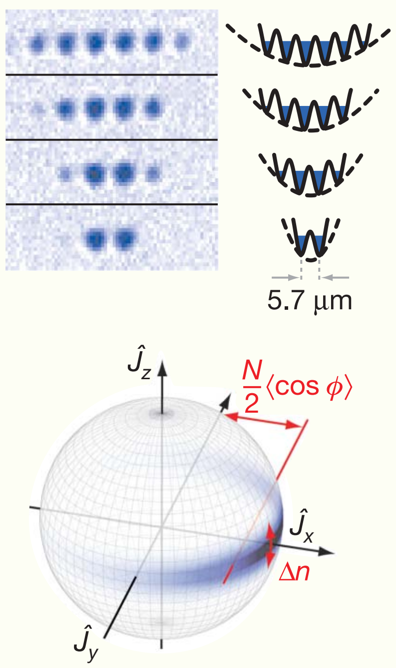
observation and manipulation of Dicke states



B. Lücke, *et.al.*, PRL 112, 155304 (2014)

boson sampling

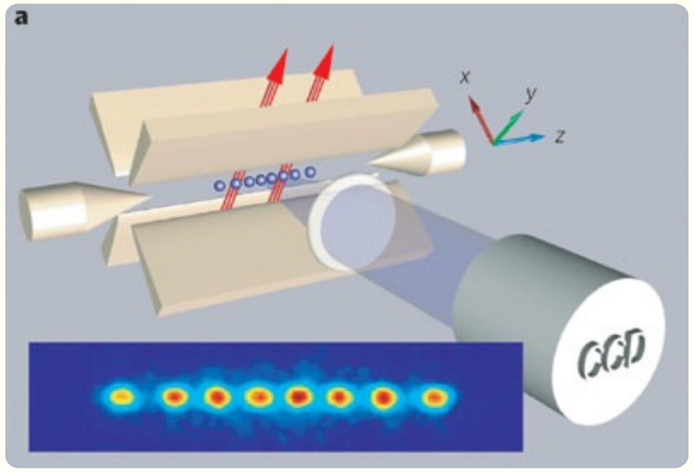
C. Shen, *et al.*, PRL 112, 050504 (2014)



ultra high-precision quantum interferometry

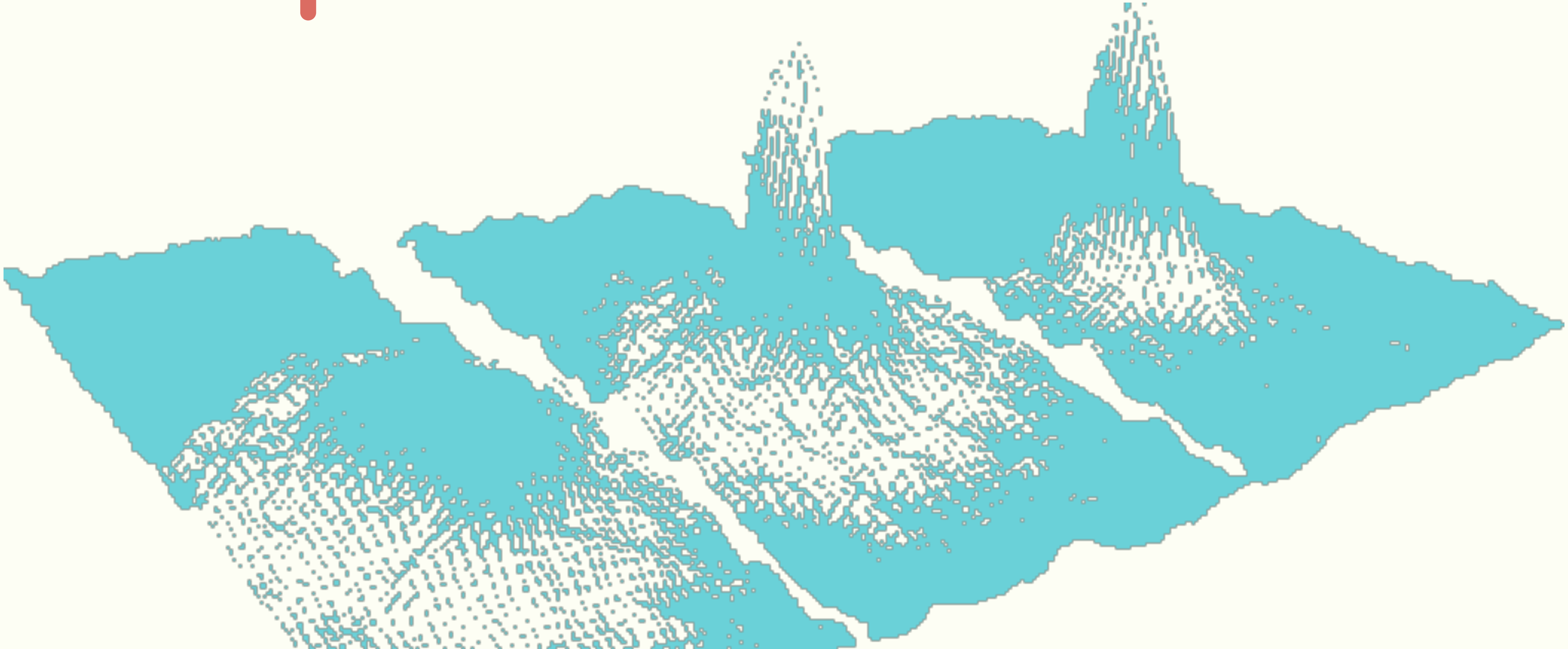
.Estève, *et al.*,
Nature 455, 1216 (2008)

multiparticle entanglement of trapped ions



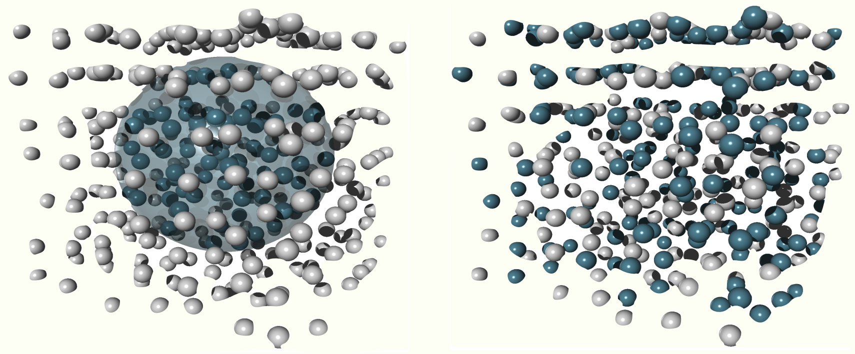
T. Monz, *et.al.*, PRL 102, 040501 (2009)

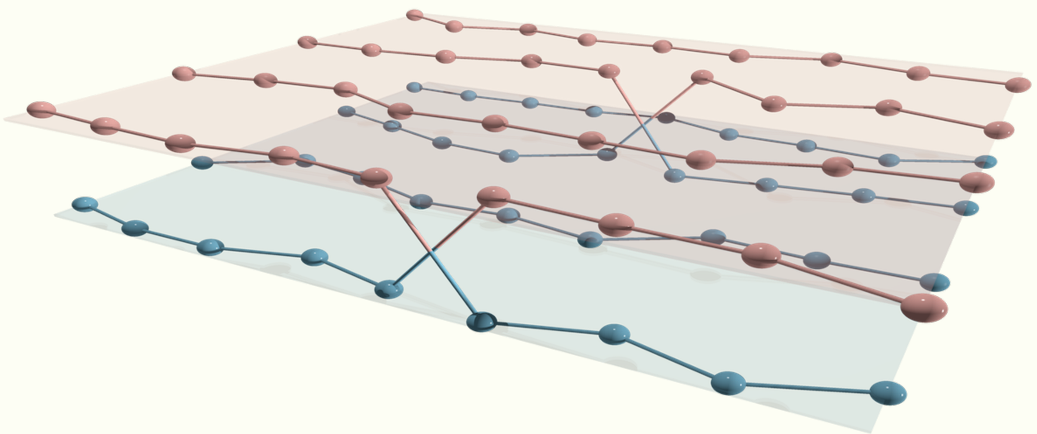
Can we quantify and optimize the entanglement of interacting atoms in the spatial continuum?



Quantifying Entanglement

bipartite Rényi entropies in the spatial continuum



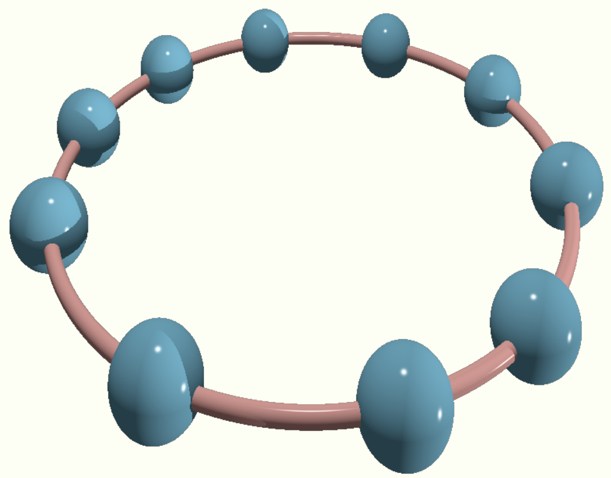


Algorithmic Development

measurement and benchmarking using path integral quantum Monte Carlo

Applications in 1d

interacting bosons and the connection between entanglement and condensate fraction



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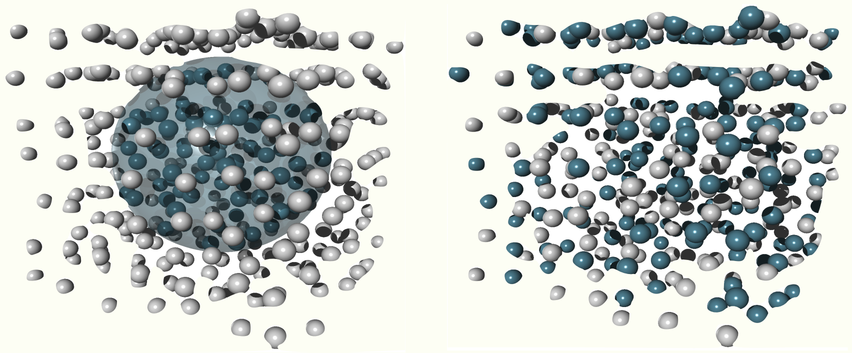

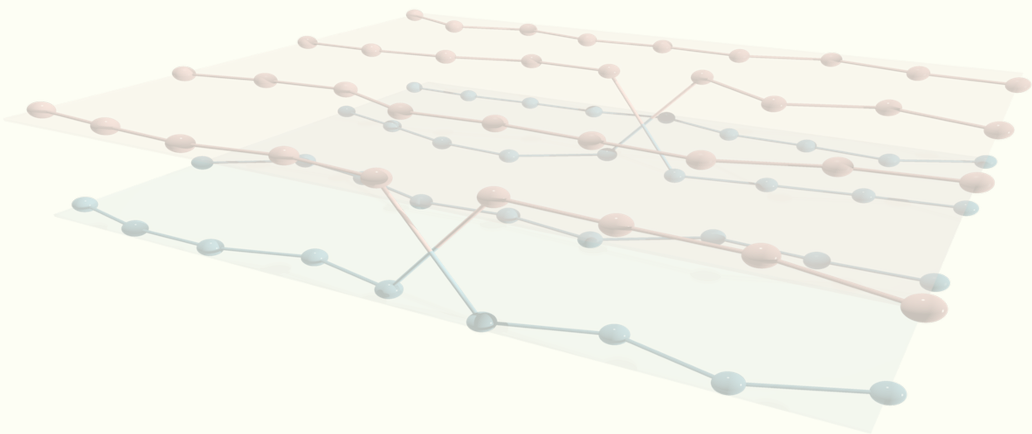

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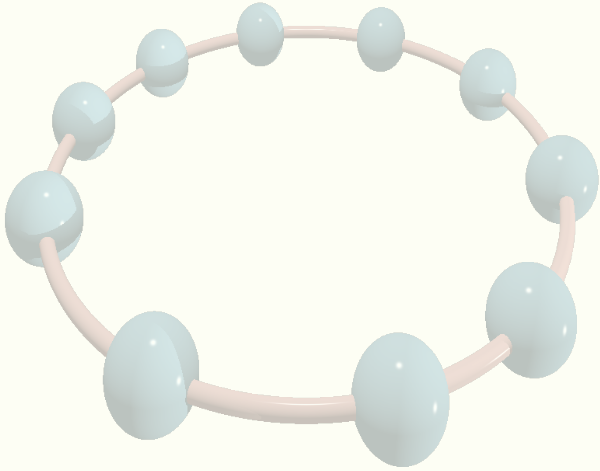


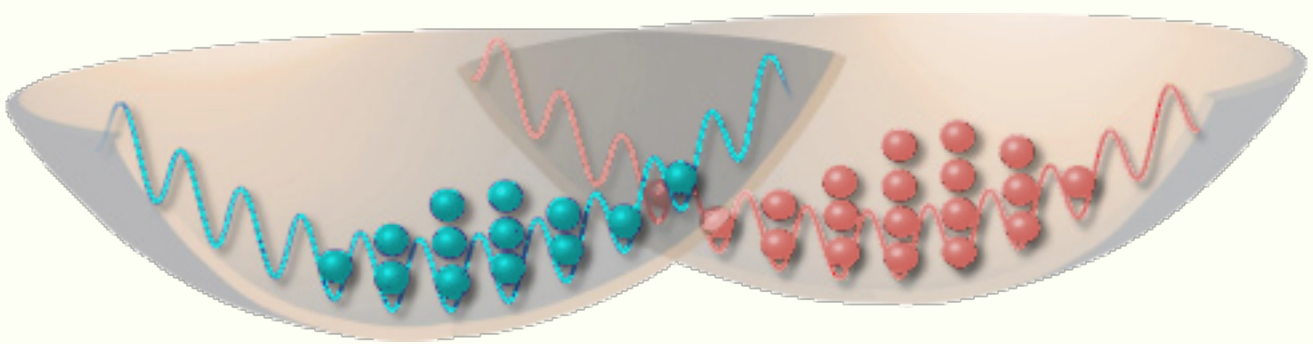
Study systems of quantum fluids and gasses

governed by the general many-body Hamiltonian

$$H = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m_i} \nabla_i^2 + U_i \right) + \sum_{i<j} V_{ij},$$

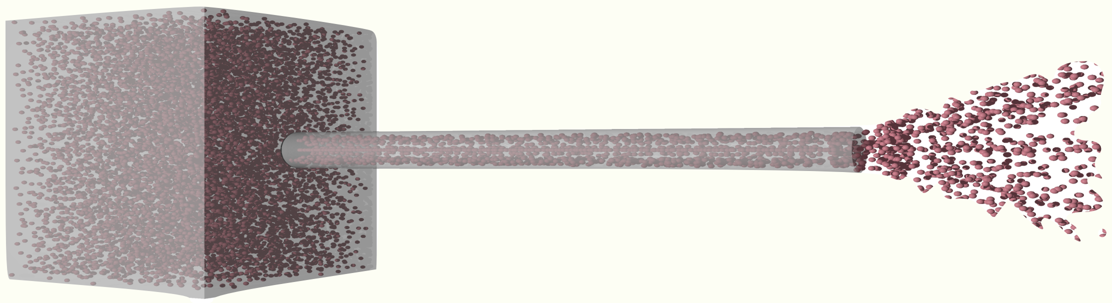
external potential

interaction potential



trapped ions with a periodic lattice potential

J. Wernsdorfer et al. PRA, 81, 043620 (2010)



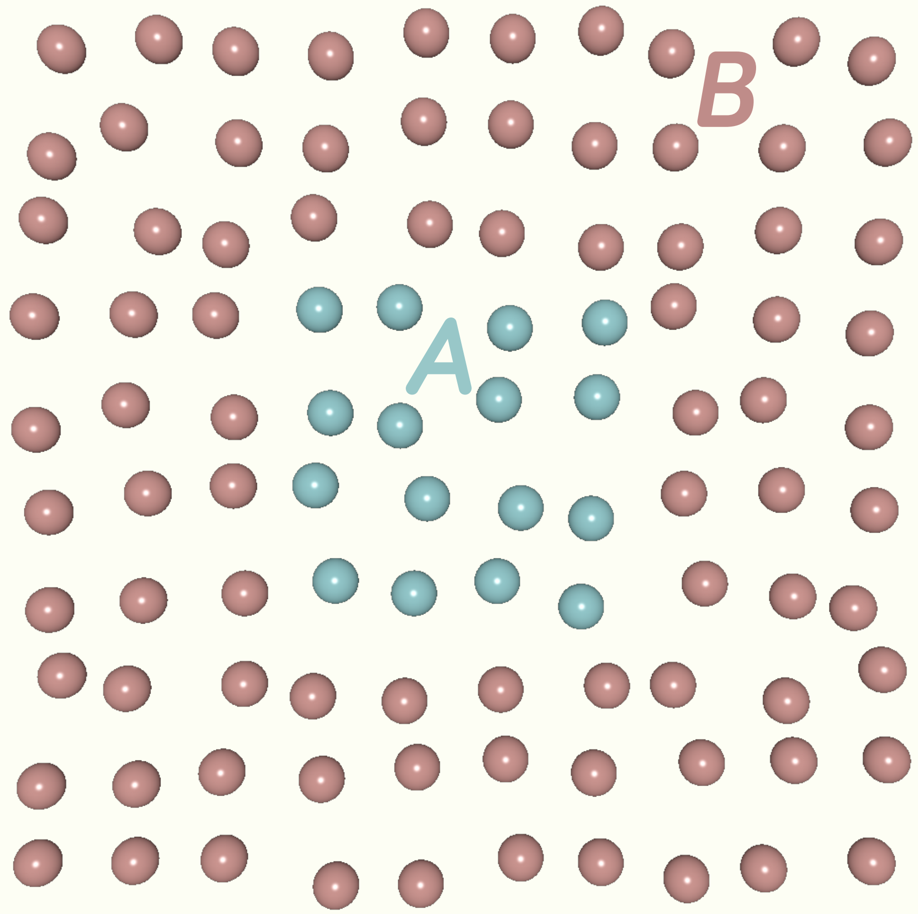
quantum nanofluids of helium-4

B. Kulchytskyy et al. PRB, 88, 064512 (2013)

Quantifying bipartite entanglement

bipartition into two subsystems: A & B

compute the reduced density matrix by tracing over region B



$$\rho_A = \operatorname*{Tr}_B \rho$$

$$\rho \equiv |\Psi\rangle\langle\Psi|$$

$$|\Psi\rangle \stackrel{?}{=} |\Psi_A\rangle \otimes |\Psi_B\rangle$$

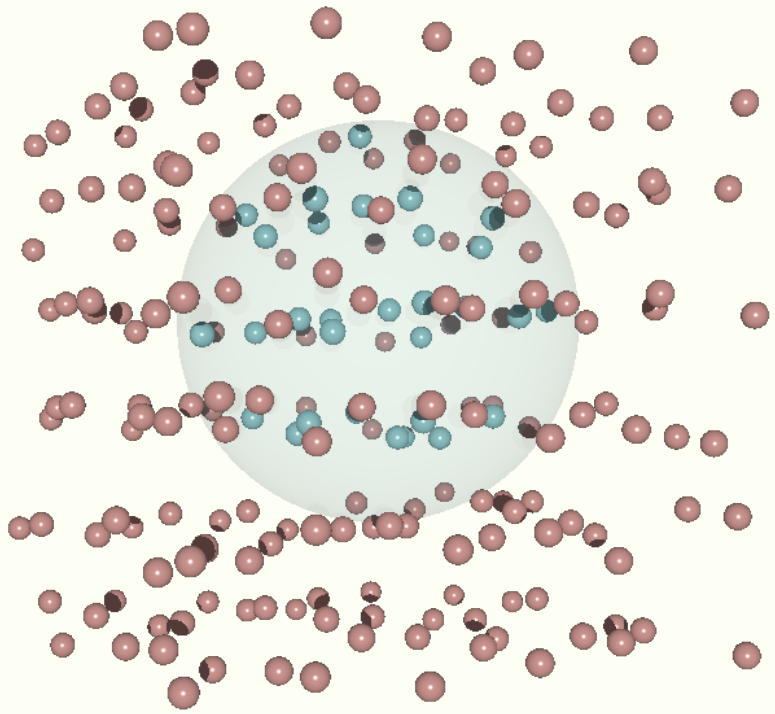
Rényi Entanglement Entropy:

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \operatorname{Tr} \rho_A^\alpha$$

reduces to von Neumann entropy when $\alpha \to 1$ $\quad S = \operatorname{Tr} \rho_A \log \rho_A$

Different bipartitions of itinerant bosons

for identical particles in the spatial continuum, various ways to partition ground state



Spatial Bipartition

Constructed from the Fock space of single-particle modes

$$|\Psi\rangle = \sum_{\boldsymbol{n}_A,\boldsymbol{n}_B} c_{\boldsymbol{n}_A\boldsymbol{n}_B} \Big|\boldsymbol{n}_A\Big\rangle \otimes \Big|\boldsymbol{n}_B\Big\rangle$$

$$\rho_A \to S(A)$$

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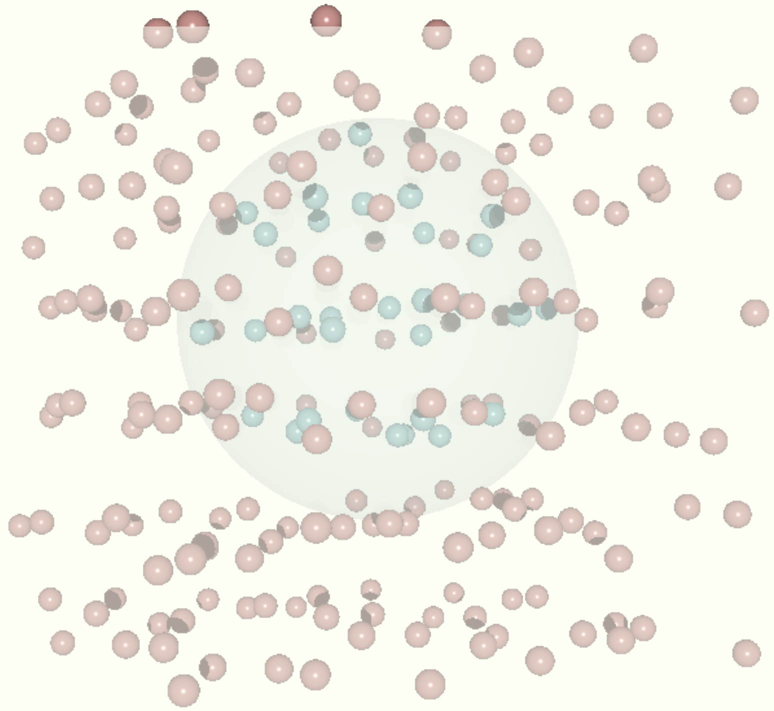


$$\rho_A \to S(A)$$

Particle Bipartition

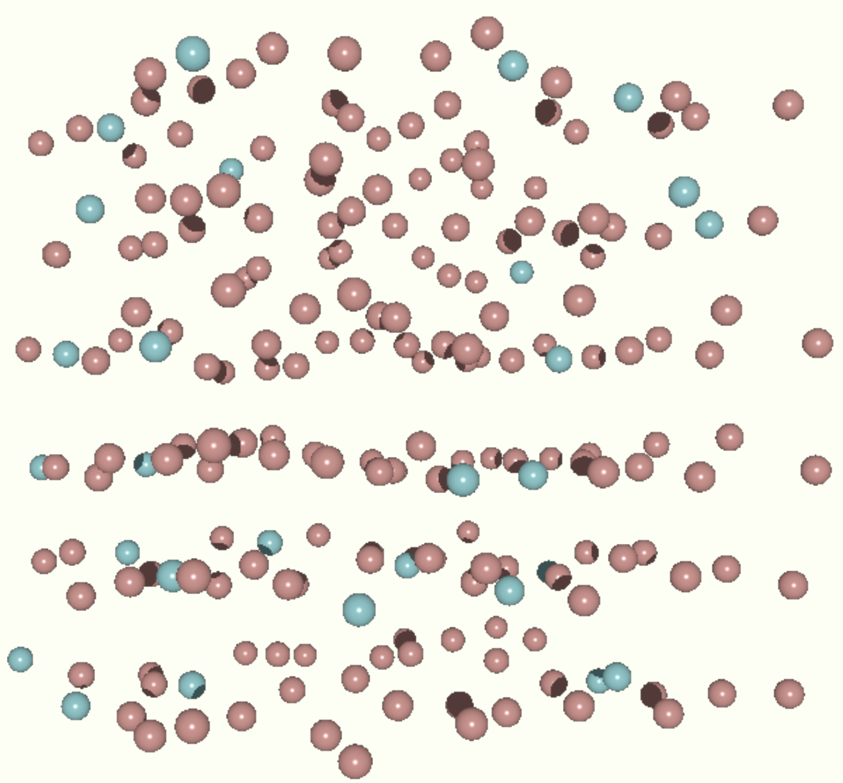
Artificially label a subset of n particles

$$|\Psi\rangle = |\boldsymbol{r}_1 \cdots \boldsymbol{r}_{\boldsymbol{N}}\rangle$$

$$\rho_n = \int d\boldsymbol{r}_n \cdots d\boldsymbol{r}_N \langle\Psi|\hat{\rho}|\Psi\rangle$$

n-body density matrix

$$\rho_n \to S(n)$$



Example: entanglement in the free Bose gas

$$|\mathrm{BEC}\rangle \equiv \frac{1}{\sqrt{N!}} \left(\phi_0^\dagger\right)^N |\mathbf{0}\rangle$$

Spatial Bipartition

entanglement is non-zero and is generated via number fluctuations

$$S_2(A) \sim \frac{1}{2} \log \ell_A$$

Particle Bipartition

Ground state is already in product-form in first quantized notation

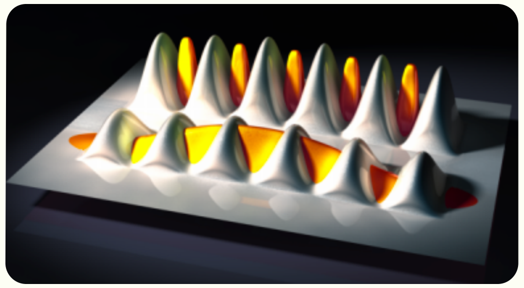
$$S_2(n) = 0$$

C. Simon, PRA 66, 052323 (2002)
W. Ding and K. Yang, PRA 80, 012329 (2009)

How do interactions change this picture?

"toy" quantum fluid: 1d Bose-Hubbard model

$$H_{\rm BH} = \sum_j \left[-t \left(b_j^\dagger b_{j+1} + \text{h.c.} \right) + \frac{U}{2} n_j \left(n_j + 1 \right) - \mu_j n_j \right]$$



E. Haller *et al.*, Nature 466, 597 (2010)

3 types of candidate ground states

$$|\text{BEC}\rangle \equiv \frac{1}{\sqrt{N!}} \left(\phi_0^\dagger \right)^N |\mathbf{0}\rangle$$

$$|\text{Mott}\rangle \equiv \prod_j b_j^\dagger |\mathbf{0}\rangle$$

$$|\text{Cat}\rangle \equiv \sum_j \frac{1}{\sqrt{L}\sqrt{N!}} \left(b_j^\dagger \right)^N |\mathbf{0}\rangle$$

State	Particle Entanglement	Spatial Entanglement
BEC	0	$1/2 \log L$
Mott	$L \log 2$	0
Cat	$\log L$	$\log 2$

Can any of this entanglement be put to use?

Accessing entanglement as a resource requires the ability to perform local physical operations on subsystems

Spatial Entanglement

particle number conservation prohibits direct measurement

Particle Entanglement

inaccessible due to the indistinguishability of particles

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The Entanglement of Particles

$$E_p(A) \equiv \sum_n P_n S(\rho_{A,n})$$

$$\rho_{A,n} \equiv \frac{1}{P_n} \hat{P}_n \rho_A \hat{P}_n$$

probability (P_n)

projection operator ($\hat{P}_n$)

$$E_p(A) < S(A)$$

$$E_p(A) > 0 \Rightarrow S(n) > 0$$

H. M. Wiseman and J. A. Vaccaro, PRL 91, 097902 (2003)

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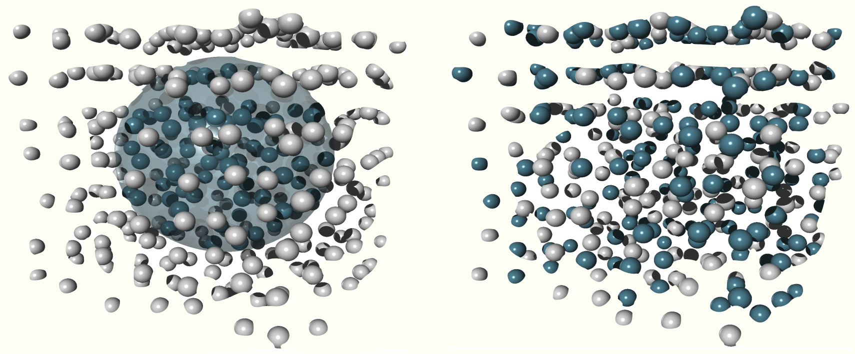
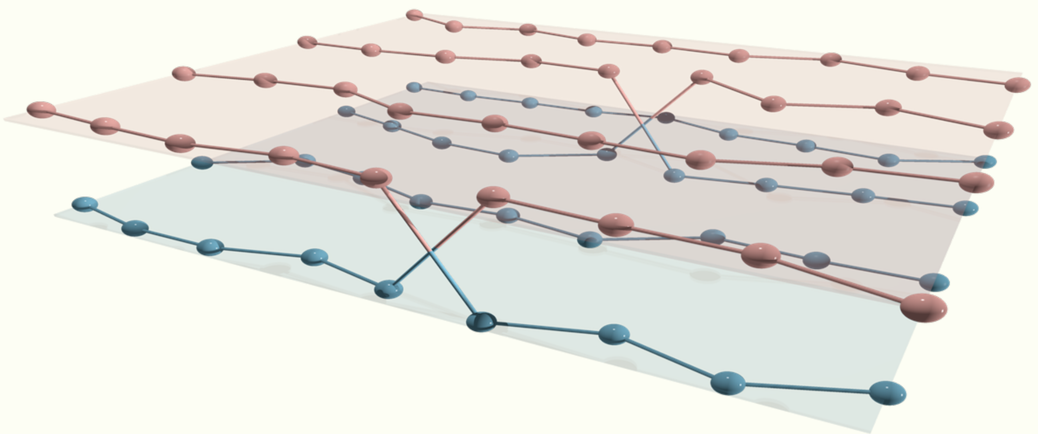


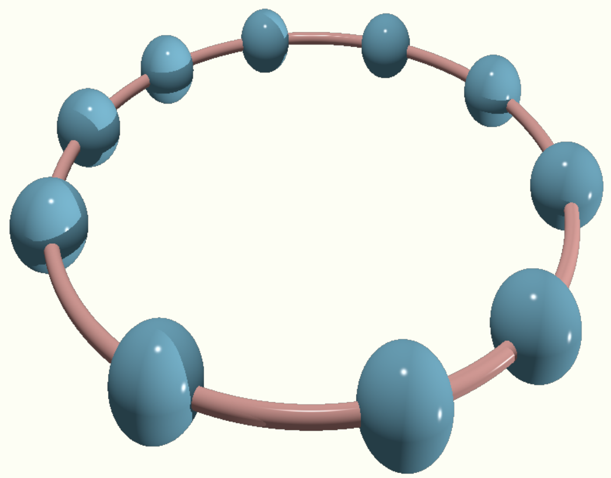
Algorithmic Development



measurement and benchmarking using path integral quantum Monte Carlo

Applications in 1d

interacting bosons and the connection between entanglement and condensate fraction



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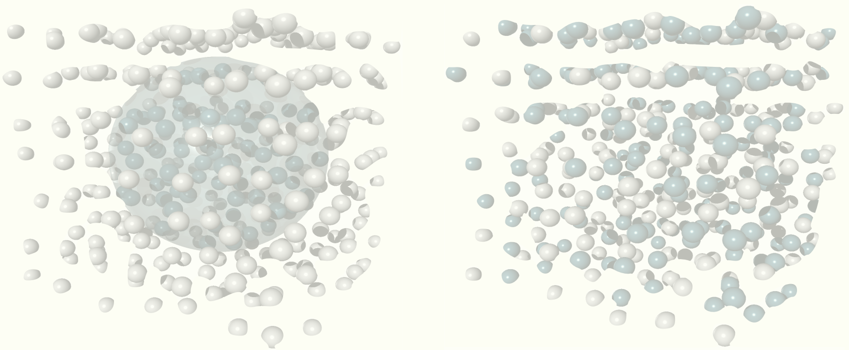

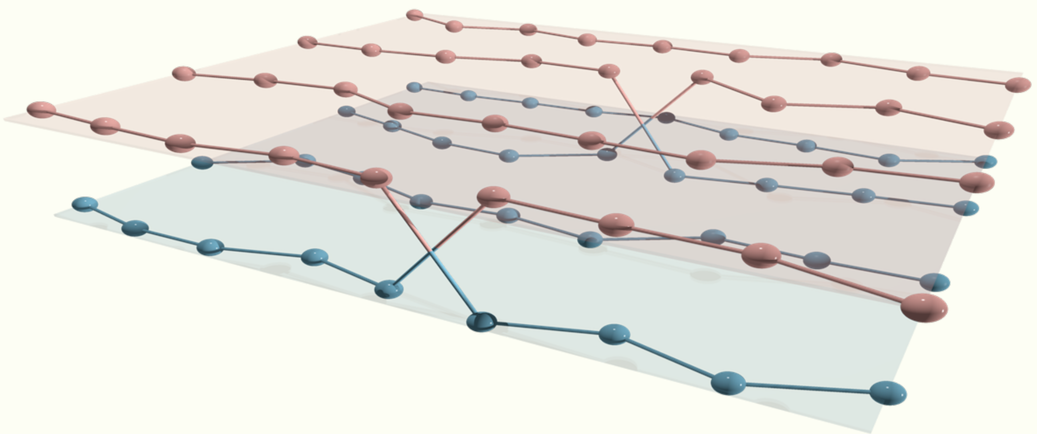

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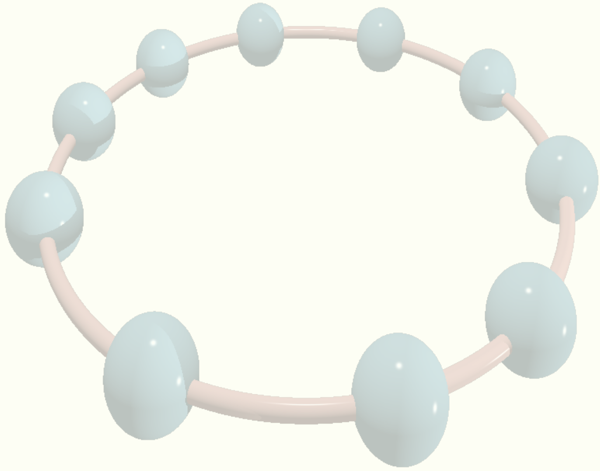
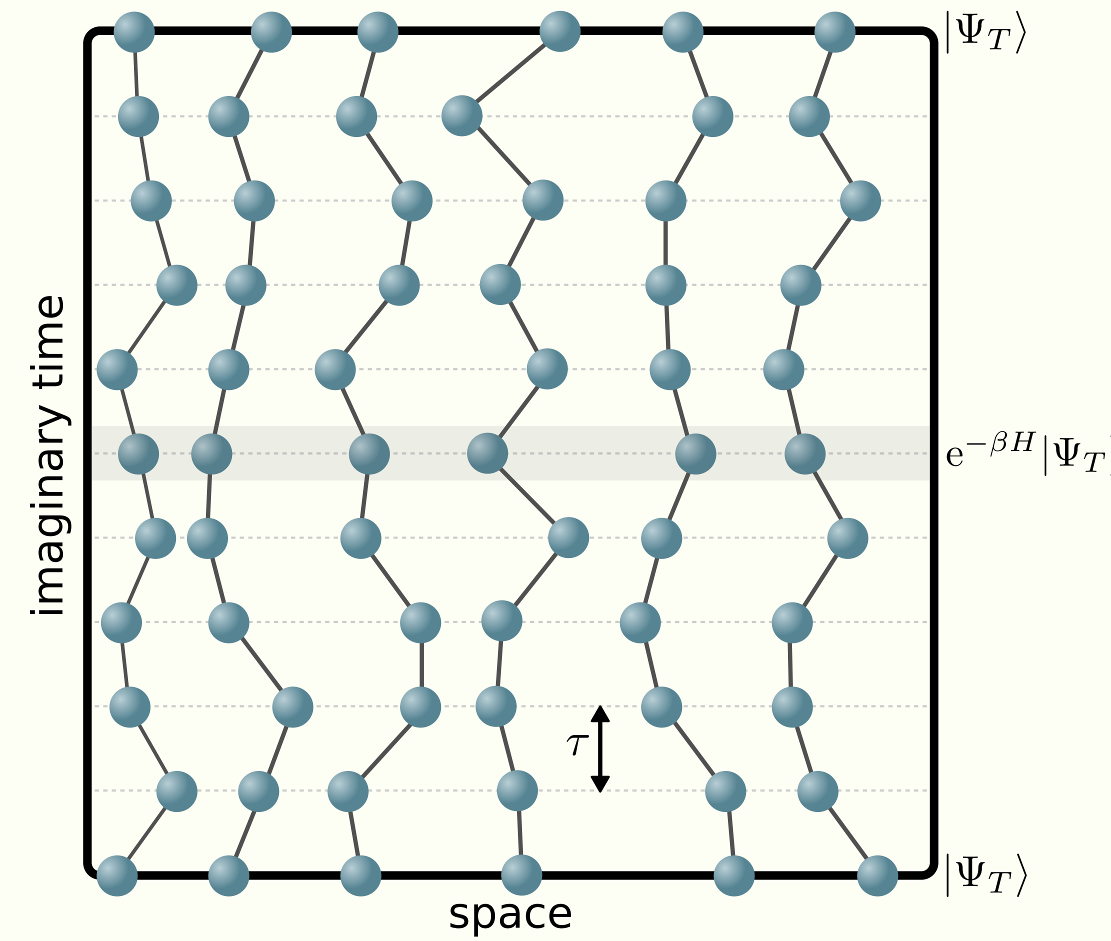


Path integral ground state quantum Monte Carlo

Description

$$H = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m_i} \nabla_i^2 + U_i \right) + \sum_{i<j} V_{ij},$$



Project

a trial wave function onto the ground state

$$\left|\Psi\right\rangle = \lim_{\beta\to\infty} \mathrm{e}^{-\beta H} \left|\Psi_T\right\rangle$$

Configurations

discrete imaginary time worldlines constructed from products of the short time propagator

$$\rho_\tau(\boldsymbol{R}, \boldsymbol{R}') = \langle \boldsymbol{R} | \mathrm{e}^{-\tau H} | \boldsymbol{R}' \rangle$$

Observables

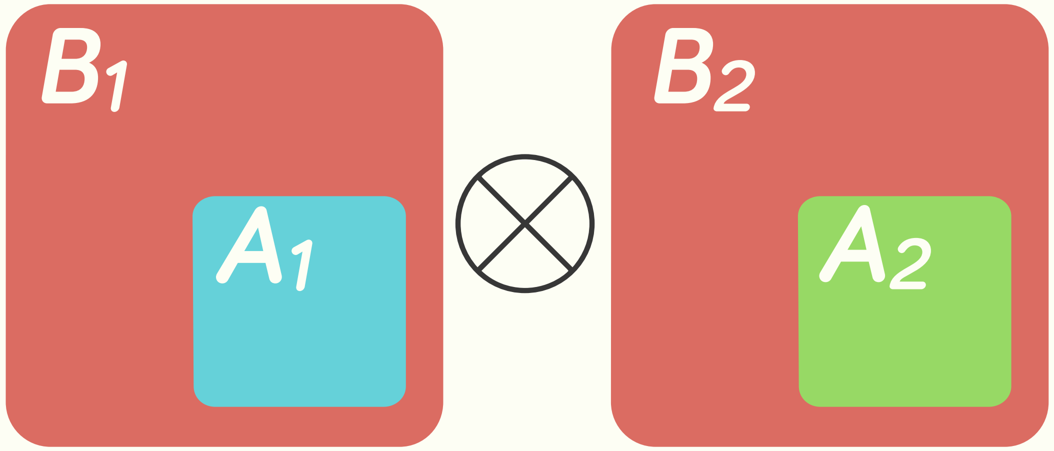
an exact method for computing ground state expectation values

$$\langle \hat{\mathcal{O}} \rangle = \lim_{\beta\to\infty} \frac{\left\langle \Psi_{\mathrm{T}} \right| e^{-\beta H} \hat{\mathcal{O}} e^{-\beta H} \left| \Psi_{\mathrm{T}} \right\rangle}{\left\langle \Psi_{\mathrm{T}} \right| e^{-2\beta H} \left| \Psi_{\mathrm{T}} \right\rangle}$$

D. M. Ceperley, RMP 67, 279 (1995)
A. Sarsa, *et. al.*, J. Chem. Phys. 113, 1366 (2000)

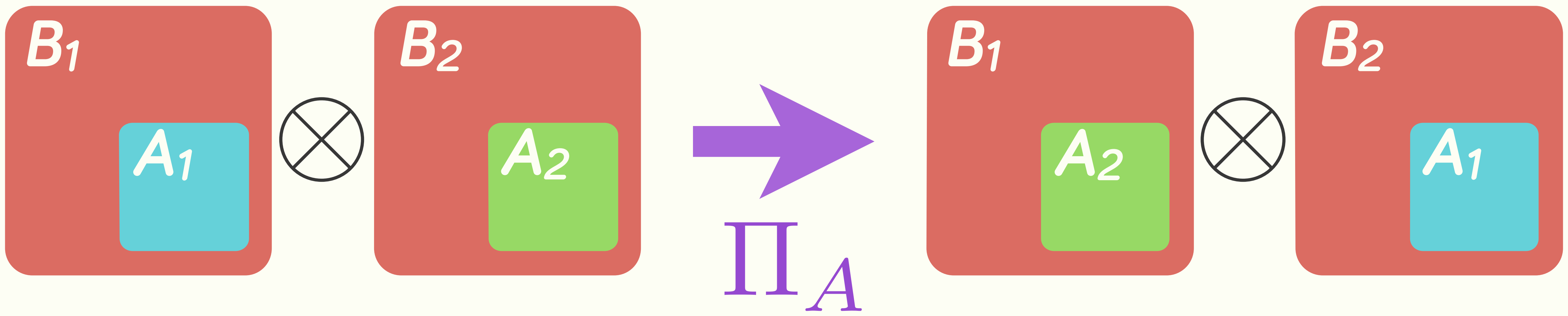
Computing Rényi entropies in Monte Carlo

Replicate the system



Computing Rényi entropies in Monte Carlo

Replicate the system *Permute (swap) the subregions*



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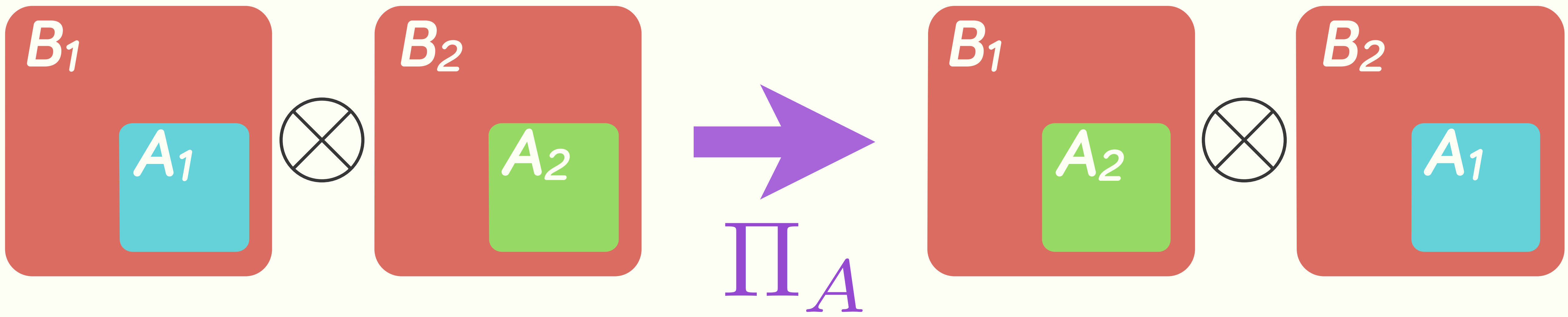


Technology imported from QFT to QMC

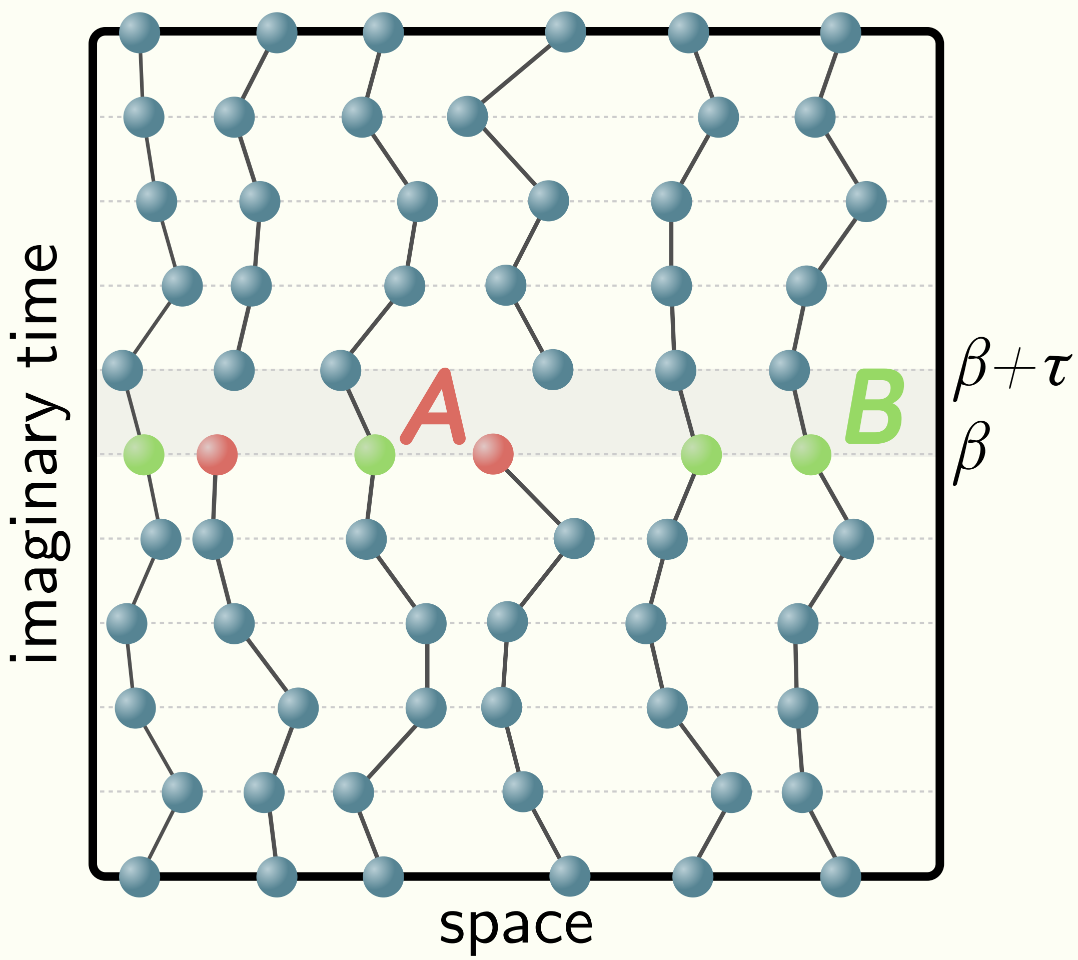
P. Calabrese and J. Cardy, J. Stat. Mech.: Theor. Exp. 2004, P06002 (2004)
M. B. Hastings, I. González, A. B. Kallin, and R. G. Melko, PRL 104, 157201 (2010)
R. Melko, A. Kallin, and M. Hastings, PRB 82, 100409 (2010)

For $\alpha = 2$ replicas, expectation value of the permutation operator is a measure of the 2nd Rényi entropy.

$$S_2 = -\log\langle\Pi_A\rangle$$

Porting to the path integral representation

Break continuous space paths at the center time slice β



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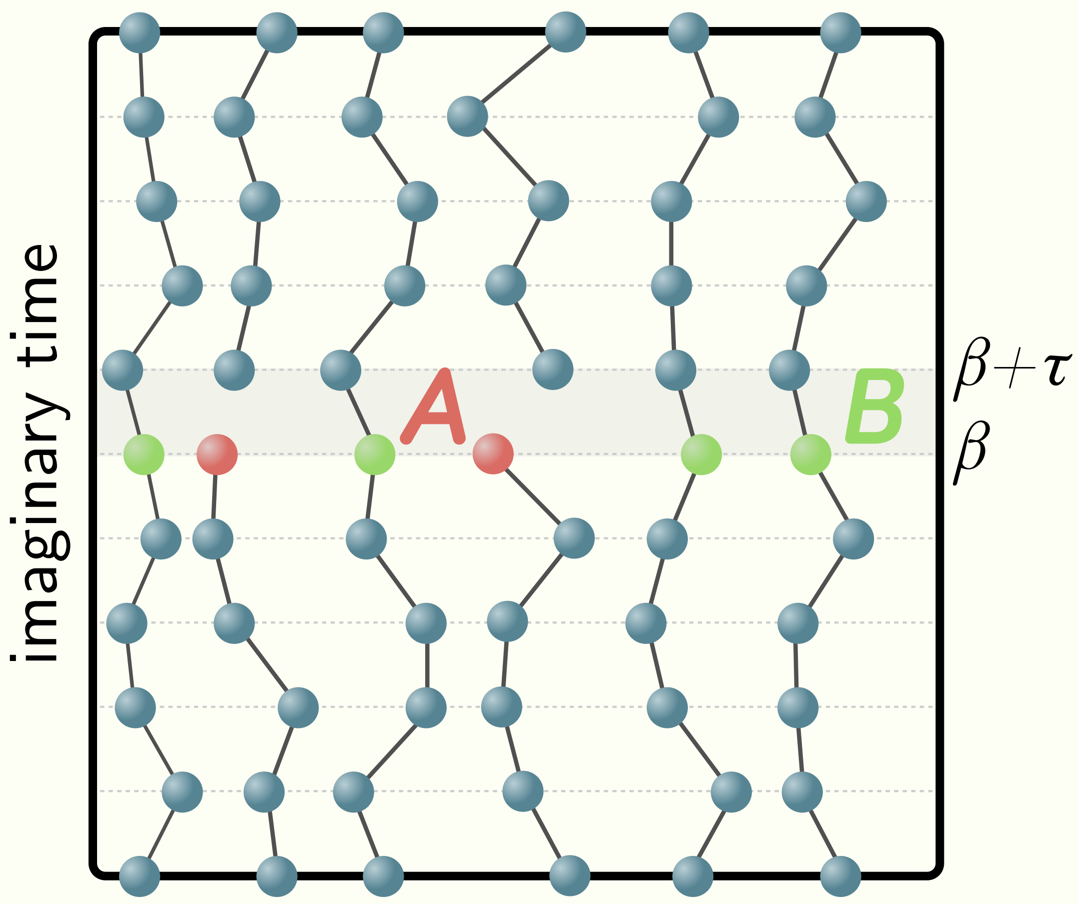
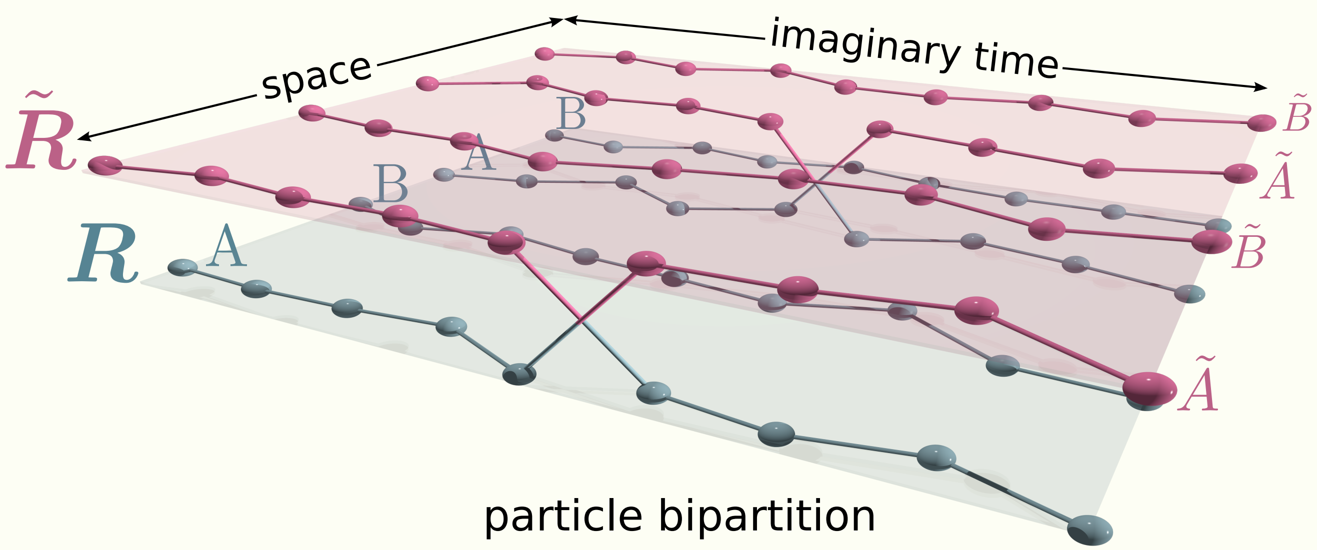
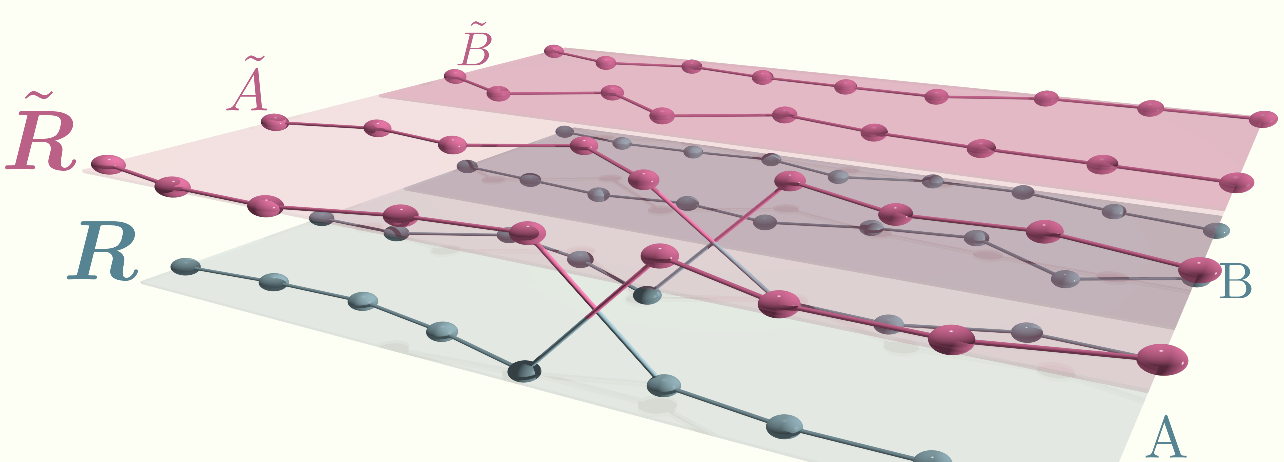


The bipartitions only exist at this time slice.
Broken links are in A.



particle bipartition



spatial bipartition

$$\langle \Pi_2^A \rangle \sim \left\langle \rho_\tau^A \left(\boldsymbol{R}^\beta \otimes \tilde{\boldsymbol{R}}^\beta ; \Pi_2^A \left[\boldsymbol{R}^{\beta+\tau} \otimes \tilde{\boldsymbol{R}}^{\beta+\tau} \right] \right) \right\rangle$$

C. Herdman *et al.* Phys. Rev. B, 89, 140501 (2014)

Benchmarking on a non-trivial model

N-Harmonium in 1d

harmonically interacting and confined bosons

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx_i^2} + \frac{1}{2} m \omega_0^2 x_i^2 + \frac{1}{2} m \omega_{\text{int}}^2 \sum_{j>i} (x_i - x_j)^2 \right]$$

exact solution can be computed using Wigner quasi-distributions for bosons or fermions C. L. Benavides-Riveros, I. V. Toranzo, and J. S. Dehesa, arXiv:1404.4447v1, (2014)

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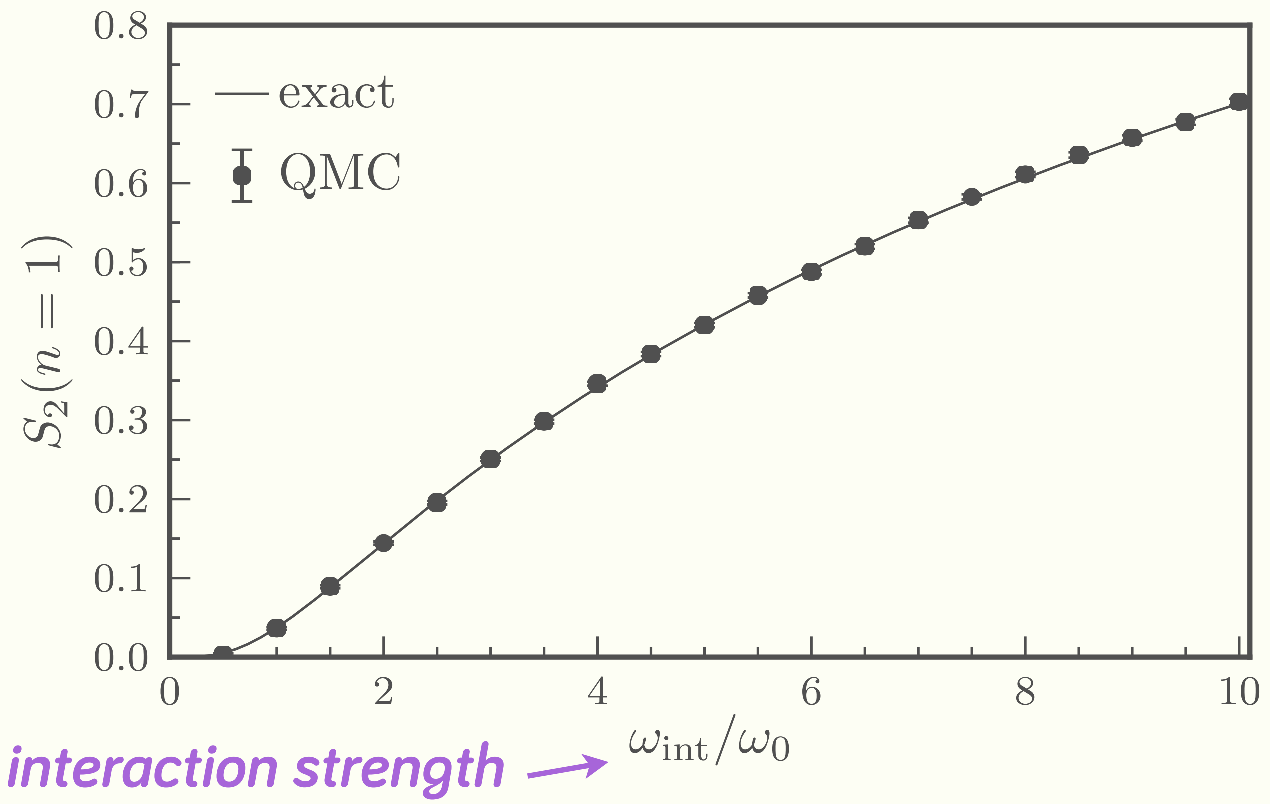
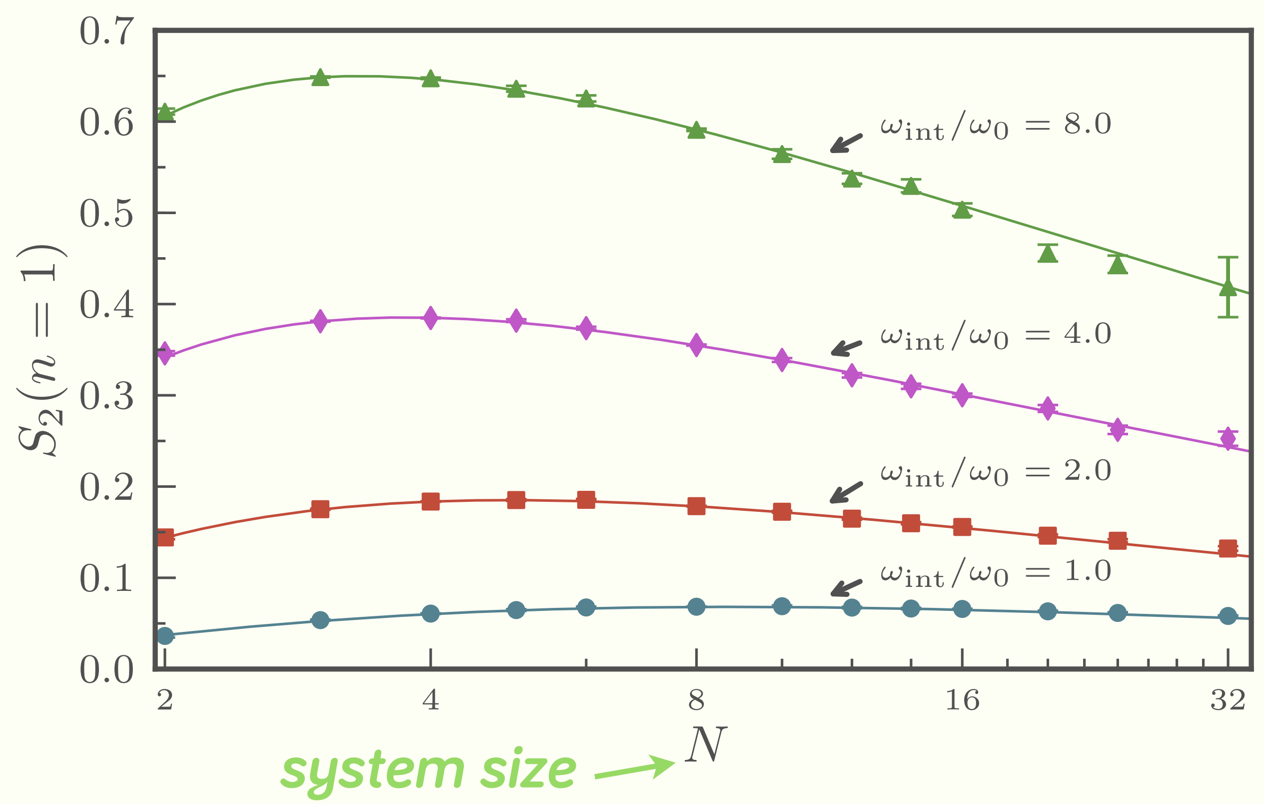
QMC Results: Particle Entanglement

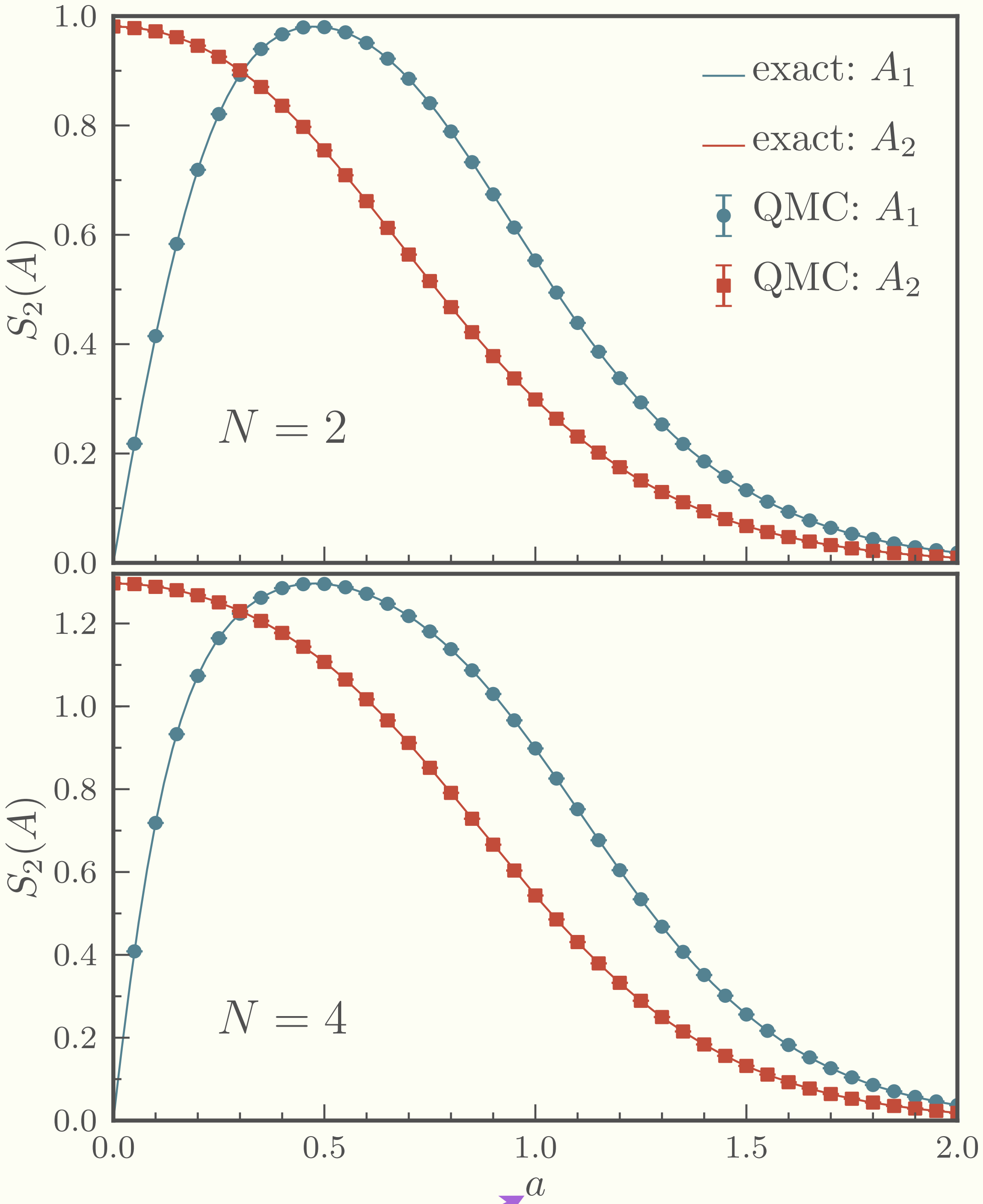
C. M. Herdman *et al.* arXiv:1404.7104





Benchmarking on a non-trivial model

Spatial Entanglement



bipartition size

C. M. Herdman *et al.* arXiv:1404.7104

Benchmarking on a non-trivial model

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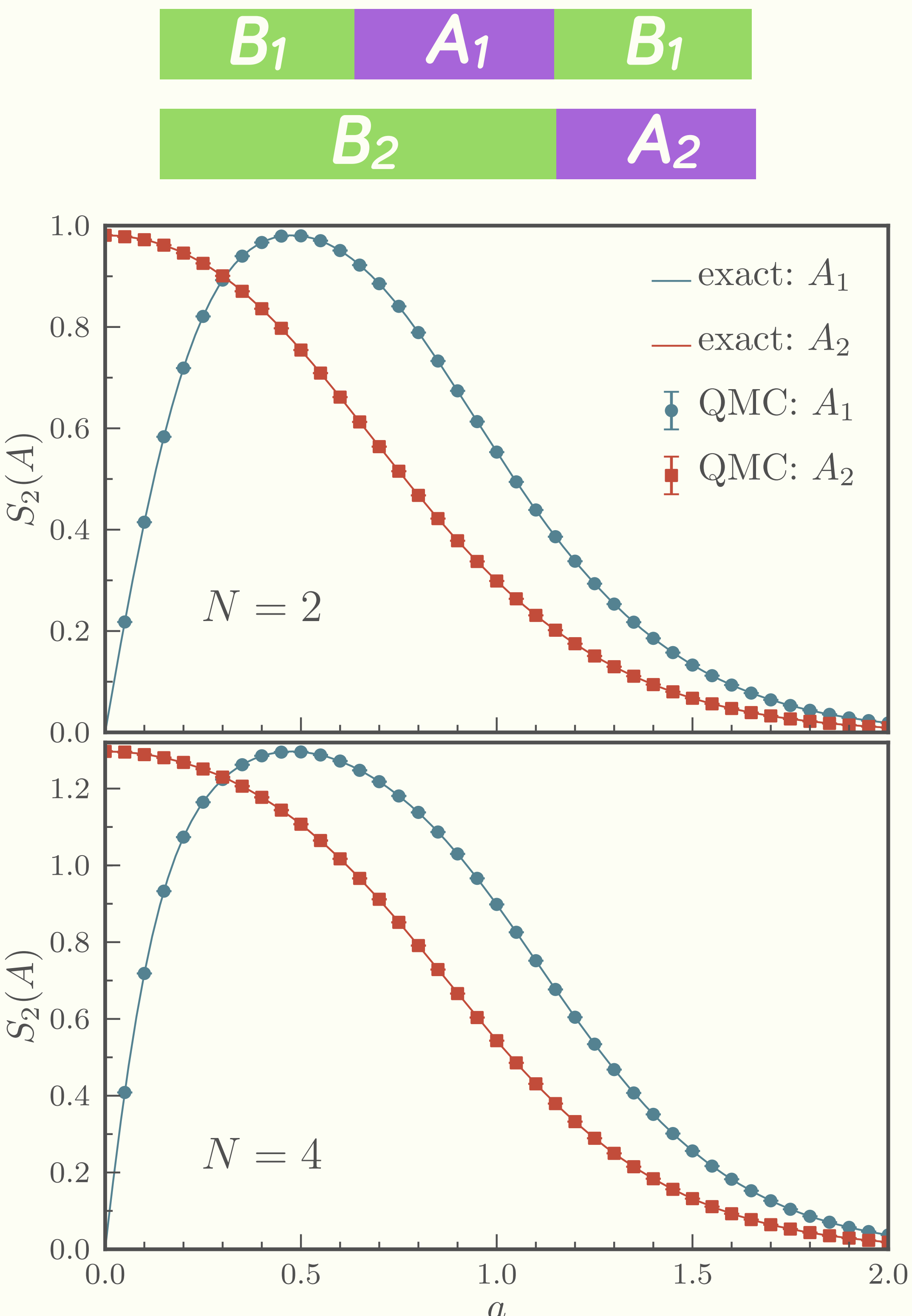
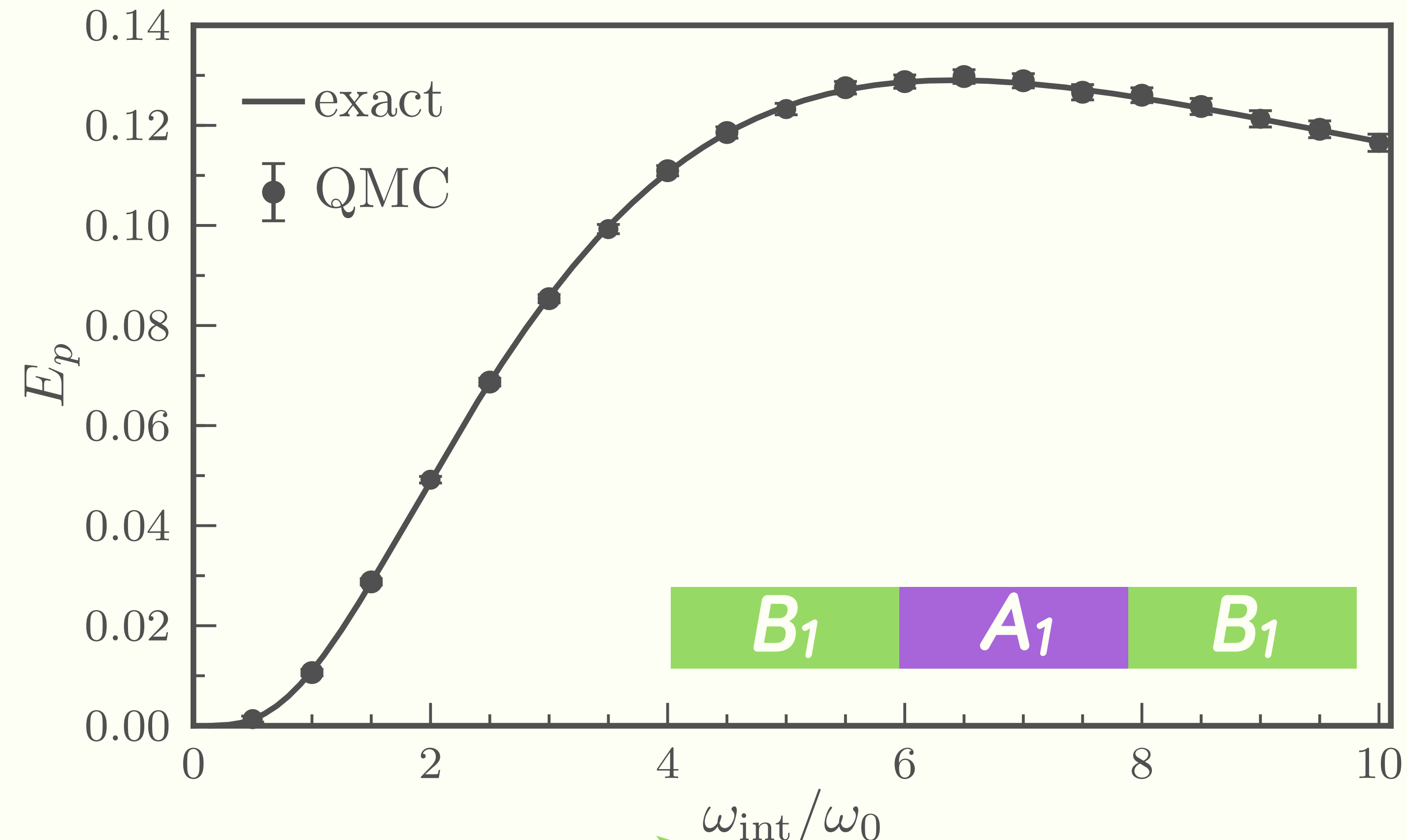


bipartition size

Entanglement of Particles



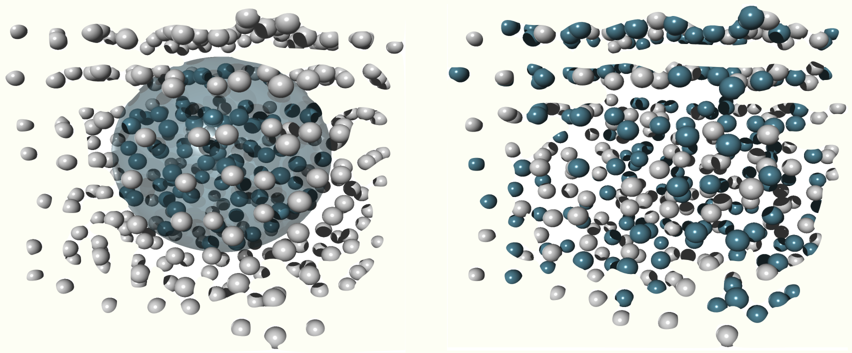
interaction strength

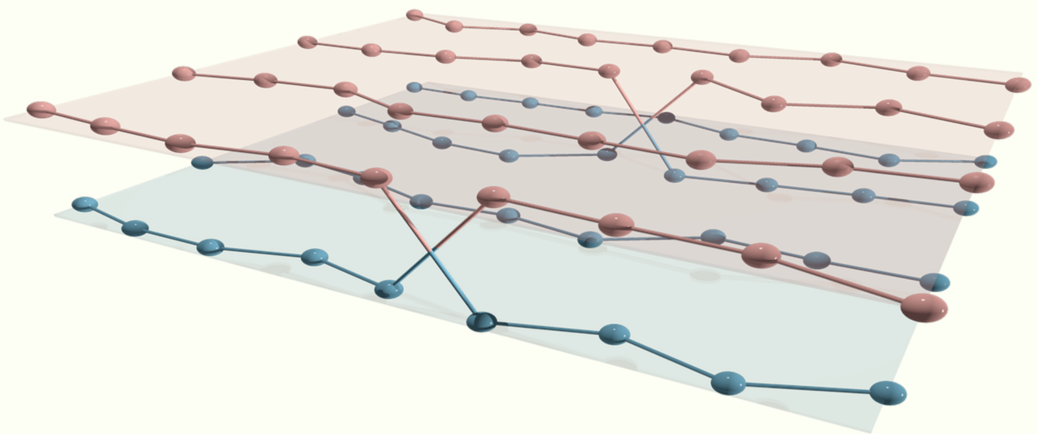
The useful entanglement is zero for non-interacting particles and peaks at some value of $\omega_{\rm int}$

C. M. Herdman *et al.* arXiv:1404.7104

Quantifying Entanglement

bipartite Rényi entropies in the spatial continuum



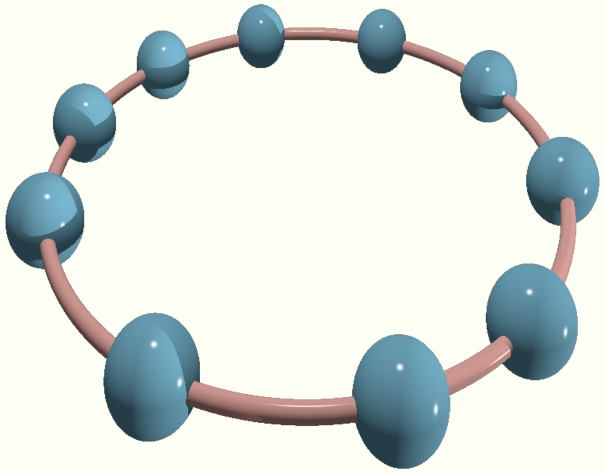


Algorithmic Development

measurement and benchmarking using path integral quantum Monte Carlo

Applications to 1d bosons

interactions and the connection between entanglement and condensate fraction



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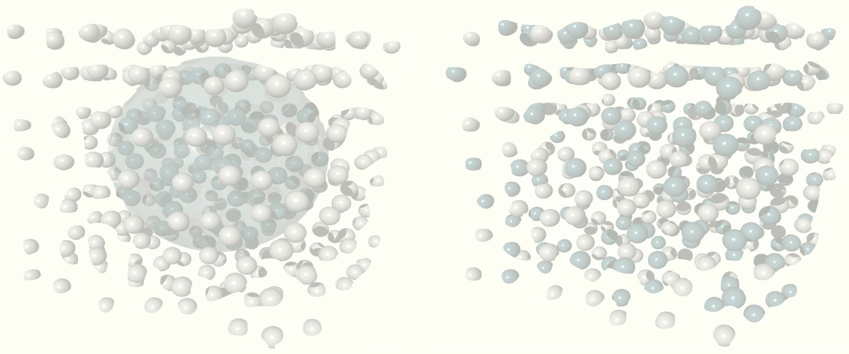

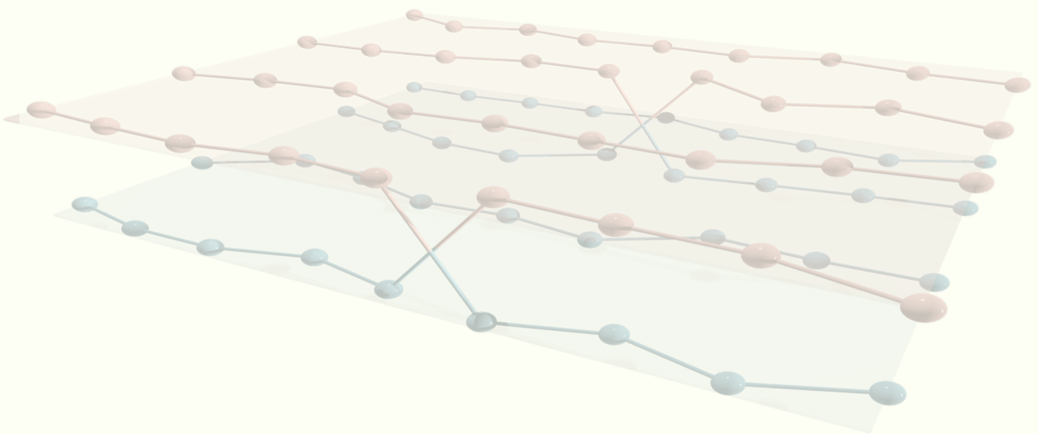

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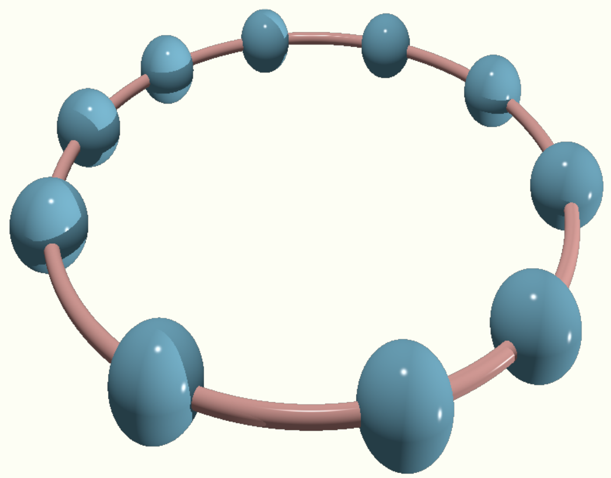
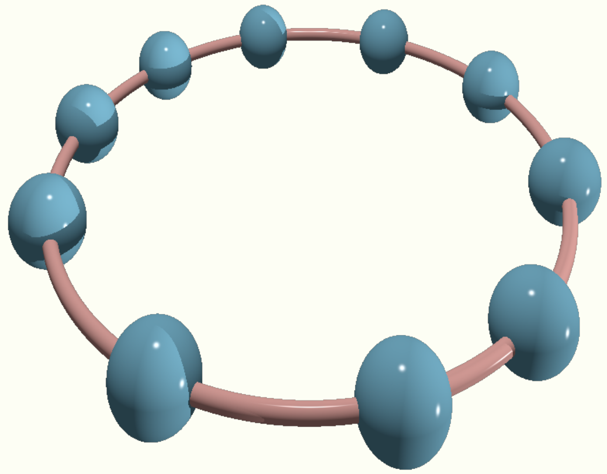


Moving towards a physically realizable system

one dimensional short-range interacting bosons

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx_i^2} + \frac{2c}{\sqrt{2\pi\sigma^2}} \sum_{j>i} \mathrm{e}^{-|x_i - x_j|^2/2\sigma^2} \right]$$



as σ → 0 we recover the Lieb-Liniger model of delta-function interacting bosons. E. H. Lieb and W. Liniger, PR 130, 1605 (1963)

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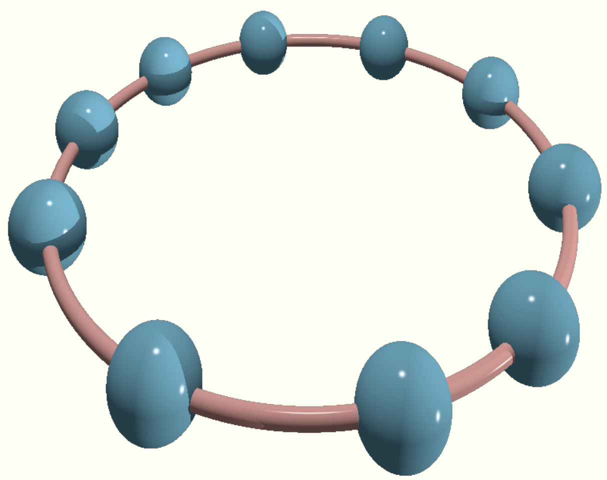

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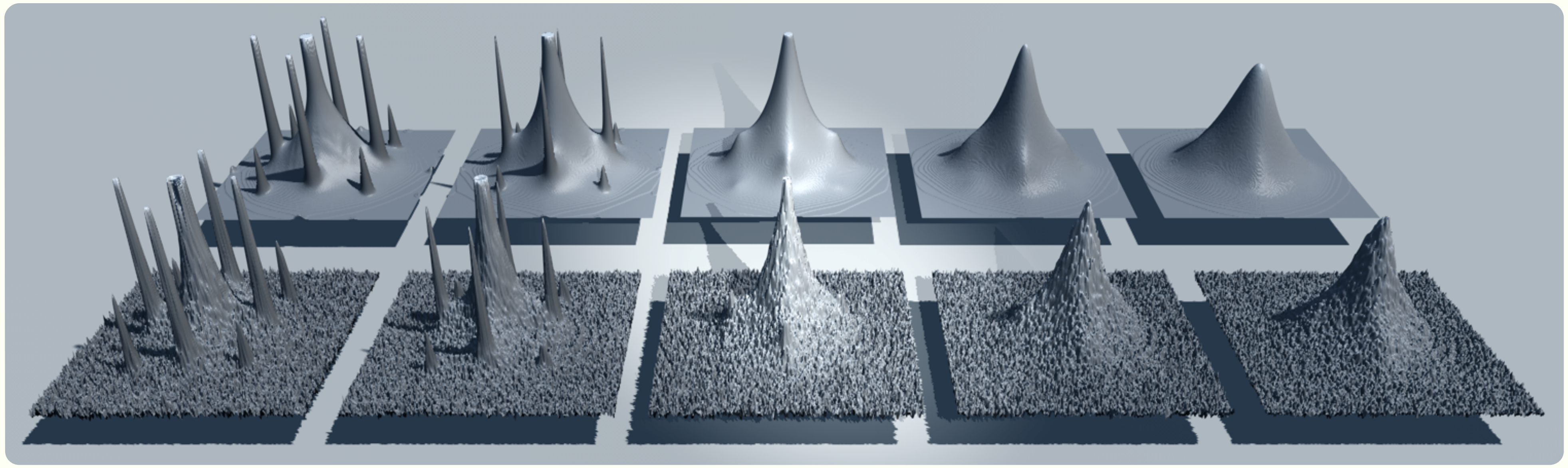
In the low energy limit, the system can be described via Luttinger liquid theory

1. *no phase transitions as a function of interaction strength*
2. *algebraic decay of all correlation functions*

Tonks-Girardeau gas: B. Paredes, *et al.*, Nature 429, 277 (2004)
T. Kinoshita, T. Wenger, and D. S. Weiss, Science 305, 1125 (2004)

Single particle entanglement is related to the condensate fraction!

the fractional population of the zero-momentum state is experimentally accessible via the momentum distribution



QMC

experiment

S. Trotzky, *et al.*, Nat. Phys. 6, 998 (2010)

- n_0 ***is the largest eigenvalue of the one-body density matrix***
- ***determines the "single-copy" entropy:*** $S_\infty = -\log n_0$
- ***fixes the binary (qubit) entropy:*** $S_{\mathrm{QB}} = -\log\left[n_0^2 + (1-n_0)^2\right]$

S_∞ **&** S_{QB} ***can be used to bound*** $S_2(n=1)$

Bounding entanglement of interacting bosons

$$S_\infty \leq S_{\mathrm{QB}} \leq S_2(n=1) \leq 2S_\infty \qquad (n_0 \leq 1/2)$$

$$S_{\mathrm{QB}} \leq S_\infty \leq S_2(n=1) \leq 2S_\infty \qquad (n_0 > 1/2)$$

Bounding entanglement of interacting bosons

$$S_\infty \leq S_{\rm QB} \leq S_2(n=1) \leq 2S_\infty \qquad (n_0 \leq 1/2)$$
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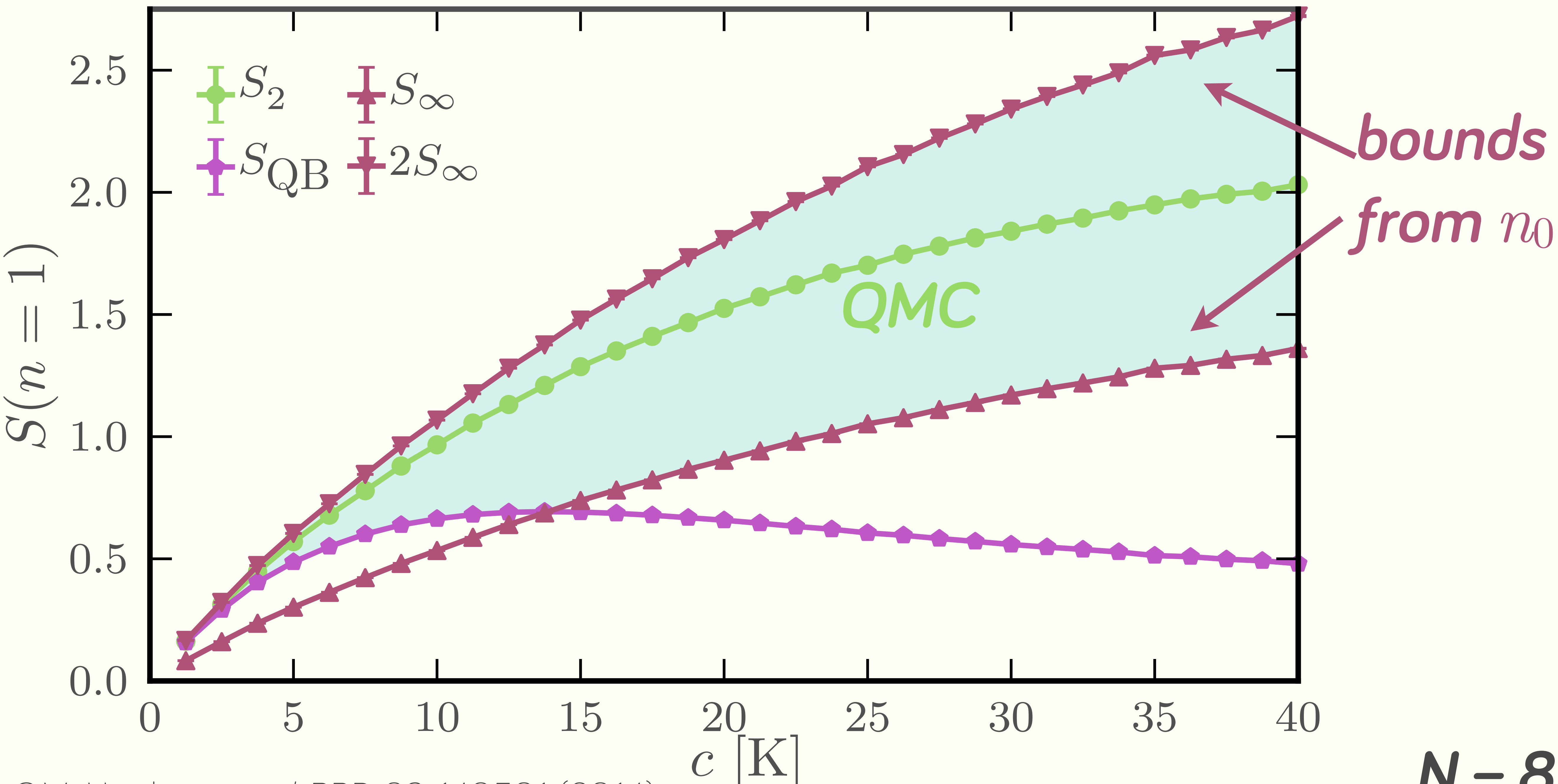


N = 8

C.M. Herdman *et al.* PRB, 89, 140501 (2014)

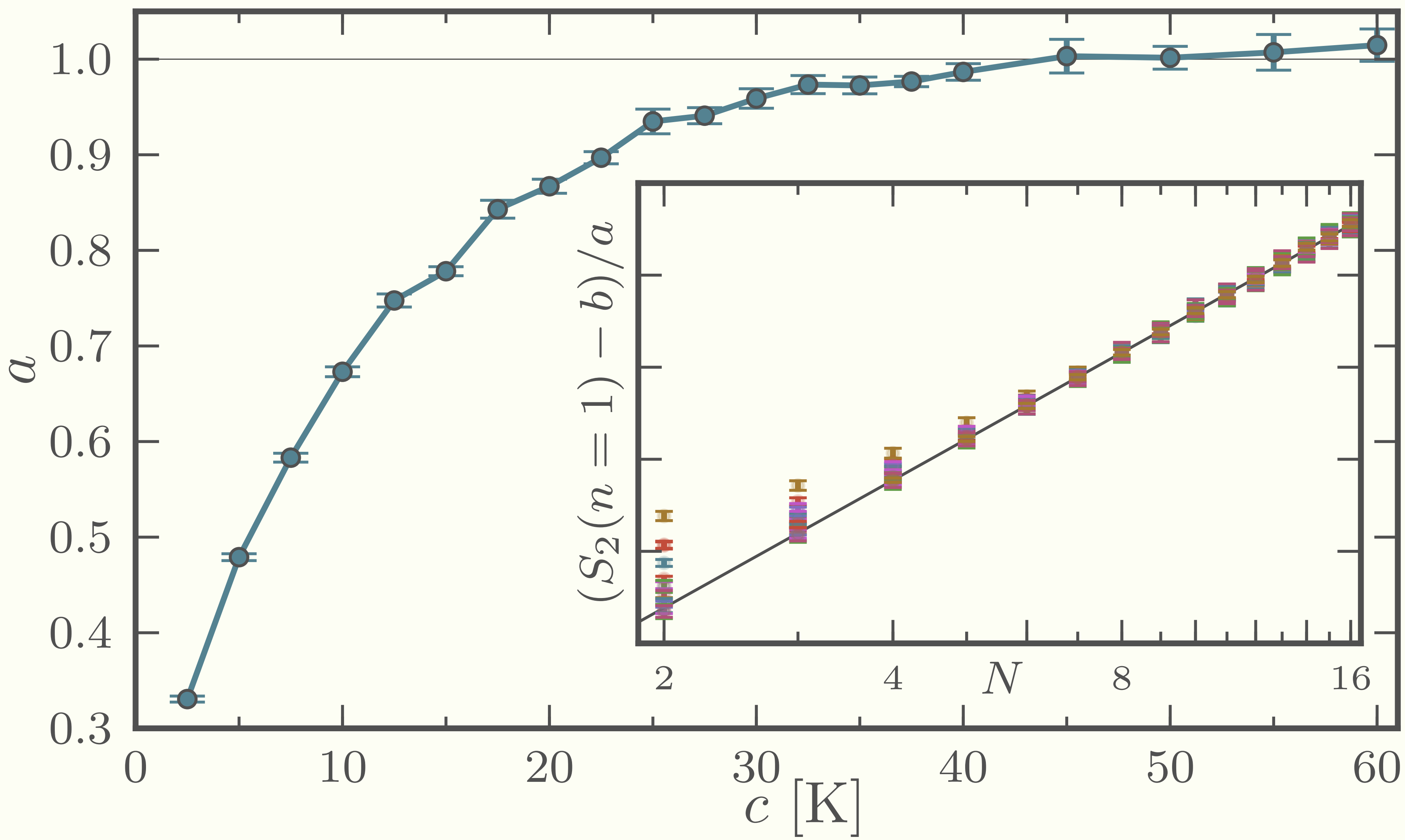
Finite size scaling and universality

Canonical Form

A universal canonical scaling function for particle entanglement entropy

$$S(n, N; a, b) = an \log N + b$$

O. Zozulya, M. Haque, and K. Schoutens, PRA 78, 042326 (2008)



Tonks-Girardeau limit

nearly perfect data collapse to log scaling for N > 8

C.M. Herdman *et al.* PRB, 89, 140501 (2014)

Can now quantify entanglement in itinerant boson systems in the spatial continuum

Experimental measurement & optimization

Bound entanglement via the condensate fraction and learn how to optimize the functional entanglement that can be transferred to a register for quantum information processing.

Applications to low dimensional quantum field theory

Scaling pre-factor of the one-particle entanglement is related to the Luttinger parameter of the effective field theory.

Computing resources and partners in research

VERMONT COMPLEX SYSTEMS CENTER

Calcul Québec

compute • calcul CANADA

VACC
Vermont Advanced Computing Core
UNIVERSITY OF VERMONT

XSEDE
Extreme Science and Engineering Discovery Environment