Analytic derivation of the sub stellar mass limit for non-zero degeneracy parameter

$$\psi = \frac{k_B T}{\mu_F}$$

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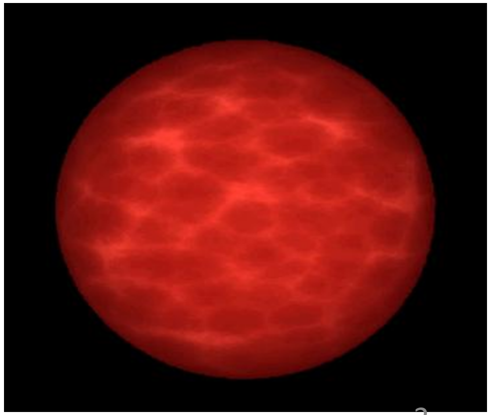
Western

What are brown dwarfs?

- These are sub stellar objects with too low a mass to sustain stable hydrogen burning and too much mass to be called a planet
- They emit their energy in the near infrared and cool quickly below the luminosity range of $\sim 10^{-4} L_{sun}$
- These are failed stars where the **electron degeneracy pressure** prevents the gravitational collapse. The stability is reached at a temperature much lower than what is necessary for stable nuclear burning.



Artist Impression (Google Image)

Basic equations of Stellar structure

- Hydrostatic equilibrium

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

$$\frac{dL_r}{dr} = -4\pi r^2 \rho\varepsilon$$

M = Mass
P= Pressure
r = Radius of the star
L = Luminosity
ρ =Density
G =Gravitational Constant
ϵ = The energy generation rate

- Energy transport equation (radiation/ convection) and a equation of state.

How to get a minimum mass?

- Numerically solving the stellar equations using the condition of quasi equilibrium in successive steps.
- An analytic model which clarifies the essential physics while also obtaining an accurate answer.

The Uncertainties

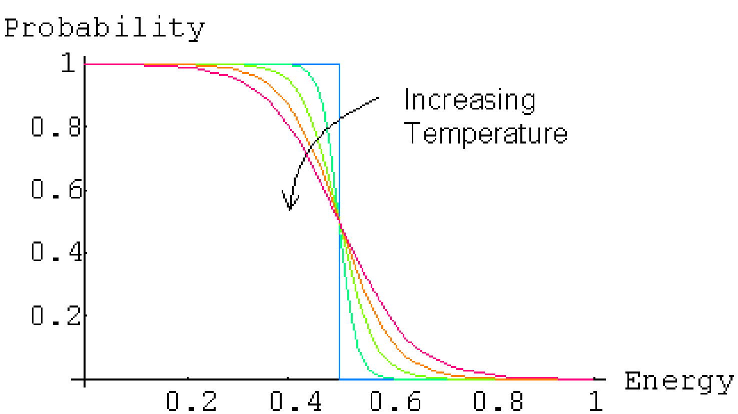
- An equation of state that incorporates the partial degeneracy of the electrons and the contribution of the ions.
- The nuclear rates suited to the low ($T < 10^6$ K) central temperature and high central density regimes of **very low mass** stars and **brown dwarfs**.
- A detailed treatment of the surface temperature and the atmospheric opacity.

Equation of state

- Sub stellar objects like brown dwarfs derive their stability from the electron degeneracy pressure.
- In the non relativistic limit, the Fermi integral is

$$P_F = a\int_0^\infty \frac{\varepsilon^{\frac{3}{2}} d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

ϵ = energy
μ = Fermi energy
a = constant
$\beta = k_B T$



In the limit **T->0** , for all $\varepsilon < \mu$ the exponent goes to zero and the integral reduces to Fermi pressure at T = 0 , i.e

$$P_F = a\frac{2}{5}\mu^{5/2}$$

Equation of state (cont)

The pressure of a degenerate Fermi gas at finite temperature

A = constants

$$\alpha = \frac{5\mu_{electron}}{2\mu_{ions}}$$

$$P_F = a\int_0^\infty \frac{\varepsilon^{\frac{3}{2}} d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$P_F = a\frac{2}{5}\mu^{\frac{5}{2}} + \frac{3}{4}a\beta^{-1}\mu^{\frac{3}{2}}\ln(1+e^{-\beta\mu}) - \frac{3}{4}a\beta^{-2}\mu^{\frac{1}{2}}Li_2(-e^{-\beta\mu}) - \frac{3}{4}a\beta^{-3}\mu^{\frac{-1}{2}}Li_3(-e^{-\beta\mu})\ldots$$

The total pressure of **the partially degenerate electron** and **ions**

$$P = \frac{2}{5}aA^{\frac{5}{2}}\left[\frac{\rho}{\mu_e}\right]^{\frac{5}{3}}\left[1+\frac{15}{8}\psi\ln(1+e^{-\beta\mu}) - \frac{15}{8}\psi^2 Li_2(-e^{-\beta\mu}) + \alpha\psi\right]$$

Li_N (z) are the polylogs of order N

The Power Law

- The P-ρ relation is a power law

$$P = \frac{2}{5} a A^{\frac{5}{2}} \left[\frac{\rho}{\mu_e} \right]^{\frac{5}{3}} \left[1 + \frac{15}{8} \psi \ln(1 + e^{-\beta\mu}) - \frac{15}{8} \psi^2 Li_2(-e^{-\beta\mu}) + \alpha\psi \right]$$

$$P = K \rho^{5/3}$$

- The equations for the hydrostatic equilibrium can be combined for polytropes to obtain ***the Lane–Emden equation.***

The surface phenomenon

- The effective surface temperature T_e can be approximated by using **the isentropic condition** to match the entropy of the photosphere and the vast convective region.

$$T_e = 1.8 \times 10^6 \rho_e^{.42} \psi^{1.545} K^{[1]}$$

$$\rho_e = \frac{1.013152 \times 10^{-5}}{\kappa_R^{1.1828}} \left(M / M_0 \right)^{1.174} \frac{\mu_e^{3.051}}{\left[1 + \frac{15}{8}\psi \ln(1 + e^{-\beta\mu}) - \frac{15}{8}\psi^2 Li_2(-e^{-\beta\mu}) + \alpha\psi \right]^{1.408} \psi^{1.088}}$$

The surface luminosity

$$L = 4\pi\sigma T_e^{\,4}$$

$$L = \frac{462.7 L_o \sigma}{\kappa_R^{\;1.1828}} \left({M}/{M_0} \right)^{1.305} \frac{\mu_e^{\;1.793}}{\left[1 + \frac{15}{8}\psi \ln(1 + e^{-\beta\mu}) - \frac{15}{8}\psi^2 Li_2(-e^{-\beta\mu}) + \alpha\psi \right]^{.3656} \psi^{4.352}}$$

Here σ is the Stefan Boltzmann constant

L_o is the Solar luminosity

M_0 is the Solar mass

The variation of ψ with time

- The energy equation

$$\frac{dE}{dt} + P\frac{dV}{dt} = T\frac{dS}{dt} = \dot{\varepsilon} - \frac{\partial L}{\partial M}$$

$$\frac{d\psi}{dt} = -\frac{9.37515\times 10^{-19}}{\kappa_R^{1.1828}\mu_e^{.8736}}\left(\frac{M_0}{M}\right)^{1.0283}\psi^{4.352}$$

E = Energy

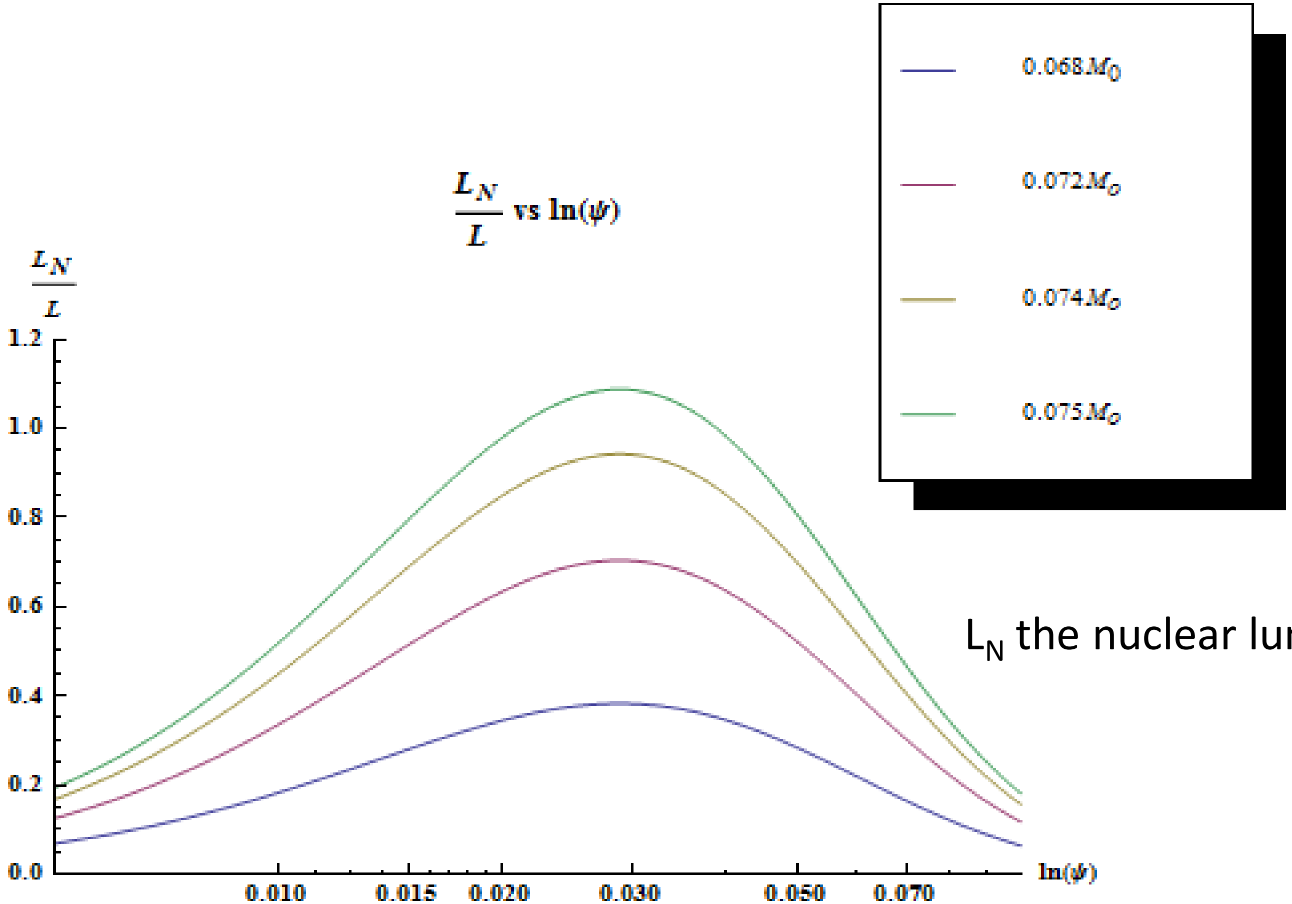
S = Entropy

$\dot{\varepsilon}$ = energy generation

ψ = degeneracy parameter

κ_R = Opacity

Luminosity Plots

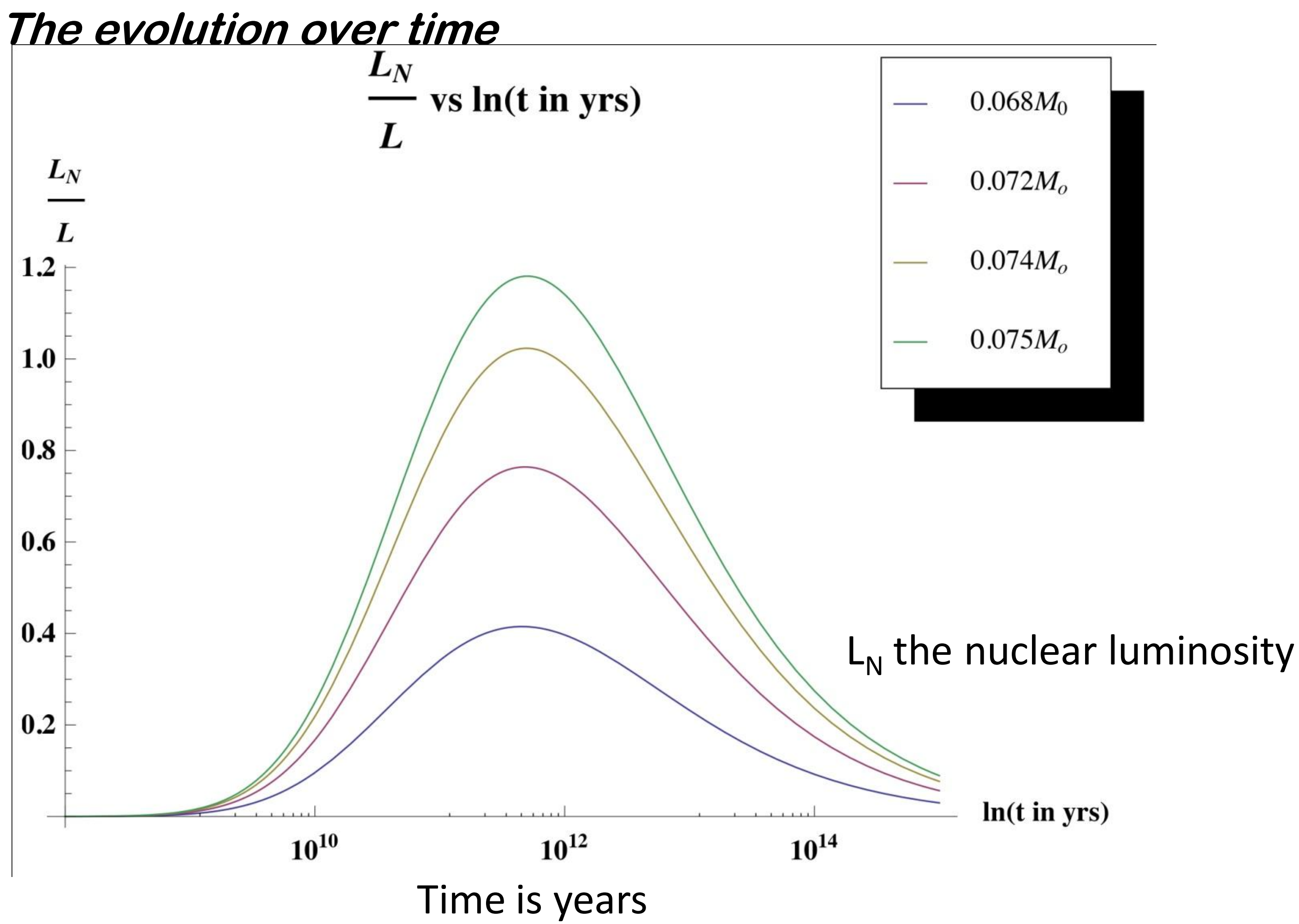


L_N the nuclear luminosity

Ψ (degeneracy parameter)

Stable nuclear burning (main sequence) is established if and when L_N /L reaches unity.

Plot 2



Stable nuclear burning (main sequence) is established if and when L_N /L reaches unity.

Conclusion

- The minimum mass for stable hydrogen fusion is 0.074 M_o.
- However objects below the minimum mass limit do sustain nuclear burning for long period but never reaches the main sequence.
- There can be a variation in the minimum mass boundary due to number of uncertainties involved.
- The minimum mass varies significantly with the hydrogen mass fraction of the stellar body.

Back Up Slides

The derivation of the Fermi integral

$$P_F = a\int_0^{\mu} \frac{\epsilon^{\frac{3}{2}} d\epsilon}{e^{\beta(\epsilon-\mu)}+1} + a\int_{\mu}^{\infty} \frac{\epsilon^{\frac{3}{2}} d\epsilon}{e^{\beta(\epsilon-\mu)}+1}, \tag{A1}$$

$$= a\int_0^{\mu} \epsilon^{\frac{3}{2}} d\epsilon - a\int_0^{\mu} \epsilon^{\frac{3}{2}} d\epsilon + a\int_0^{\mu} \frac{\epsilon^{\frac{3}{2}} d\epsilon}{e^{\beta(\epsilon-\mu)}+1} + a\int_{\mu}^{\infty} \frac{\epsilon^{\frac{3}{2}} d\epsilon}{e^{\beta(\epsilon-\mu)}+1}, \tag{A2}$$

$$= a\int_0^{\mu} \epsilon^{\frac{3}{2}} d\epsilon - a\int_0^{\mu} \frac{\epsilon^{\frac{3}{2}} d\epsilon}{e^{-\beta(\epsilon-\mu)}+1} + a\int_{\mu}^{\infty} \frac{\epsilon^{\frac{3}{2}} d\epsilon}{e^{\beta(\epsilon-\mu)}+1}. \tag{A3}$$

Let us substitute $x = -\beta(\epsilon - \mu)$ in the second term and $x = \beta(\epsilon - \mu)$ in the third term and we arrive at

$$P_F = a\int_0^{\mu} \epsilon^{\frac{3}{2}} d\epsilon - \frac{a}{\beta}\int_0^{\beta\mu} \frac{(\mu - \frac{x}{\beta})^{\frac{3}{2}}}{e^x+1} dx + \frac{a}{\beta}\int_0^{\infty} \frac{(\mu + \frac{x}{\beta})^{\frac{3}{2}}}{e^x+1} dx \tag{A4}$$

The surface temperature

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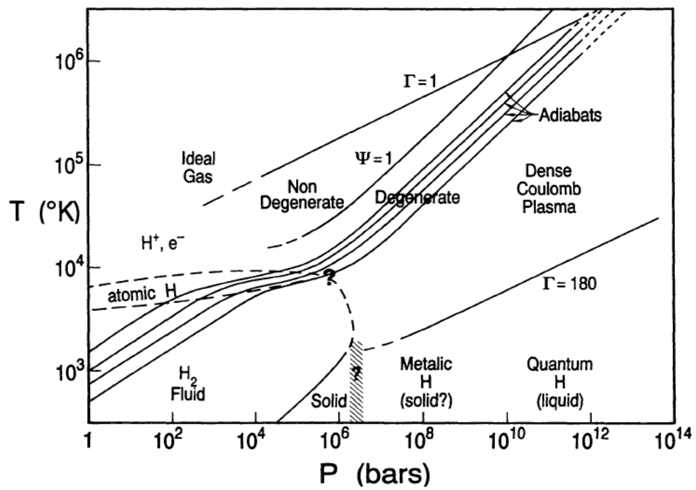

Figure 2 Phase diagram for hydrogen, with superimposed adiabats for a variety of brown dwarf entropies. These adiabats apply irrespective of mass and are truncated at the central pressure, which does depend strongly on mass (Equation 12).

$$\sigma_1 = -1.594 \ln \frac{1}{\psi} + 12.43$$

$$\sigma_2 = 1.032 \ln \frac{T}{\rho^{.42}} - 2.438$$

The Nuclear luminosity

$$L_N = 7.33127 L_o \left(M \big/ M_0 \right)^{11.98} \frac{\psi^{6.316}}{\left[1 + \frac{15}{8}\psi \ln(1 + e^{-\beta\mu}) - \frac{15}{8}\psi^2 Li_2(-e^{-\beta\mu}) + \alpha\psi \right]^{16.466}}$$

The Lame- Emden Equation

- Two equation for the hydrostatic equilibrium can be combined to get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right]=-\theta^n$$

$$\xi=\frac{r}{a}$$

$$a=\left[(n+1)\frac{K}{4\pi G}\rho_c^{\frac{1-n}{n}}\right]^{\frac{1}{2}}$$