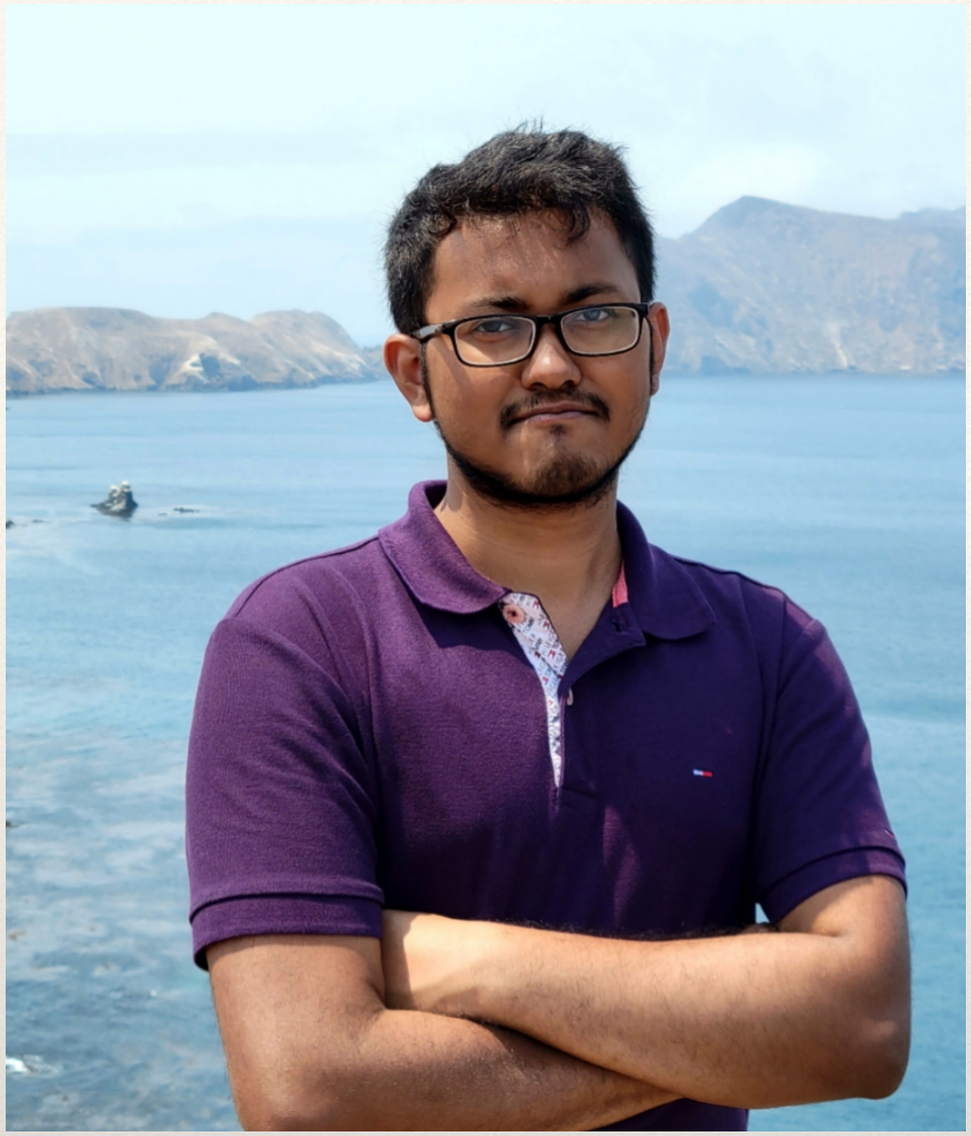
Quantization of the Volume of a Grain of Space

Tunneling Between Euclidean and Lorentzian Tetrahedra

Loops ‘22
July 22nd, 2022

Hal M. Haggard,
Bard College



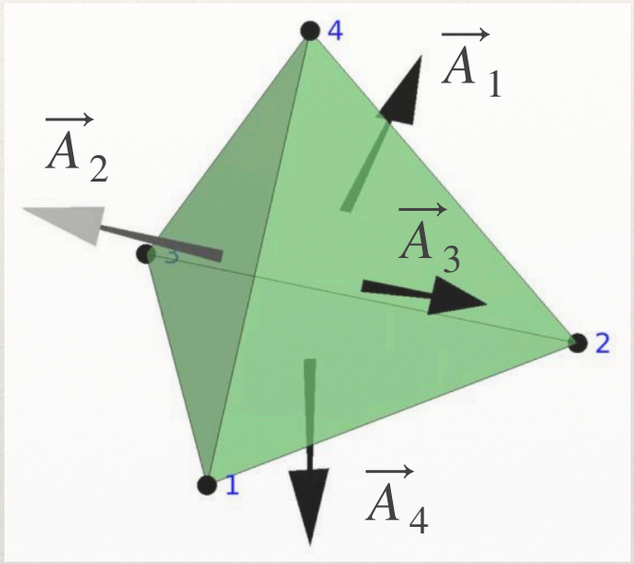
Antu Santanu

Charles Doran

Today's talk is based on unpublished work with Antu Santanu, and Charles Doran

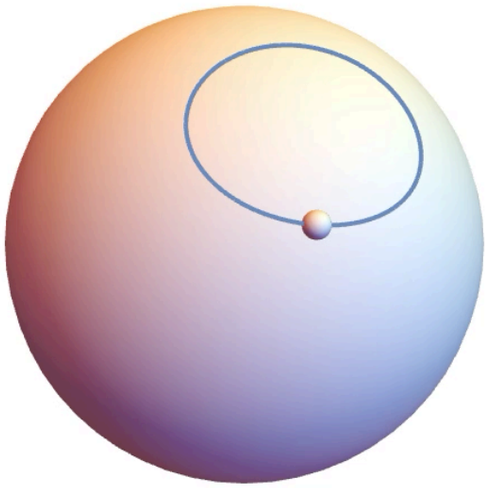
It is a stunning fact that discrete geometries, just as continuous spacetime, can be made dynamical.

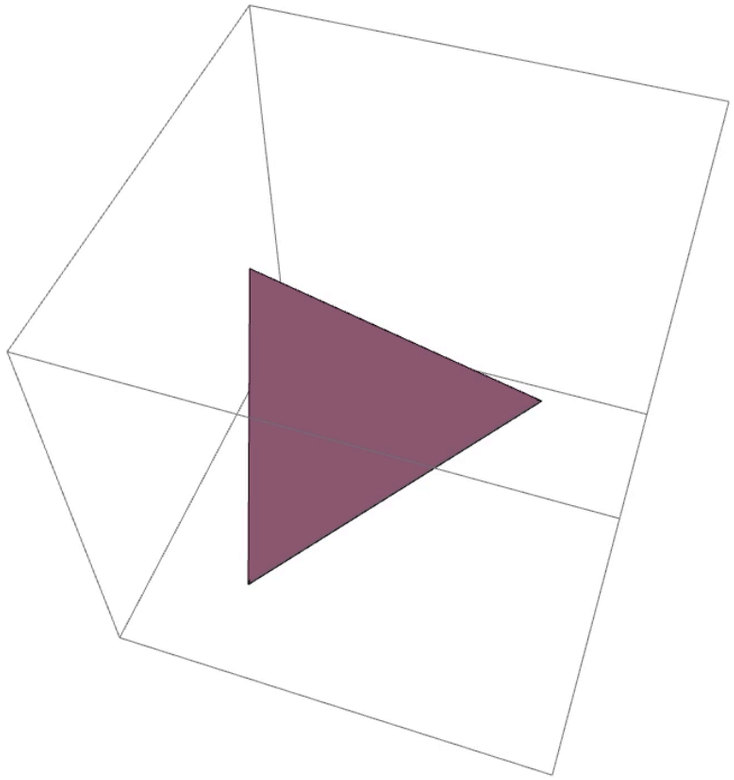
Their shapes evolve according to the choice of variables and the dynamical Hamiltonian. This means that even a tetrahedron has:

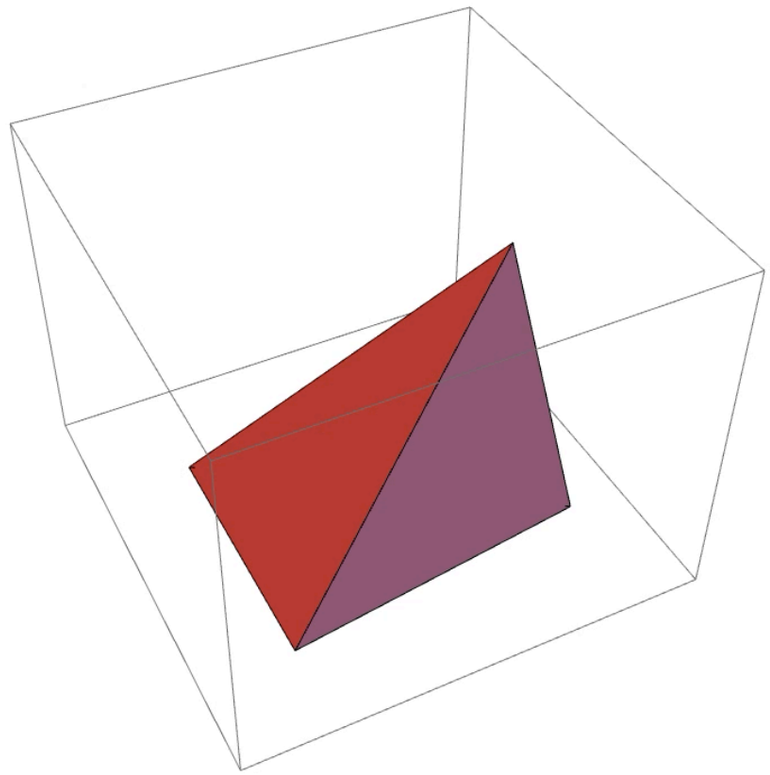


- A phase space of shapes
- Trajectories within that phase space
- Quantization of geometrical observables on this phase space, such as the face areas and the tetrahedral volume

The flow along the curves of constant volume V is integrable and describes a family of tetrahedra with different shapes, but equal volumes





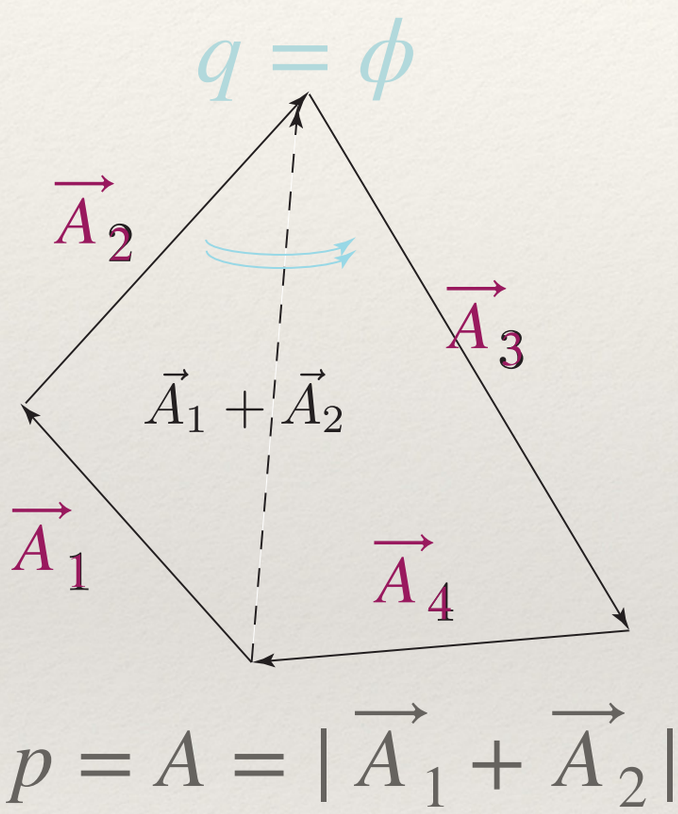


At left: the phase space of shapes with a constant volume contour; at right: two different views of the same tetrahedron as it undergoes the volume flow (click to play)

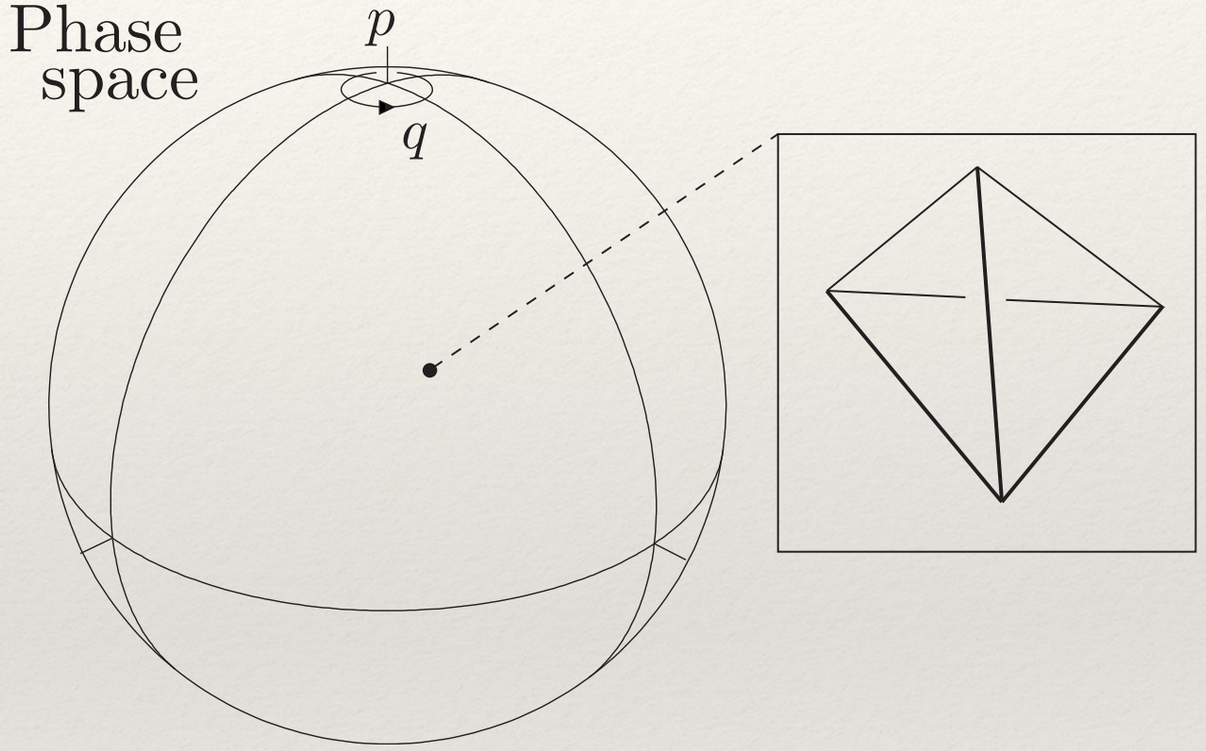
The phase space for a tetrahedron with areas A_1 … A_4 satisfies

closure: $\overrightarrow{A}_1 + \overrightarrow{A}_2 + \overrightarrow{A}_3 + \overrightarrow{A}_4 = 0.$

We adopt canonical coordinates by rearranging closure as vector addition:



$\Leftrightarrow$

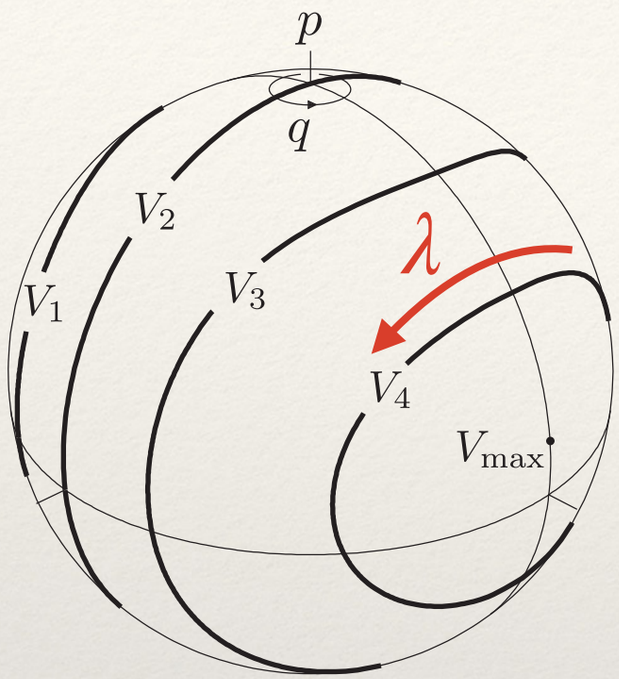


$p = A = |\overrightarrow{A}_1 + \overrightarrow{A}_2|$

Because we understand the vectors as elements of $\mathfrak{so}(3)$, the coords are naturally equipped with a Poisson bracket:

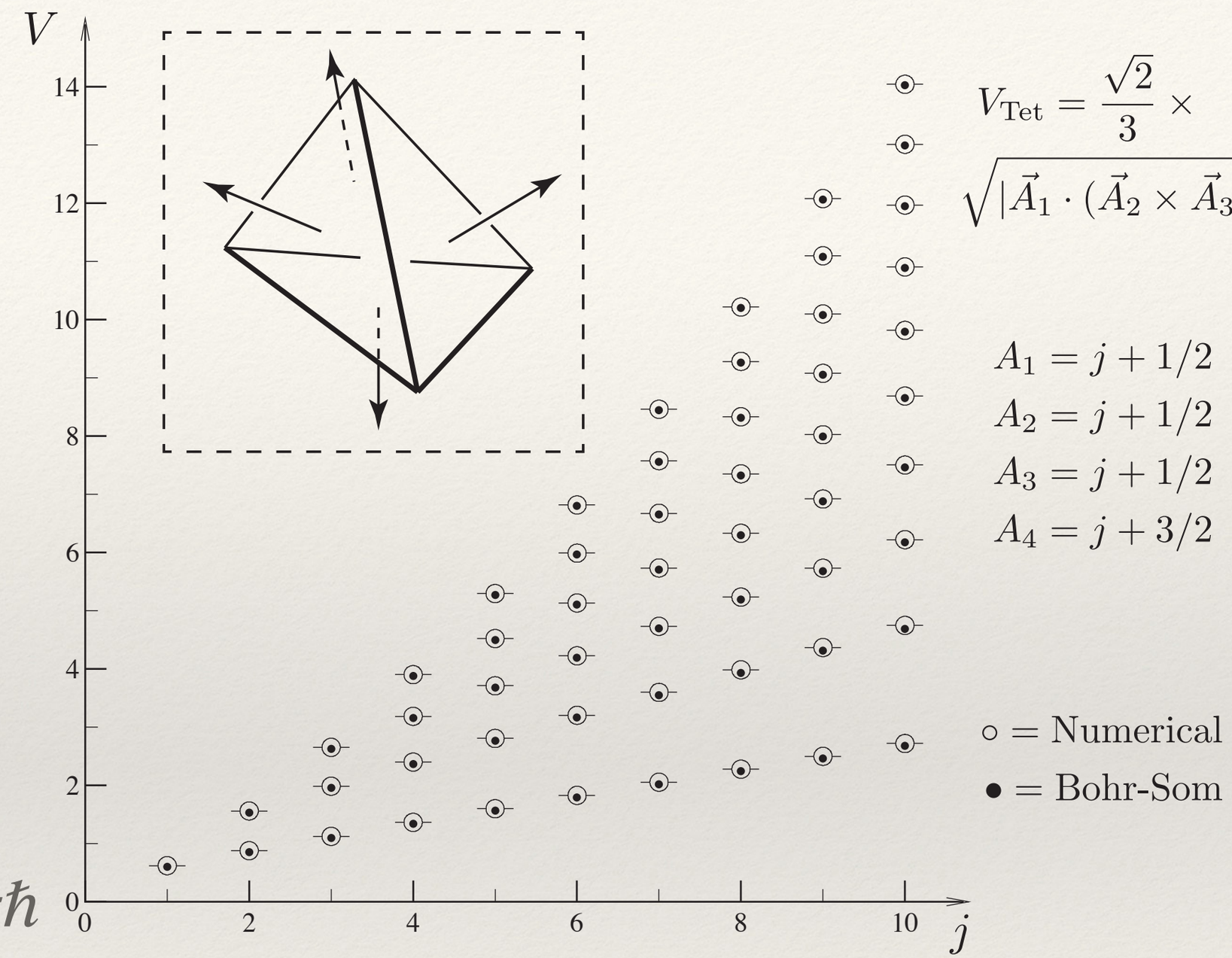
$$\{q, p\} = \{\phi, A\} = 1.$$

E. Bianchi & I found a semiclassical quantization of volume eigenvalues:



$$a(V_n) = \oint V d\lambda$$

$$= \left(n + \frac{1}{2}\right) 2\pi\hbar$$



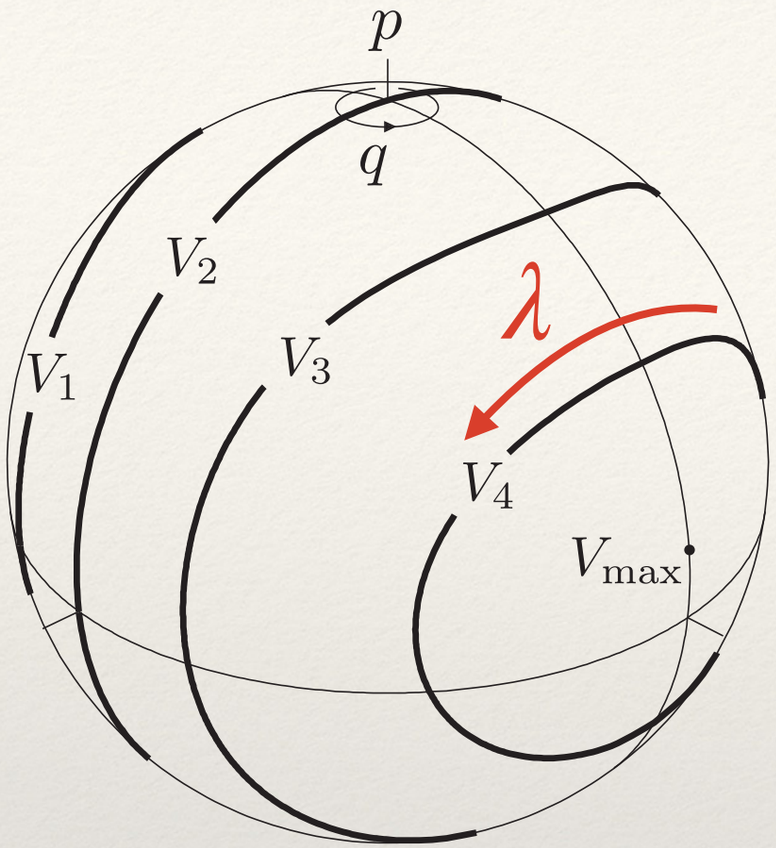
$$V_{\text{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$

$$A_1 = j + 1/2$$
$$A_2 = j + 1/2$$
$$A_3 = j + 1/2$$
$$A_4 = j + 3/2$$

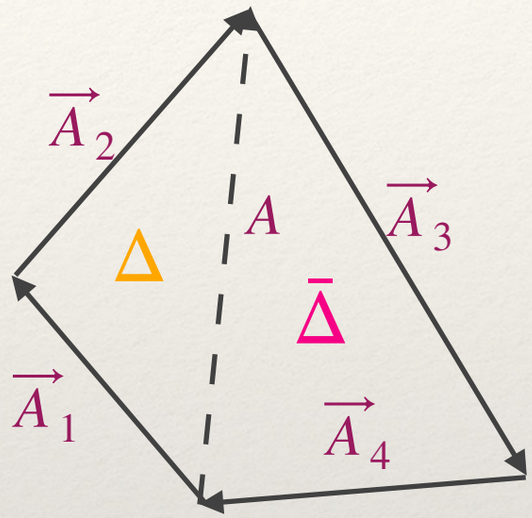
Here the Bohr-Sommerfeld values (solid dots) are compared to the numerical eigenvalues from a full quantum treatment (open circles).

Bianchi & HMH PRL 2011

The integrable volume evolution relates tetrahedra to the powerful mathematics of elliptic curves:



$$Q = V^2 = \frac{8}{9}\frac{\Delta\bar{\Delta}}{A}\sin\phi$$



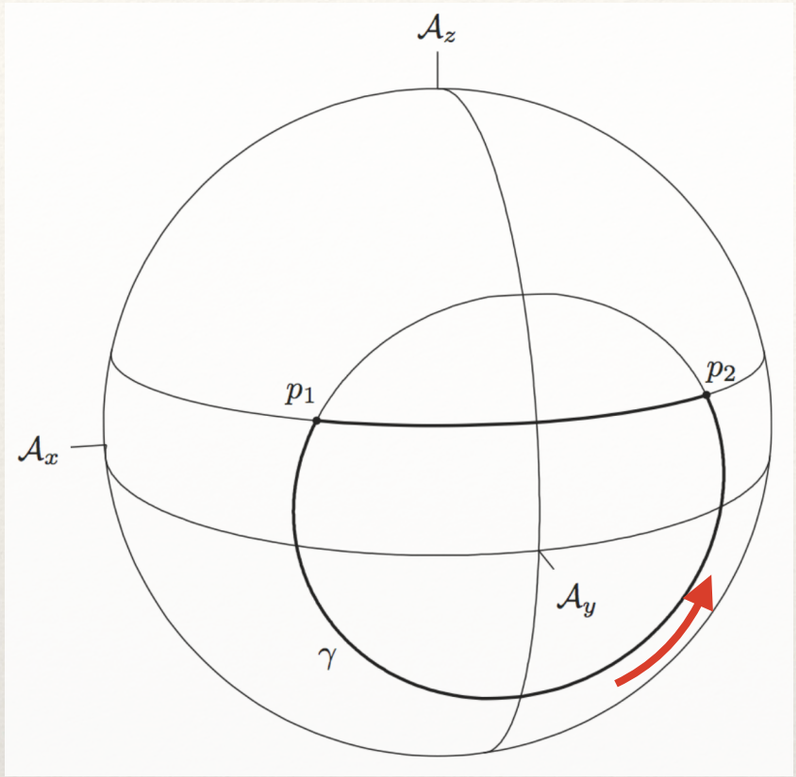
$$\bar{r}_1^2 = (A_1 - A_2)^2$$
$$\bar{r}_2^2 = (A_3 - A_4)^2$$
$$\bar{r}_3^2 = (A_1 + A_2)^2$$
$$\bar{r}_4^2 = (A_3 + A_4)^2$$

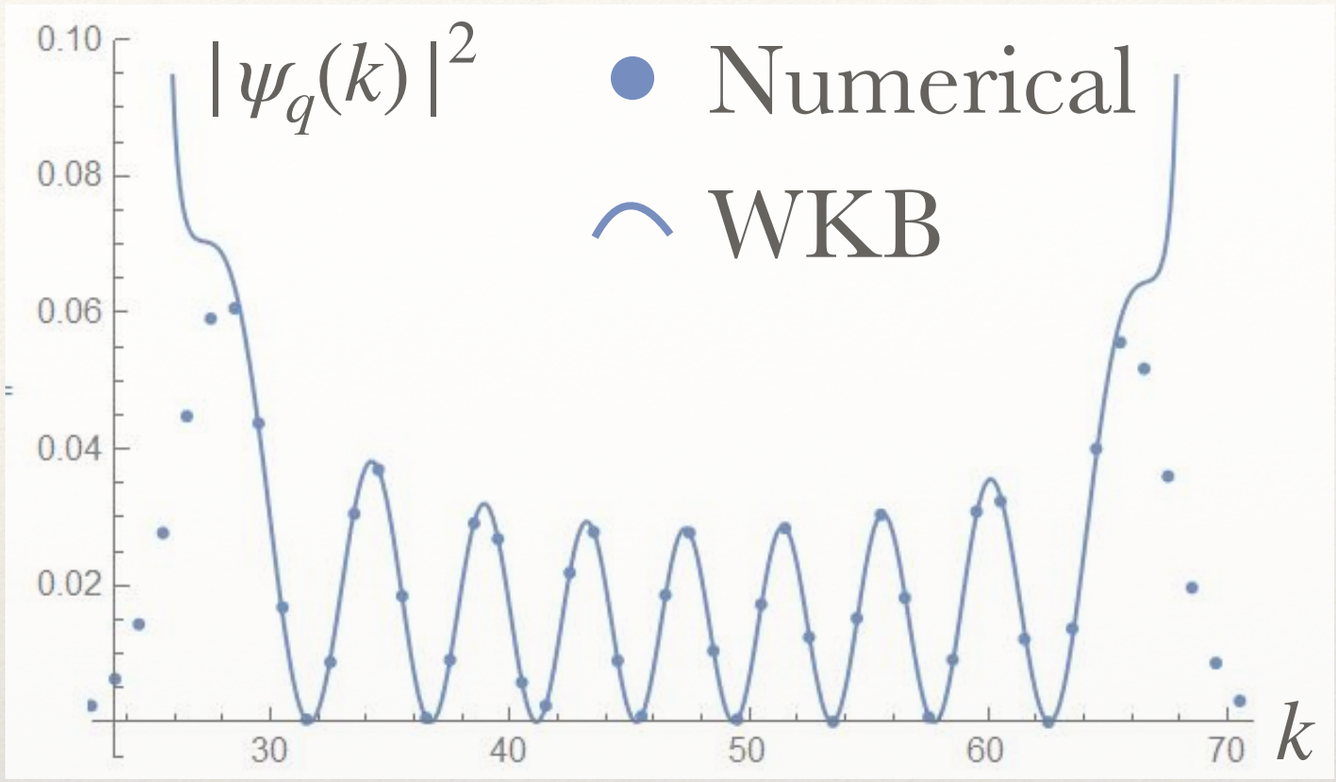
$$\frac{dA^2}{d\lambda} = \frac{1}{9}\sqrt{(4\Delta)^2(4\bar{\Delta})^2 - A^2(18Q)^2}$$

$$= \frac{1}{9}\sqrt{[A^2 - \bar{r}_1^2][A^2 - \bar{r}_2^2][\bar{r}_3^2 - A^2][\bar{r}_4^2 - A^2] - A^2(18Q)^2}$$

The solutions, $A^2(\lambda)$, are ratios of Jacobi elliptic functions.

A new result: Antu and I have been able to find the WKB volume eigenstates as well



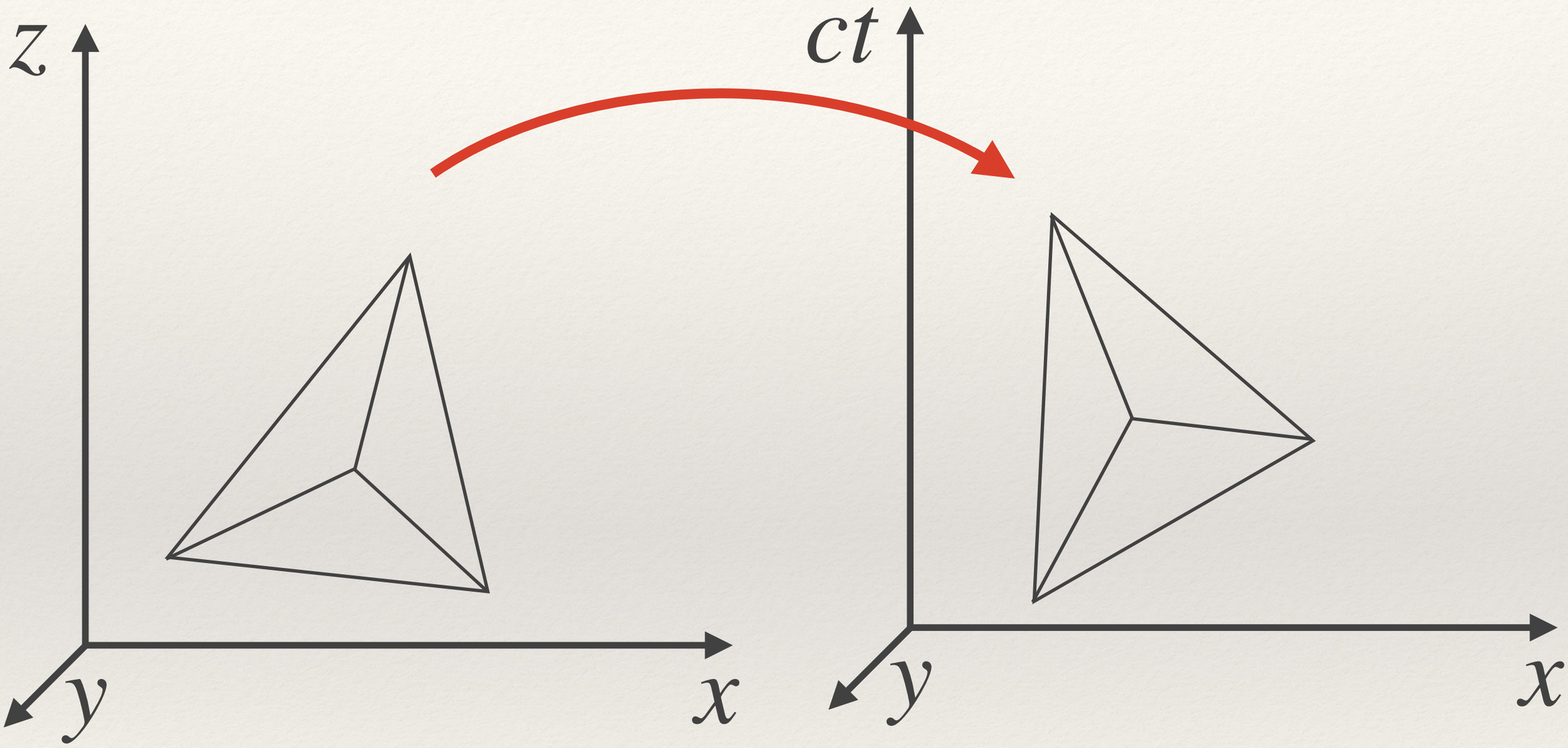


$$\psi_q(k) = \sqrt{\frac{4A}{gK(m)}} \frac{\eta}{|[(16\Delta\bar{\Delta})^2 - (18AQ)^2]^{1/4}|} \cos\left(\frac{1}{2}(\Delta S_q - \Delta S_A) - \frac{\pi}{4}\right)$$

where

$$S_{q,m} = 18gq\left[\frac{\lambda_{p_m}}{9g} - \sum_i \left(\frac{r_1}{r_1 - \bar{r}_i}\lambda_{p_m} - \frac{\bar{r}_i(r_2 - r_1)}{(r_2 - \bar{r}_i)(r_1 - \bar{r}_i)}\Pi\left(\alpha_i^2, \mathrm{am}\left(\frac{\lambda_{p_m}}{9g}, m\right), m\right)\right)\right]$$

The elliptic curves perspective connects the geometry of Euclidean tetrahedra to that of Lorentzian tetrahedra: that is, it describes tunneling between spacelike and timelike geometry!



The key is to complexify variables and to consider the complex actions that arises on the surface $V = \text{const}$.

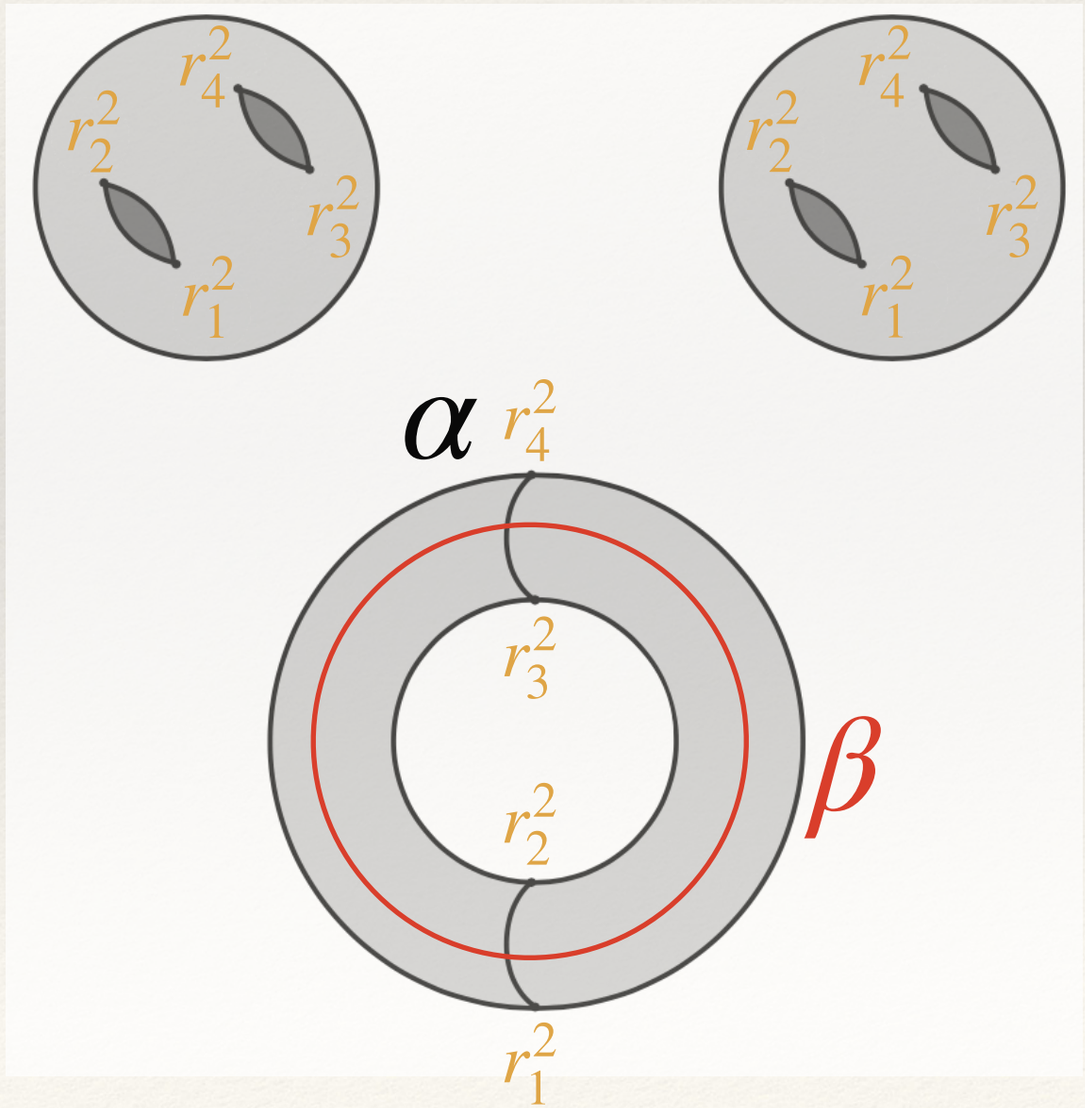
It turns out that this surface is a torus and that the action and dual action are computed around the α and β cycles of the torus

$$\frac{dA^2}{d\lambda} = \frac{1}{9}\sqrt{(4\Delta)^2(4\bar{\Delta})^2 - A^2(18Q)^2}$$

$$= \frac{1}{9}\sqrt{[A^2 - \bar{r}_1^2][A^2 - \bar{r}_2^2][\bar{r}_3^2 - A^2][\bar{r}_4^2 - A^2] - A^2(18Q)^2}$$

Bohr-Sommerfeld action:

$$a = \oint_\alpha p dq = \left(n + \frac{1}{2}\right) 2\pi\hbar$$



The dual action,

$$a^D = \oint_\beta p dq,$$

characterizes tunneling.

In fact these two actions are related solutions of a 3rd order Picard-Fuchs equation that we have found explicitly

$$D(r_i, Q)\frac{d^3a}{dQ^3} + c_2(r_i, Q)\frac{d^2a}{dQ^2} + c_1(r_i, Q)\frac{da}{dQ} = 0.$$

Conclusions

- We have an explicit description of the tunneling between Lorentzian and Euclidean tetrahedra. This may allow us to realize the Speziale-Rovelli symmetry principle explicitly.
- In other examples, Picard-Fuchs equations have allowed the computation of spectra to all orders in $\hbar$.
- Discreteness of geometrical observables, such as the volume of a tetrahedron, is not independent of the allowed tunneling between related geometries, e.g. Lorentzian & Euclidean tetrahedra of the same V.

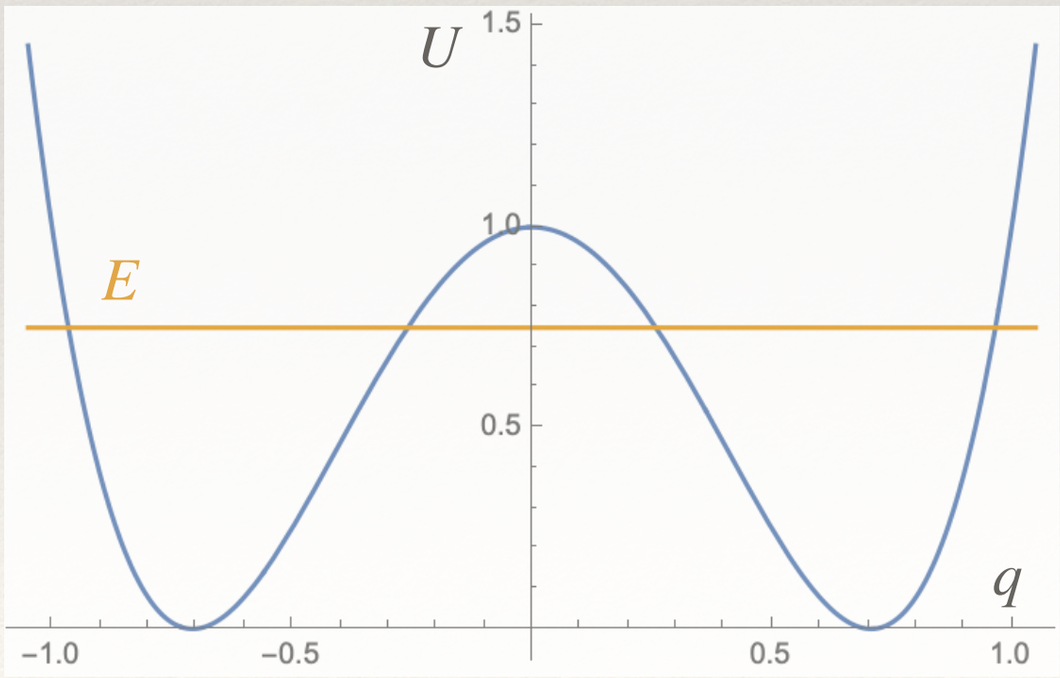
The End

Appendix: Semiclassical quantization and the geometry of spacetime

Much has been written about the fact that quantum mechanics is most smoothly formulated in $\mathbb{C}$.

Intriguingly, the complexification of classical mechanics is much less familiar to most. Nonetheless, this complexification encodes intriguing observations.

To better situate what I say about quantum gravity, I will first take you through an anharmonic oscillator example.



Constructive resurgence is a technology that leverages classical tools to inform you about perturbative/non-perturbative relations in quantum mechanics

Consider the conserved energy of a symmetric quartic oscillator

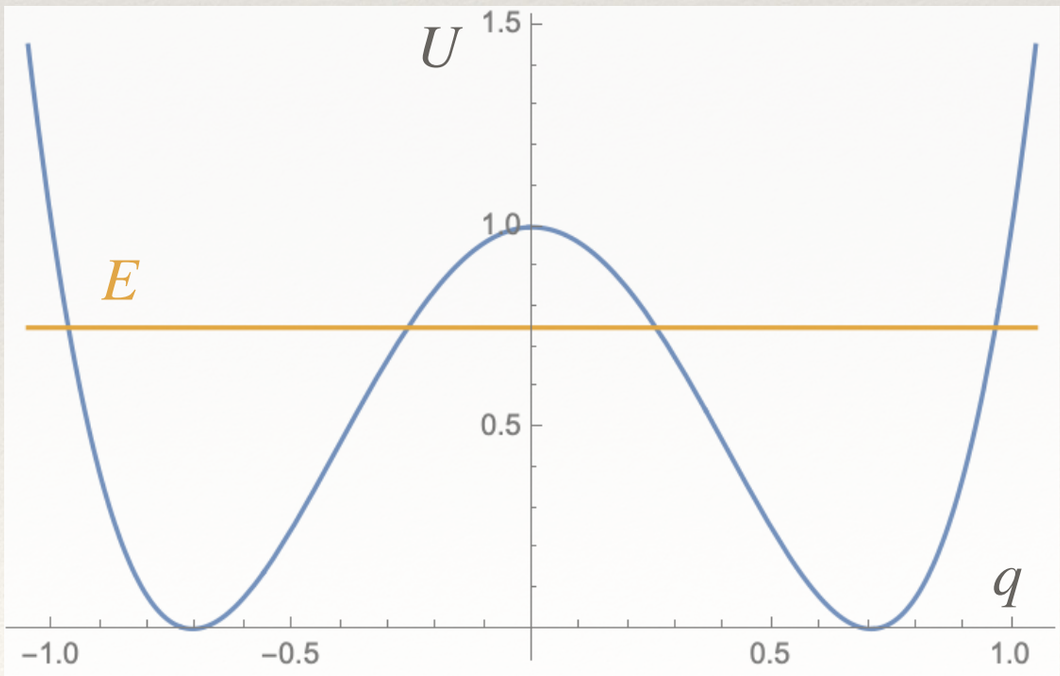
$$E = K + U = p^2/2m + (2q^2 - 1)^2.$$

The momentum is given by $p = \sqrt{2m[E - (2q^2 - 1)^2]}$ and branches at the four turning points where $(2q_*^2 - 1)^2 = E$. The classical action and period are:

$$a = \oint p dq,$$

and

$$T = \frac{da}{dE} = \oint \frac{m}{p} dq.$$



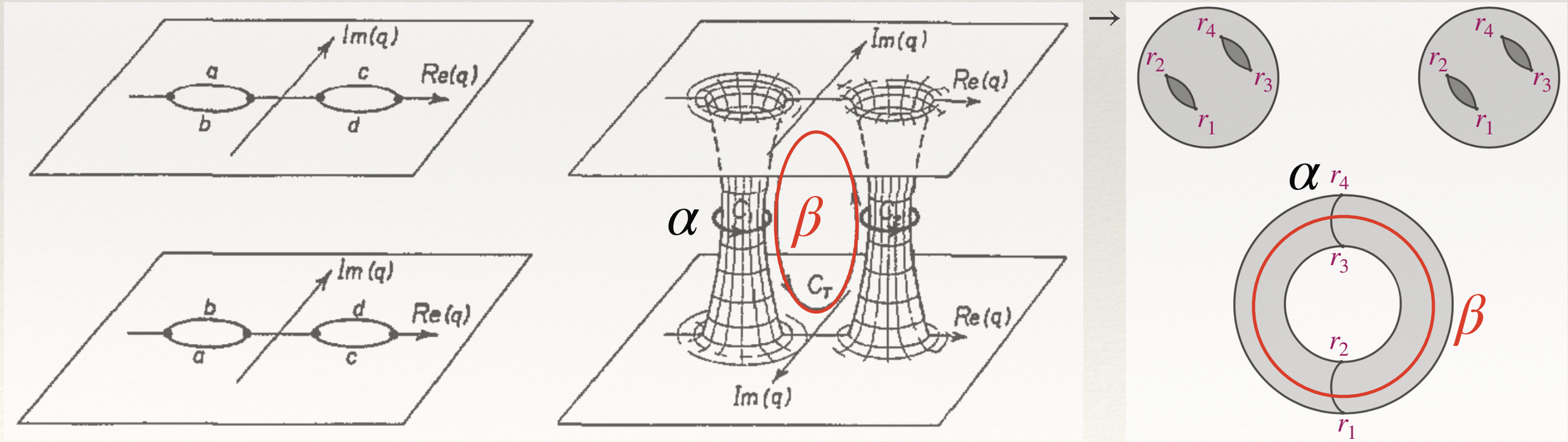
Complexifying variables and extending branch cuts between these branch points gives a Riemann surface of genus 1.

The two topologically distinct cycles give action integrals

$$a = \oint_{\alpha} p dq \qquad \text{and} \qquad a^D = \oint_{\beta} p dq,$$

sith the 2nd often referred to as the ‘dual action’.

stereographically close to Riemann spheres



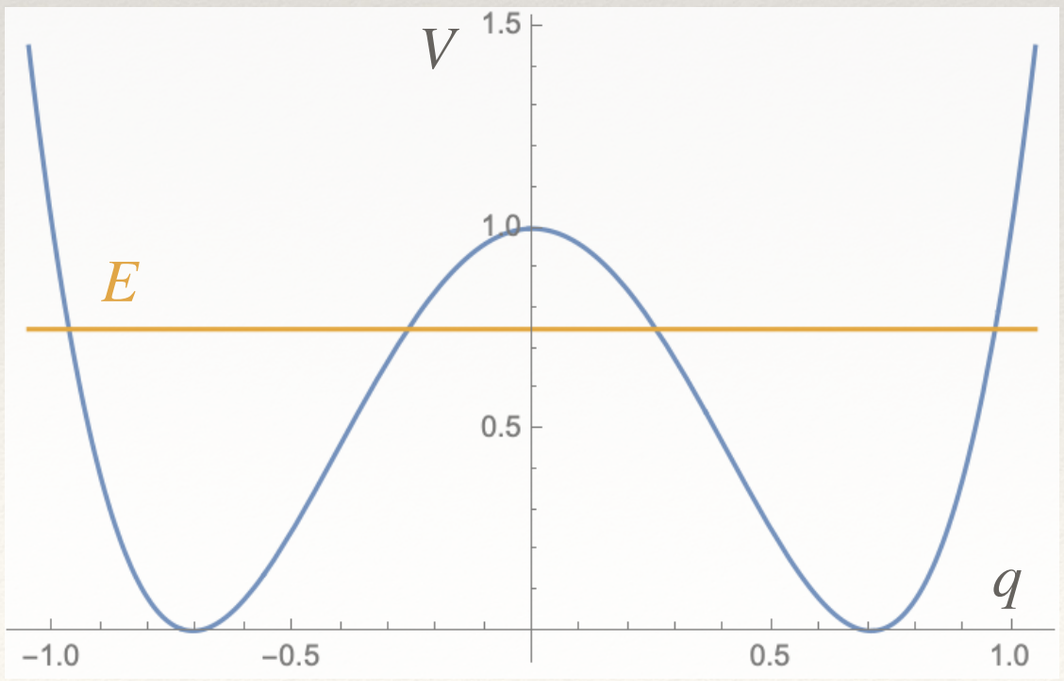
Constructive Resurgence — There is a remarkable second version of resurgence, often called constructive resurgence

Results are detailed and explicit, but largely restricted to phase spaces representable by (genus 1) Riemann surfaces

Canonical Example

$V(q) = (2q^2 - 1)^2,\ p = \sqrt{2m[E - V]}$

$$p = \sqrt{2m[E - (2q^2 - 1)^2]}$$



Volume Evolution Example

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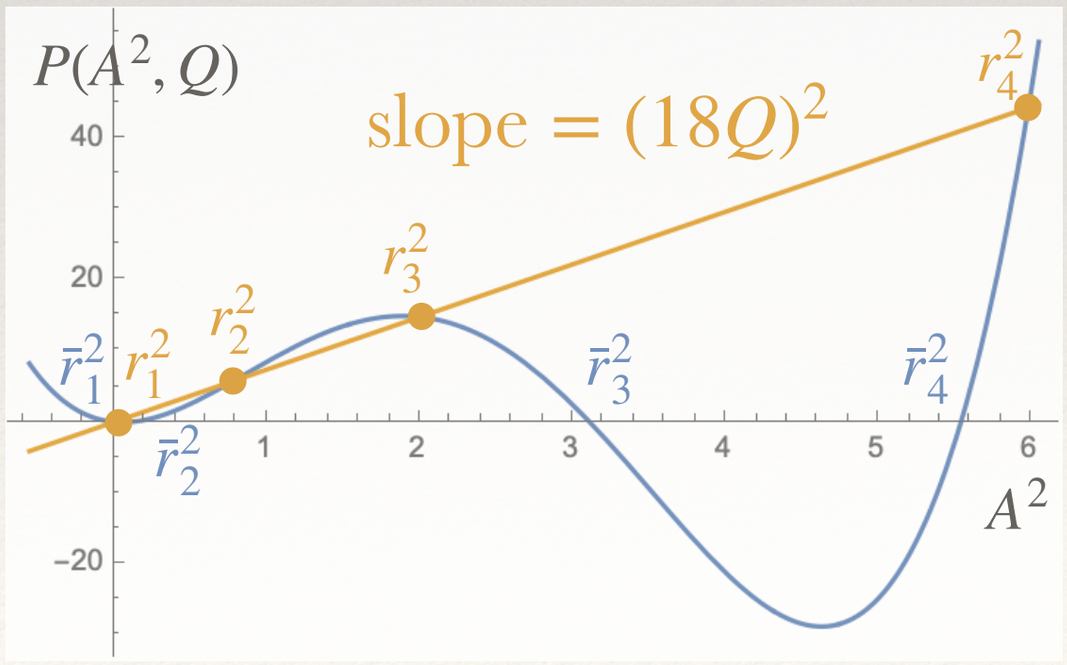

A4

In both examples we can view the phase space as a genus 1 torus by complexifying variables and gluing along branch cuts

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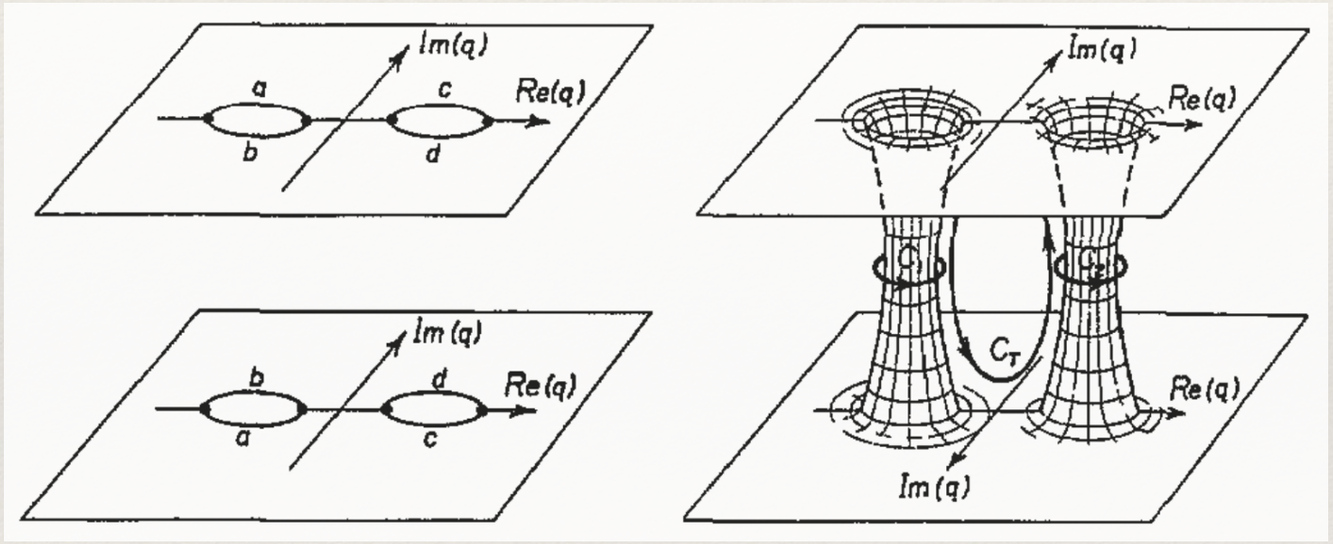

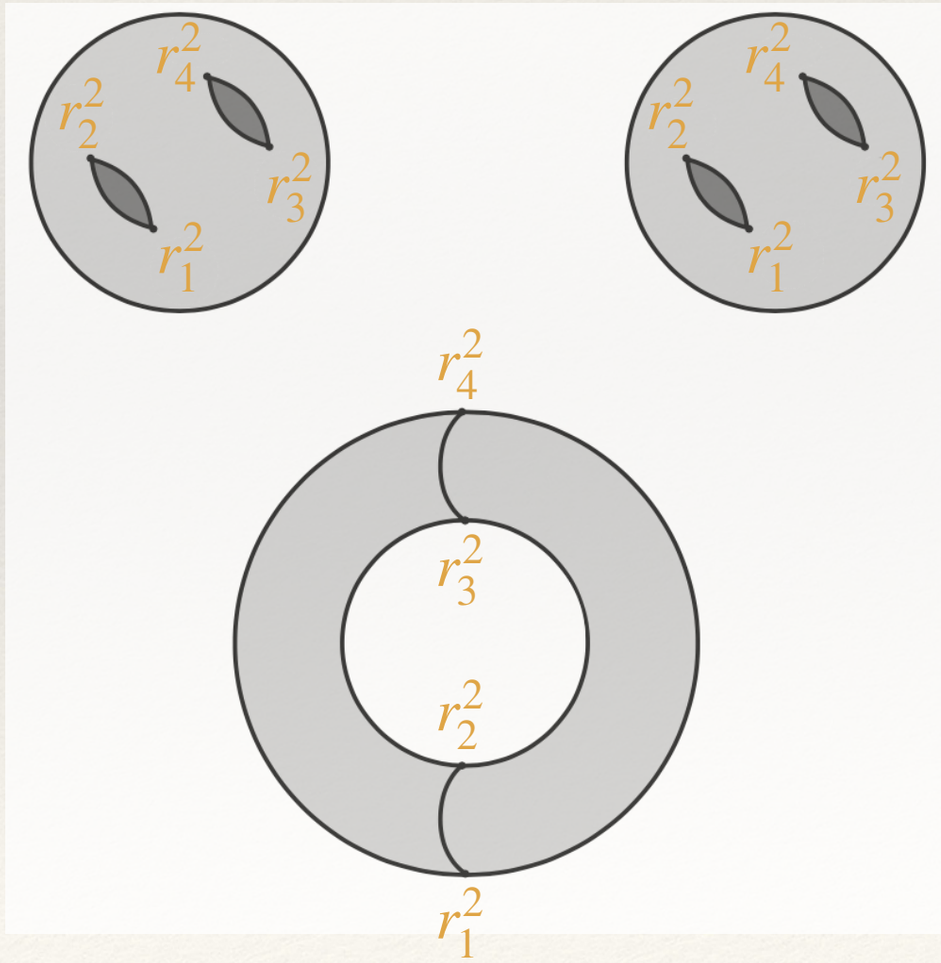

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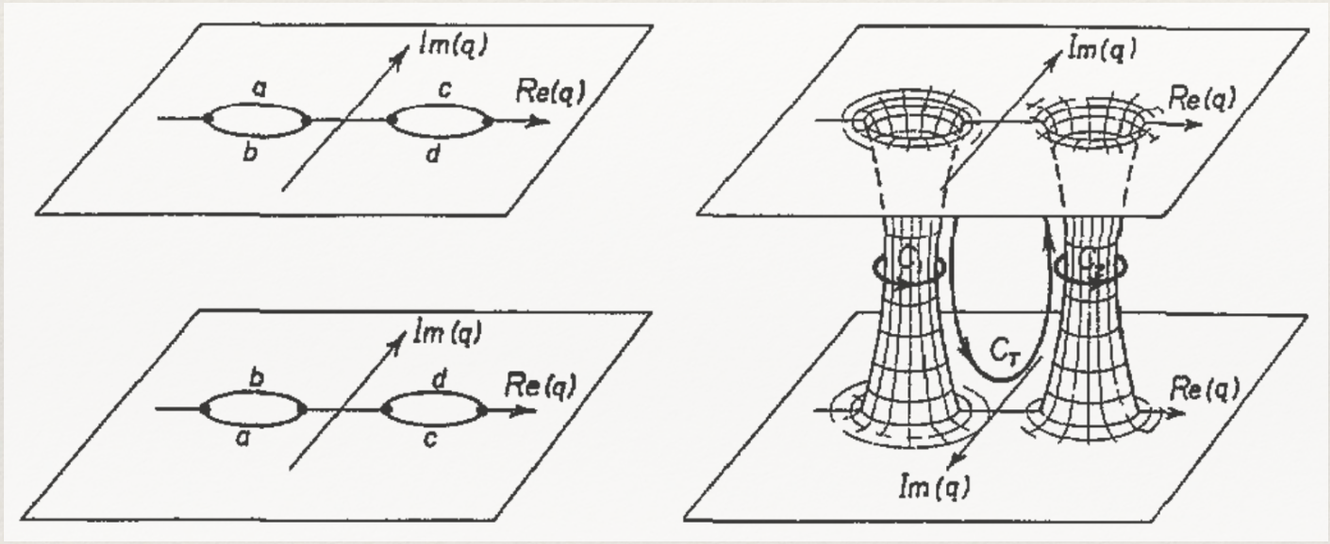
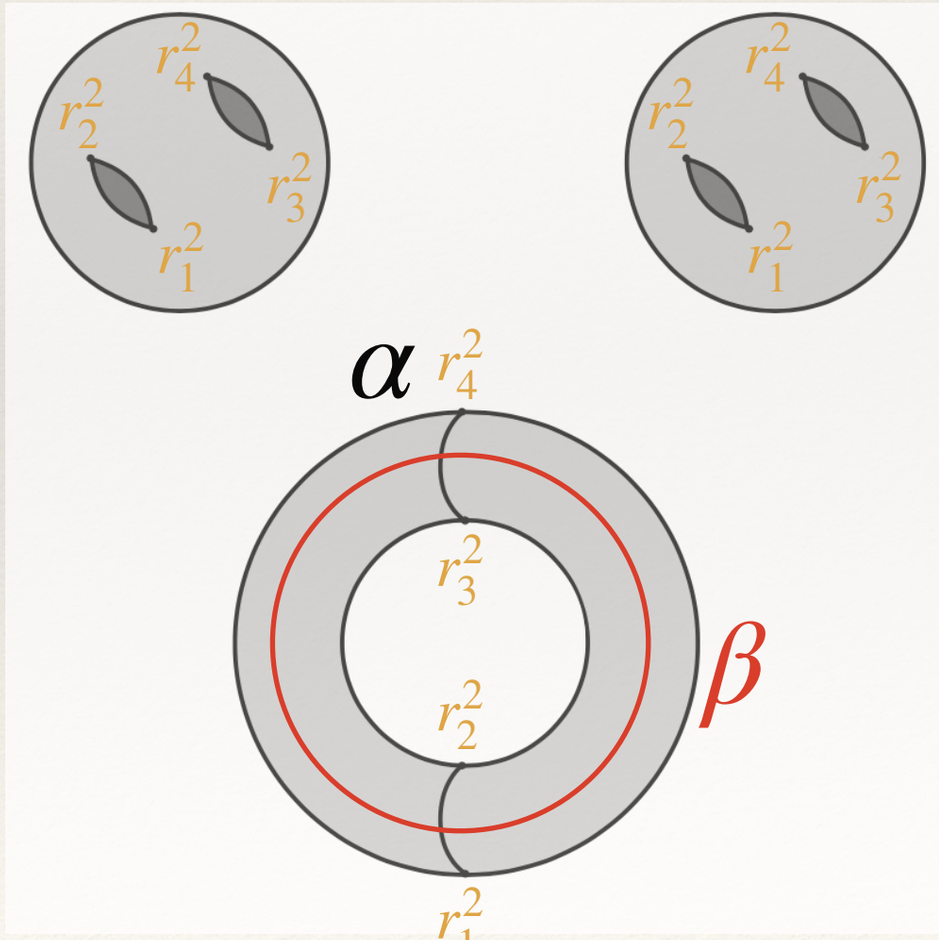


Bohr-Somm: $a = \oint_\alpha pdq = \left(n + \frac{1}{2}\right) 2\pi\hbar$

Tunneling-splitting: $a^D = \oint_\beta pdq$

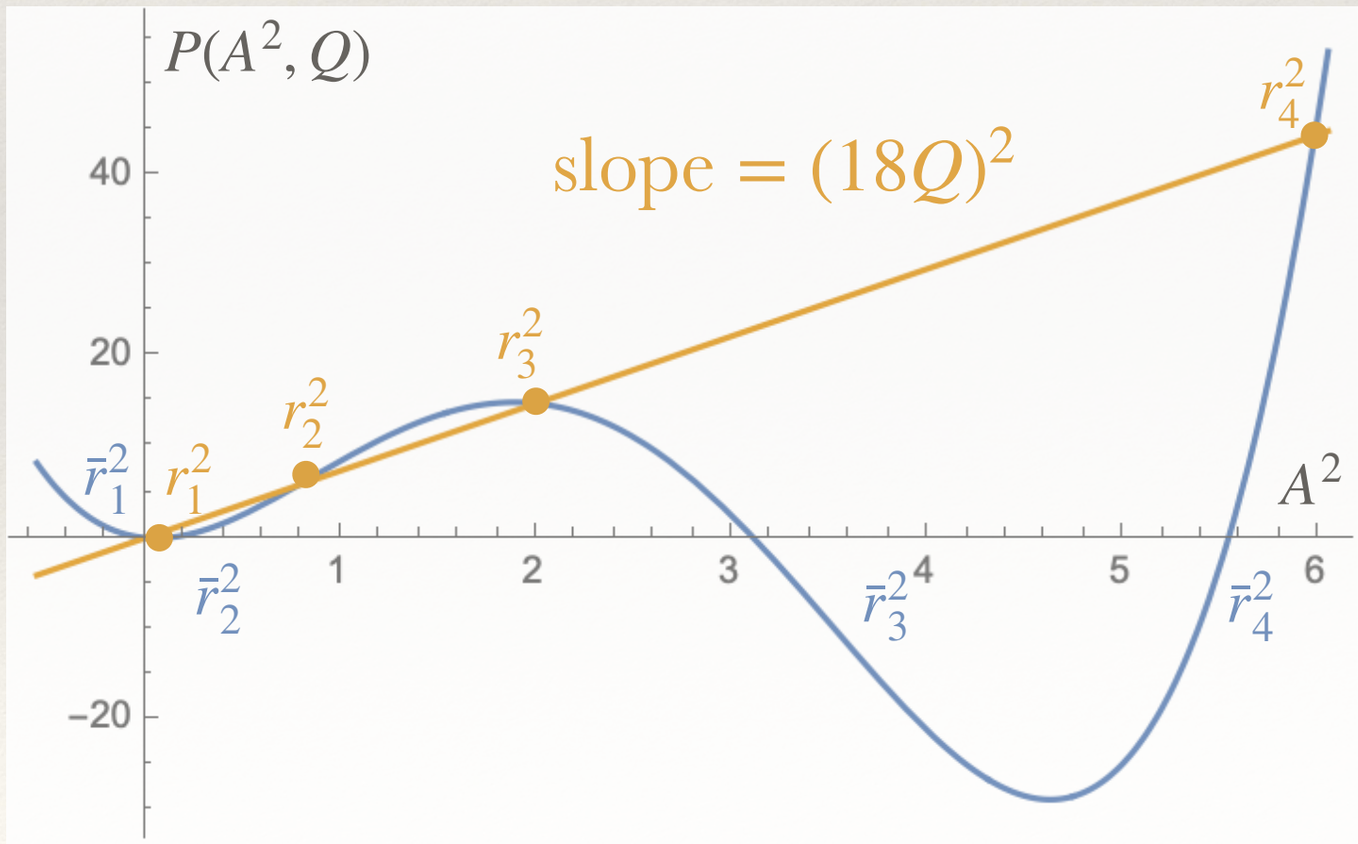


Antu and I have found an explicit 3rd order Picard-Fuchs equation for the volume evolution

$$D(r_i, Q)\frac{d^3a}{dQ^3} + c_2(r_i, Q)\frac{d^2a}{dQ^2} + c_1(r_i, Q)\frac{da}{dQ} = 0.$$

Here $D(r_i, Q)$ is the quartic discriminant, an 8th order polynomial in Q that vanishes iff r_i coalesce and c_1 and c_2 are respectively of 10th and 12th order in Q.

The evident constant solution has a nice explanation: it is due to punctures in the Riemann surface.



The Picard-Fuchs coefficients are below: here $Q_b = 18Q$, $s_0 = r_1r_2r_3r_4$, … , $s_3 = r_1 + r_2 + r_3 + r_4$ are symm root functions

```
D = (4 (Qb^2 - s1) (12 s0 + s2^2) + (3 (Qb^2 - s1)^2 + 32 s0 s2) s3 + (-Qb^2 + s1) s2 s3^2 - 9 s0 s3^3)
   (27 Qb^8 - 256 s0^3 - 108 Qb^6 s1 + 162 Qb^4 s1^2 - 108 Qb^2 s1^3 + 27 s1^4 - 144 Qb^4 s0 s2 + 288 Qb^2 s0 s1 s2 - 144 s0 s1^2 s2 + 128 s0^2 s2^2 + 4 Qb^4 s2^3 -
     8 Qb^2 s1 s2^3 + 4 s1^2 s2^3 - 16 s0 s2^4 + 2 (Qb^2 - s1) (9 (Qb^2 - s1)^2 s2 - 8 s0 (12 s0 + 5 s2^2)) s3 +
     (6 s0 (Qb^2 - s1)^2 - 144 s0^2 s2 - (Qb^2 - s1)^2 s2^2 + 4 s0 s2^3) s3^2 - 2 (Qb^2 - s1) (2 (Qb^2 - s1)^2 - 9 s0 s2) s3^3 + 27 s0^2 s3^4)

c2 =
 (18 (243 Qb^12 s3 + 3 Qb^10 (180 (12 s0 + s2^2) - 378 s1 s3 - 27 s2 s3^2 - 4 s3^4) +
      3 Qb^8 (-684 s1 (12 s0 + s2^2) + (675 s1^2 + 3024 s0 s2 + 68 s2^3) s3 + 117 s1 s2 s3^2 - 3 (255 s0 + 11 s2^2) s3^3 + 12 s1 s3^4 + 4 s2 s3^5) +
      Qb^6 (-8 (12 s0 + s2^2) (-351 s1^2 + 72 s0 s2 - 2 s2^3) - 12 s1 (135 s1^2 + 2160 s0 s2 + 44 s2^3) s3 + 2 (1008 s0^2 - 297 s1^2 s2 + 1860 s0 s2^2 - 4 s2^4) s3^2 +
        4 s1 (1617 s0 + 65 s2^2) s3^3 + (-24 s1^2 - 1618 s0 s2 + s2^3) s3^4 - 32 s1 s2 s3^5 + 180 s0 s3^6) +
      Qb^4 (-8 s1 (12 s0 + s2^2) (189 s1^2 - 72 s0 s2 + 2 s2^3) + (405 s1^4 + 8 (1632 s0^3 + 2916 s0 s1^2 s2 - 1392 s0^2 s2^2 + 45 s1^2 s2^3 + 118 s0 s2^4)) s3 +
        2 s1 (243 s1^2 s2 + 4 (-540 s0^2 - 873 s0 s2^2 + s2^4)) s3^2 - 2 (2817 s0 s1^2 - 2784 s0^2 s2 + 93 s1^2 s2^2 + 208 s0 s2^3) s3^3 -
        s1 (24 s1^2 - 2994 s0 s2 + s2^3) s3^4 + 3 (8 s1^2 s2 + 3 s0 (-63 s0 + 5 s2^2)) s3^5 - 324 s0 s1 s3^6) -
      (s0 (48 s1 - 32 s2 s3 + 9 s3^3) - s1 (3 s1 s3 + s2 (-4 s2 + s3^2))))
   (256 s0^3 - s0^2 (128 s2^2 + 192 s1 s3 - 144 s2 s3^2 + 27 s3^4) + 2 s0 (s1^2 (72 s2 - 3 s3^2) + 2 s2^3 (4 s2 - s3^2) + s1 s2 s3 (-40 s2 + 9 s3^2)) +
     s1^2 (-27 s1^2 + s2^2 (-4 s2 + s3^2) + 2 s1 (9 s2 s3 - 2 s3^3))) +
   Qb^2 (36 864 s0^4 - 48 s0^3 (320 s2^2 + 480 s1 s3 - 288 s2 s3^2 + 45 s3^4) +
     s1^2 (162 s1^3 s3 + 12 s1 s2^2 s3 (4 s2 - s3^2) - s2^3 (-4 s2 + s3^2)^2 + 9 s1^2 (12 s2^2 - 21 s2 s3^2 + 4 s3^4)) +
     s0^2 (864 s1^2 (8 s2 + 3 s3^2) + 18 s1 (128 s2^2 s3 - 192 s2 s3^3 + 33 s3^5) + s2 (768 s2^3 - 1024 s2^2 s3^2 + 540 s2 s3^4 - 81 s3^6)) +
     2 s0 (648 s1^4 + 6 s2^4 (-4 s2 + s3^2)^2 + s1^3 (-2592 s2 s3 + 522 s3^3) + s1 s2^2 s3 (-592 s2^2 + 256 s2 s3^2 - 27 s3^4) +
       3 s1^2 (64 s2^3 + 468 s2^2 s3^2 - 189 s2 s3^4 + 18 s3^6)))))

c1 =
 (972 Qb^2 (80 (Qb^2 - s1)^3 (12 s0 + s2^2) + 8 (128 s0^3 + 3 (Qb^2 - s1)^4 + 168 s0 (Qb^2 - s1)^2 s2 - 64 s0^2 s2^2 + 2 (Qb^2 - s1)^2 s2^3 + 8 s0 s2^4) s3 -
      8 (Qb^2 - s1) (-24 s0^2 + 3 (Qb^2 - s1)^2 s2 - 50 s0 s2^2) s3^2 - 8 (42 s0 (Qb^2 - s1)^2 - 32 s0^2 s2 + (Qb^2 - s1)^2 s2^2 + 4 s0 s2^3) s3^3 +
      (Qb^2 - s1) ((Qb^2 - s1)^2 - 184 s0 s2) s3^4 + ((Qb^2 - s1)^2 s2 + 4 s0 (-6 s0 + s2^2)) s3^5 + 21 s0 (Qb^2 - s1) s3^6)) /
  ((4 (Qb^2 - s1) (12 s0 + s2^2) + (3 (Qb^2 - s1)^2 + 32 s0 s2) s3 + (-Qb^2 + s1) s2 s3^2 - 9 s0 s3^3)
    (27 Qb^8 - 256 s0^3 - 108 Qb^6 s1 + 162 Qb^4 s1^2 - 108 Qb^2 s1^3 + 27 s1^4 - 144 Qb^4 s0 s2 + 288 Qb^2 s0 s1 s2 - 144 s0 s1^2 s2 + 128 s0^2 s2^2 +
     4 Qb^4 s2^3 - 8 Qb^2 s1 s2^3 + 4 s1^2 s2^3 - 16 s0 s2^4 + 2 (Qb^2 - s1) (9 (Qb^2 - s1)^2 s2 - 8 s0 (12 s0 + 5 s2^2)) s3 +
     (6 s0 (Qb^2 - s1)^2 - 144 s0^2 s2 - (Qb^2 - s1)^2 s2^2 + 4 s0 s2^3) s3^2 - 2 (Qb^2 - s1) (2 (Qb^2 - s1)^2 - 9 s0 s2) s3^3 + 27 s0^2 s3^4))
```