

Analytical Studies of Power Spectra and Non-Gaussianity of Loop Quantum Cosmological perturbations

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Content

- Introduction
- UAA Method
- Applications to LQC
- Conclusions

1. Introduction

- The equations of the mode functions of cosmological (scalar, vector and tensor) perturbations in LQC, including the modified ones (mLQCs), can be always cast in the form

$$\frac{d^2 \mu_k(\eta)}{d\eta^2} + \Omega^2(k, \eta) \mu_k(\eta) = 0,$$

$\Omega^2(k, \eta)$: depending on the models and approaches [A. Ashtekar, P. Singh (2011); B.-F. Li, P.Singh, AW (2021)].

- In fact, all the second-order linear homogeneous ODE's can be cast in the above form, including the Schrodinger equation, gravitational waves and quasi-normal modes of black holes.
- To solve the above equation, one normally uses either numerical computations or analytical approximations, such as the JWKB approximation.

1. Introduction (Cont.)

- Numerical computations are usually very expensive, and often high-performance computational resources are needed [I. Agullo, A. Morris (2015)]. In addition, it is also difficult to explore the relevant physics over the whole parameter space of the model.
- JWKB method has been extensively studied, and widely applied to various areas of physics, including cosmology [S. Winitzki (2005)], provided the adiabatic condition

$$\left| \frac{3\Omega'^2}{4\Omega^4} - \frac{\Omega''}{2\Omega^3} \right| \ll 1$$

is satisfied.

- In LQC/mLQCs, this condition is often violated.

1. Introduction (Cont.)

➤ For example, in the hybrid and dressed metric approaches:

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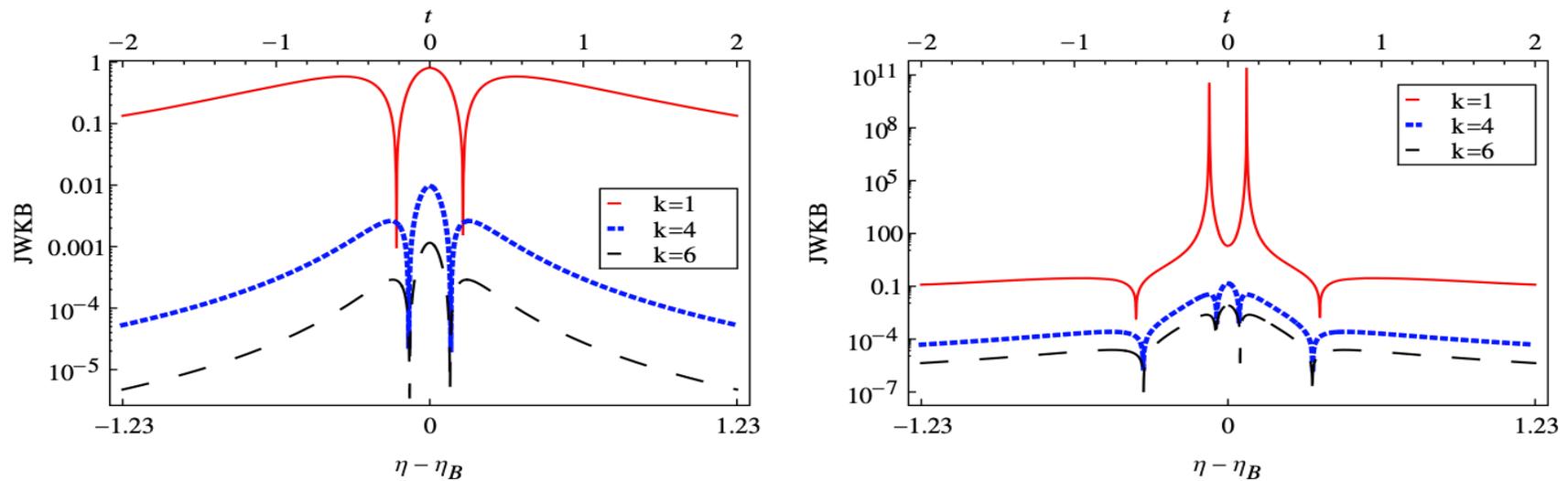


FIG. 3. The JWKB criterion is violated near the time of the bounce at $t = 0$. The left panel shows the result for the hybrid approach and the right panel shows the result for the dressed metric approach. Note that we used units where $m_{\text{Pl}} = 1$ and set $a_B = 1$ in these figures.

1. Introduction (Cont.)

- In the last couple of years, we have applied the *uniform asymptotic approximation* (UAA) method [F.W.J. Olver (1997)] to LQC/mLQCs, and obtained analytically the mode functions of cosmological perturbations, which allows us to explore various properties of the perturbations over the whole phase space of parameters involved in the models.

2. The Uniform Asymptotic Approximation (UAA) Method

- Let us rewrite the differential equation as,

$$\frac{d^2 \mu_k(y)}{dy^2} = \left[\lambda^2 g(y) + q(y) \right] \mu_k(y) \quad (2.1)$$

$g(y)$, $q(y)$: two unspecified functions

λ : a large positive dimensionless parameter and serves as a bookmark

- Then, we expand it as,

$$\mu_k(y) = \sum_{n=0}^{\infty} \frac{\mu_k^{(n)}(y)}{\lambda^n}.$$

2. The UAA Method (Cont.)

- The reason to introduce two functions, $g(y)$ and $q(y)$, is to use this extra degree of freedom to minimize the errors, by writing them in the form

$$g = g(y, a_n), \quad q = g(y, b_n)$$

a_n, b_n : a set of parameters, which will be chosen to minimize the errors.

2.1 The Liouville Transformation

- When $\lambda^2 g(y) + q(y) \simeq \text{Constant}$, Eq.(2.1) has the well-approximated solution,

$$\mu_k(y) \simeq a_k e^{\sqrt{\lambda^2 g(y) + q(y)}} + b_k e^{-\sqrt{\lambda^2 g(y) + q(y)}},$$

a_k, b_k : Constant

- However, in general the above solution is too crude. Motivated by such considerations, we may try some kind of transformations that lead the new function, say, $U(\xi)$ in terms of a new variable, ξ to satisfy the above condition. This leads us to **the well-known Liouville transformation**:

$$U(\xi) \equiv \dot{y}^{-1/2} \mu_k, \quad \dot{y} \equiv \frac{dy}{d\xi} > 0. \quad (2.2)$$

2.1 The Liouville Transformation (Cont.)

➤ Then, Eq.(2.1) takes the form,

$$\frac{d^2 U(\xi)}{d\xi^2} = [\lambda^2 (\dot{y}^2 g) + \psi(\xi)] U(\xi), \quad (2.3)$$

where

$$\psi(\xi) \equiv \dot{y}^2 q + \dot{y}^{1/2} \frac{d^2}{d\xi^2} (\dot{y}^{-1/2}) = \dot{y}^2 q - \dot{y}^{3/2} \frac{d^2}{dy^2} (\dot{y}^{1/2}) \equiv \psi(y). \quad (2.4)$$

➤ If $\left| \frac{\psi(\xi)}{\lambda^2 \dot{y}^2 g} \right| \ll 1$ we can ignore the $\psi(\xi)$ term in Eq.(2.3) to find the first-order approximate solution.

2.1 The Liouville Transformation (Cont.)

- To characterize the errors, we introduce *the error control function*,

$$F(\xi) \equiv \int \frac{\psi(\xi)}{|\dot{y}^2 g|^{1/2}} d\xi$$

and the quantity,

$$\mathcal{V}_{\xi_1, \xi_2} \equiv \int_{\xi_1}^{\xi_2} \frac{|\psi(\xi)|}{|\dot{y}^2 g|^{1/2}} d\xi$$

2.1 The Liouville Transformation (Cont.)

➤ Our goal now is to choose $y^2 g(\xi)$ properly, so that:

✓ the resulting equation,

$$\frac{d^2 U(\xi)}{d\xi^2} = \lambda^2 y^2 g U(\xi), \quad (\psi(\xi) = 0)$$

can be solved explicitly (possibly in terms of known special functions);

✓ the first-order approximation can be as close to the exact solution as possible, which can be fulfilled by minimizing the error control function

$$\frac{\partial F(\xi, a_n, b_n)}{\partial a_n} = 0, \quad \frac{\partial F(\xi, a_n, b_n)}{\partial b_n} = 0 \quad \Leftrightarrow \quad g = g(\xi, a_n), \quad q = q(\xi, b_n),$$

2.1 The Liouville Transformation (Cont.)

✓ We also require,

$$\mathcal{V}_{\alpha,\beta}(F) < \infty, \quad \forall \alpha, \beta \in (\alpha_1, \alpha_2),$$

where (α_1, α_2) is the interval we are interested in, which (one-to-one) corresponds to (a_1, a_2) of y , and can be finite or infinite.

- The choice of $\dot{y}^2 g(\xi)$ sensitively depends on the properties of the functions $g(y)$ and $q(y)$ near their singularities and zeros. These singularities and zeros are often referred to as the *poles* and *turning points*.
- In particular, depending on the *number and nature of the turning points*, the choices of $\dot{y}^2 g(\xi)$ will be different.
- In the following, we shall consider only the cases with **zero**, **one** and **two** turning points.

2.2 Approximate Solutions

- For these cases we choose $y^2 g(\xi)$ as [F.W.J. Olver, 1956; 1975; 1997]

$$y^2 g = \begin{cases} \operatorname{sgn}(g), & \text{zero turning point,} \\ \xi, & \text{one turning point,} \\ \operatorname{sgn}(g) (\xi_0^2 - \xi^2), & \text{two turning points,} \end{cases} \quad \operatorname{sgn}(g) = \begin{cases} +1, & g > 0, \\ -1, & g < 0. \end{cases}$$

- For the cases with more than two turning points, see, for example, J.-L. Zhang (1991).

2.2 Approximate Solutions (Cont.)

- Then, the first-order solutions of the equation,

$$\frac{d^2 U(\xi)}{d\xi^2} = \lambda^2 y^2 g U(\xi), \quad (\psi(\xi) = 0)$$

are given by [Zhu, AW, Cleaver, Kirstein, and Sheng, 2014, 2016],

$$U(\xi) = \begin{cases} a_+ e^{\lambda \sqrt{\text{sgn}(g)} \xi} + a_- e^{-\lambda \sqrt{\text{sgn}(g)} \xi}, & \text{zero turning point} \\ a_+ \text{Ai}(\lambda^{2/3} \xi) + a_- \text{Bi}(\lambda^{2/3} \xi), & \text{one turning point} \\ a_+ W\left(\frac{1}{2} \xi^2, \sqrt{2\lambda} \xi\right) + a_- W\left(\frac{1}{2} \xi^2, -\sqrt{2\lambda} \xi\right), & \text{two turning points} \end{cases}$$

a_{\pm} : constants;

Ai , Bi : Airy functions;

W : modified parabolic cylindrical function

2.2 Approximate Solutions (Cont.)

➤ where

$$\xi_0^2 = \pm \frac{2}{\pi} \left| \int_{y_1}^{y_2} \sqrt{g(y)} dy \right|$$

“+”: $y_{1,2}$ are real; “-”: $y_{1,2}$ are complex
 $y_{1,2}$: the two turning points (roots) of $y(y) = 0$.

- High order solutions can be obtained by recursion relations [Zhu, AW, Cleaver, Kirstein, and Sheng, 2014, 2016].
- For example, for the one-turning point case, we have

2.2 Approximate Solutions (Cont.)

$$U(\xi) = \alpha_k \left[\text{Ai}(\lambda^{2/3}\xi) \sum_{s=0}^n \frac{A_s(\xi)}{\lambda^{2s}} + \frac{\text{Ai}'(\lambda^{2/3}\xi)}{\lambda^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{\lambda^{2s}} + \epsilon_3^{(2n+1)} \right] \\ + \beta_k \left[\text{Bi}(\lambda^{2/3}\xi) \sum_{s=0}^n \frac{A_s(\xi)}{\lambda^{2s}} + \frac{\text{Bi}'(\lambda^{2/3}\xi)}{\lambda^{4/3}} \sum_{s=0}^{n-1} \frac{B_s(\xi)}{\lambda^{2s}} + \epsilon_4^{(2n+1)} \right],$$

$\epsilon_{(3,4)}^{(2n+1)}$: errors, which is related to the associated error control function ψ_{ξ_1, ξ_2} , and

$$A_{s+1}(\xi) = -\frac{1}{2}B'_s(\xi) + \frac{1}{2} \int \psi(v)B_s(v)dv,$$

$$B_s = \begin{cases} \frac{1}{2\xi^{1/2}} \int_0^\xi \{\psi(v)A_s(v) - A_s''(v)\} \frac{dv}{v^{1/2}}, & \xi > 0, \\ \frac{1}{2(-\xi)^{1/2}} \int_\xi^0 \{\psi(v)A_s(v) - A_s''(v)\} \frac{dv}{(-v)^{1/2}}, & \xi < 0, \end{cases}$$

$$A_0(\lambda, \xi) = 1$$

3. Applications to LQC

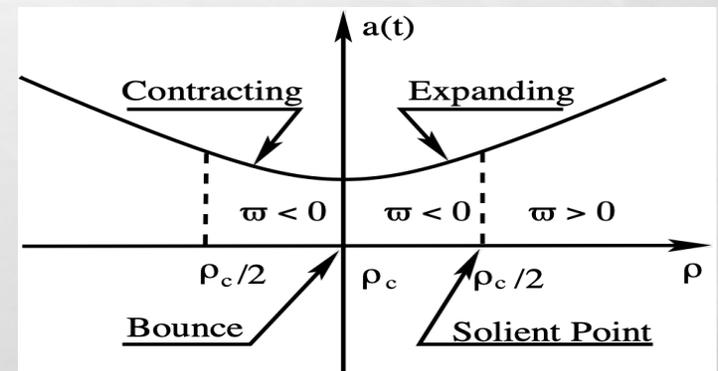
- Power Spectra in Deformed Algebra Approach [M. Bojowald, et al., 2008; T. Cailleteau et al., 2012; A. Barrau, et al., 2015]:

✓ Scalar:

$$\Omega_S^2(\eta) = \omega k^2 - \frac{z_S''(\eta)}{z_S}, \quad \omega \equiv 1 - \frac{2\rho}{\rho_c}, \quad z_S \equiv a \frac{\dot{\phi}(t)}{H}, \quad d\eta = \frac{dt}{a(t)}$$

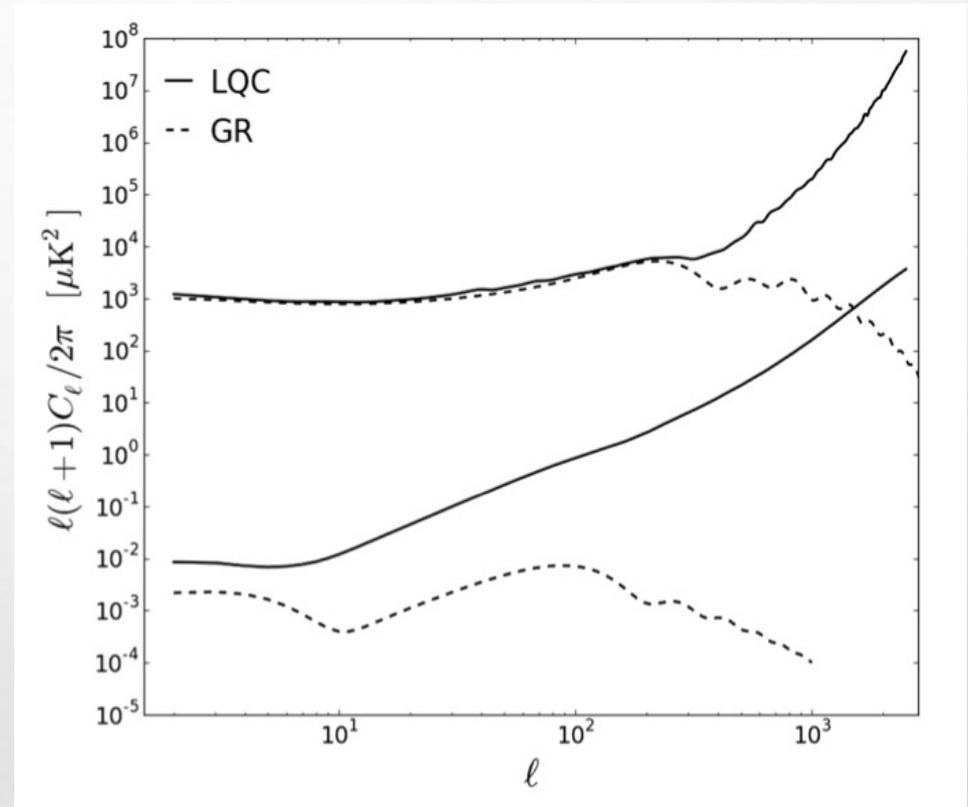
✓ Tensor:

$$\Omega_T^2(\eta) = \omega k^2 - \frac{z_T''(\eta)}{z_T}, \quad z_T \equiv \frac{a}{\sqrt{\omega}}$$



3. Applications to LQC (Cont.)

- Imposing *the Minkowski vacuum initial conditions at remote past of the quantum bounce*, it was found that the power spectra of both scalar and tensor perturbations are **inconsistent with observations** [B. Bolliet et al, PRD93 (2016) 124011].

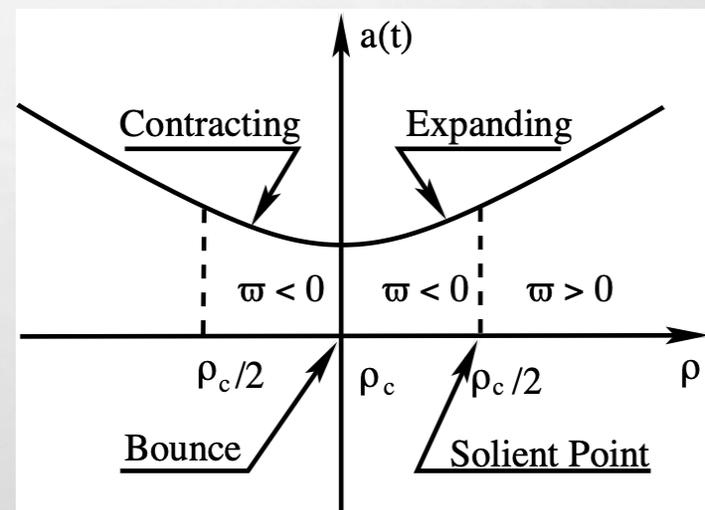


B. Bolliet et al., PRD93 (2016) 124011

3. Applications to LQC (Cont.)

➤ We revisited this problem by asking [B.-F. Li, T. Zhu, AW, K. Kirsten, G. Cleaver, Q. Sheng, PRD99 (2019) 103536]:

- Do exist initial moment t_i and conditions for which the resultant power spectra are consistent with observations?
- The answer is positive:
 - ✓ The initial moment is chosen at the silent point
 - ✓ After obtaining the analytical solutions by using UAA method, we are able to identify **uniquely** the initial conditions that are consistent with observations.



3. Applications to LQC (Cont.)

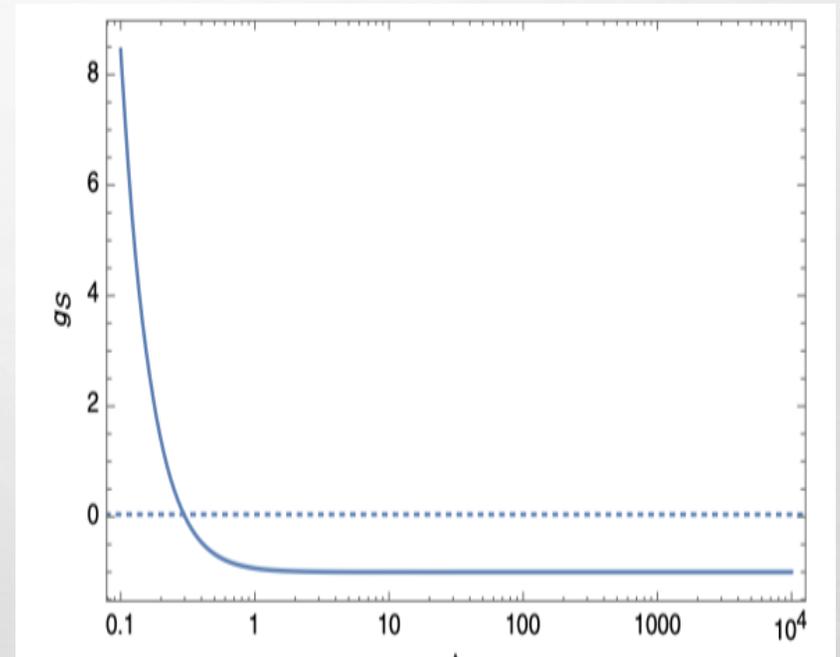
- To make the error control function

$$F(\xi) \equiv \int \frac{\psi(\xi)}{|y^2 g|^{1/2}} d\xi$$

be finite, we must choose

$$q(t) = -\frac{1}{4k^2 t^2}$$

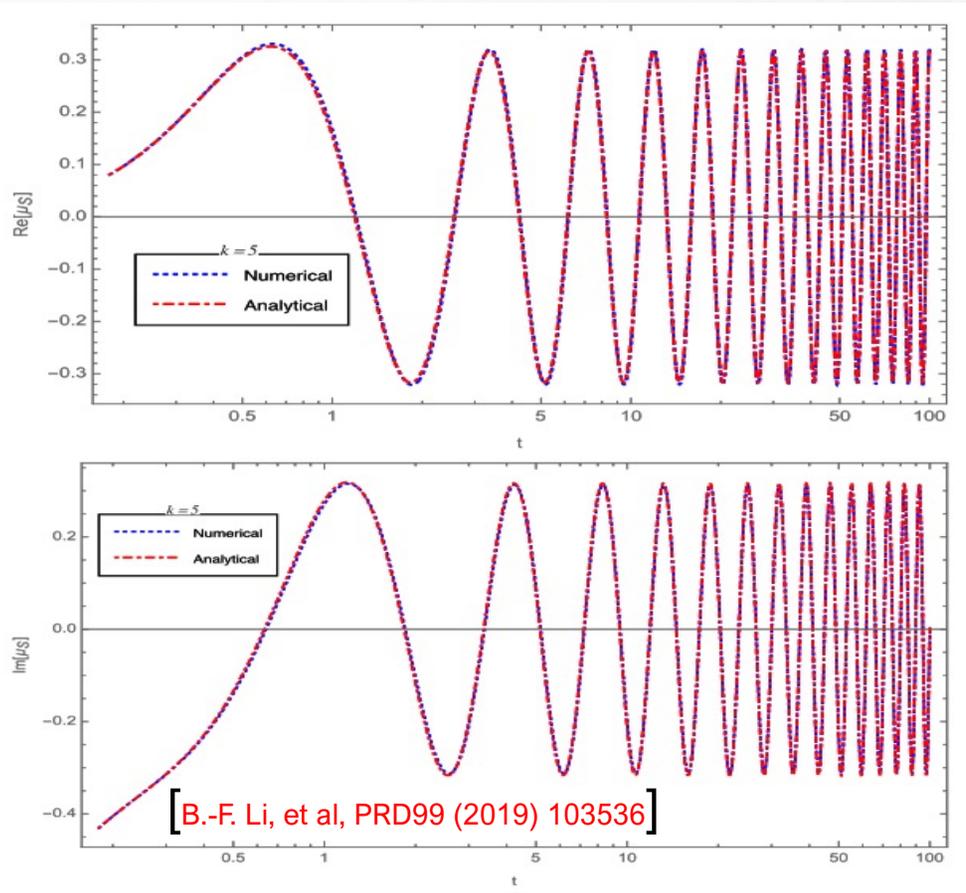
Then, $g(t)$ has only one turning point. So, to the first-order approximation, it is the linear combination of the Airy functions.



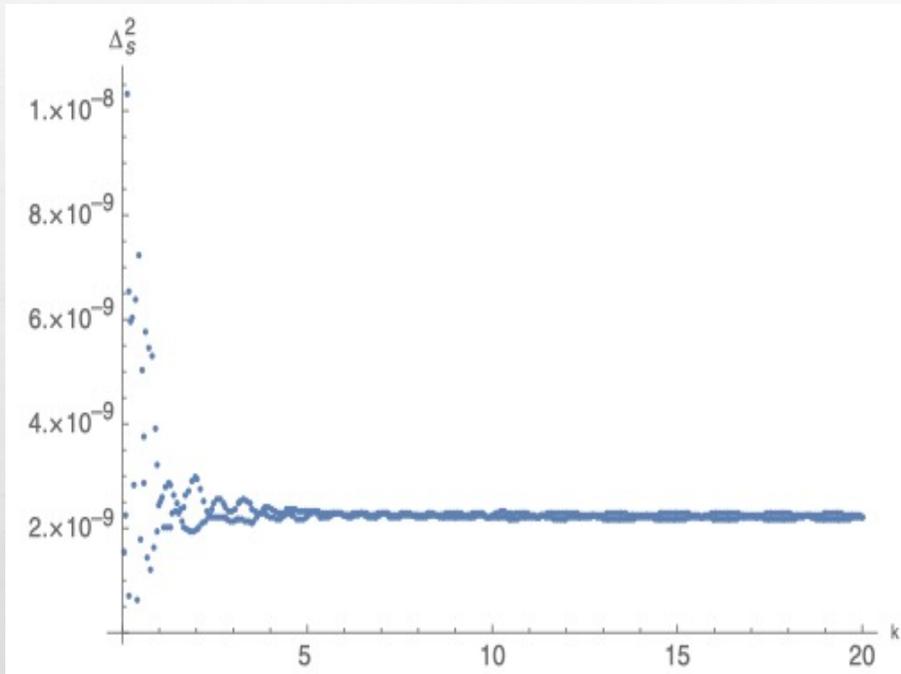
3. Applications to LQC (Cont.)

- From the figure, it can be seen that even to the first-order, the numerical (exact) solution can be described well by the analytical approximate solution.
- The consistent initial conditions are

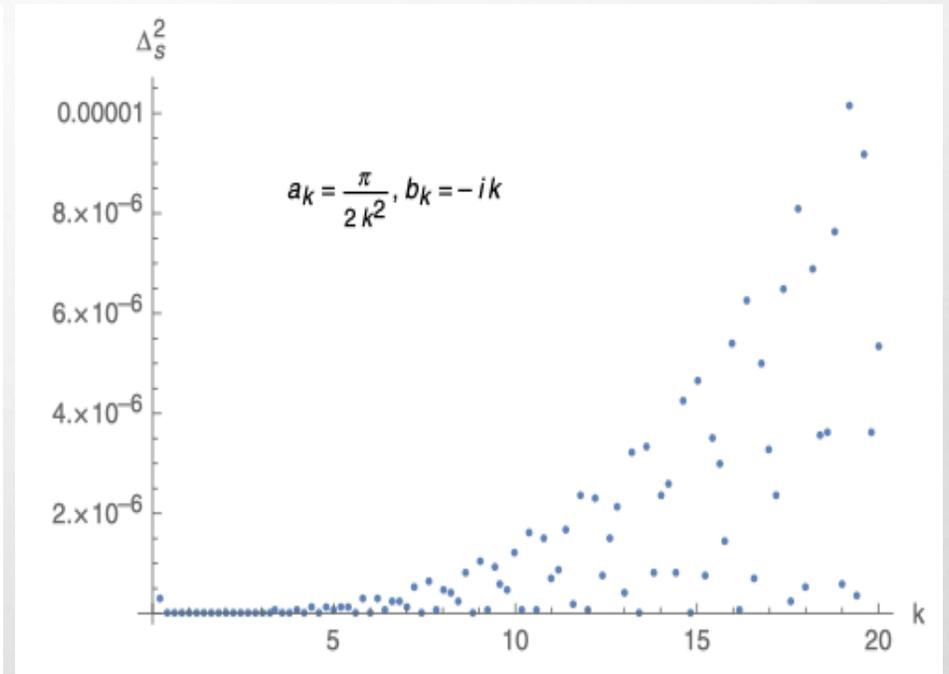
$$a_k = \sqrt{\frac{\pi}{2k}}, \quad b_k = -i\sqrt{\frac{\pi}{2k}}$$



3. Applications to LQC (Cont.)



[Consistent with Observations]



[Similar what was obtained previously]

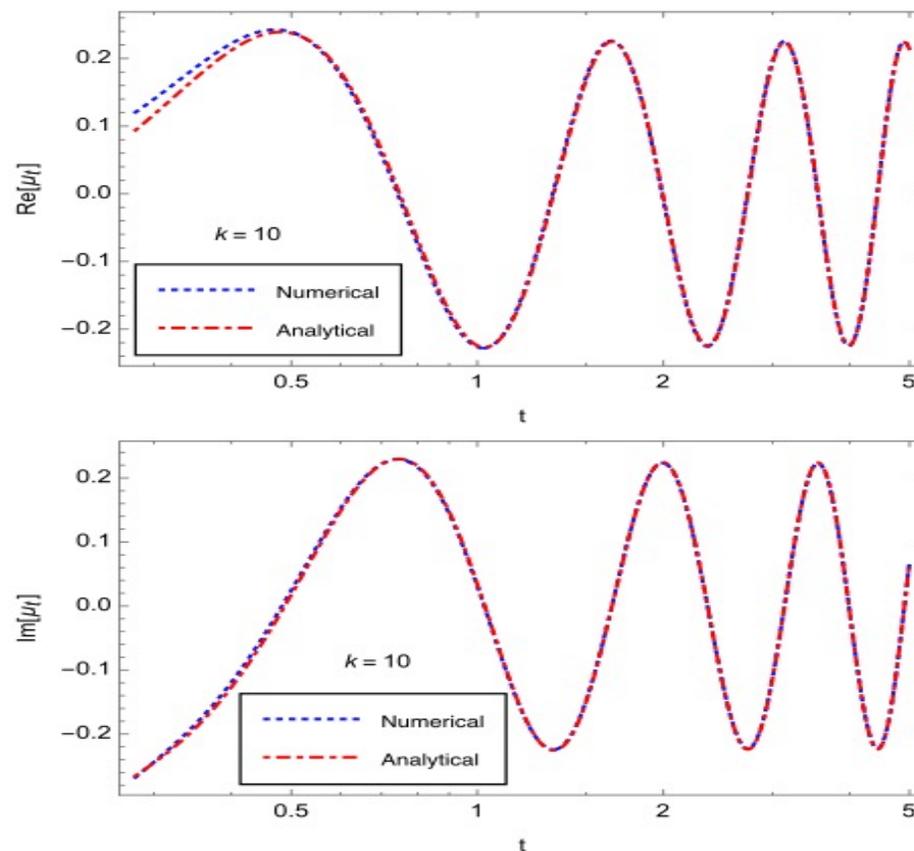
3. Applications to LQC (Cont.)

- For tensor perturbations, a pole appears at the silent point, and to minimize the errors, now $q(t)$ must be chosen (uniquely) as

$$q_T(y) = -\frac{2^{1/3}\gamma_B}{k^2(1-\gamma_B t^2)^2}$$

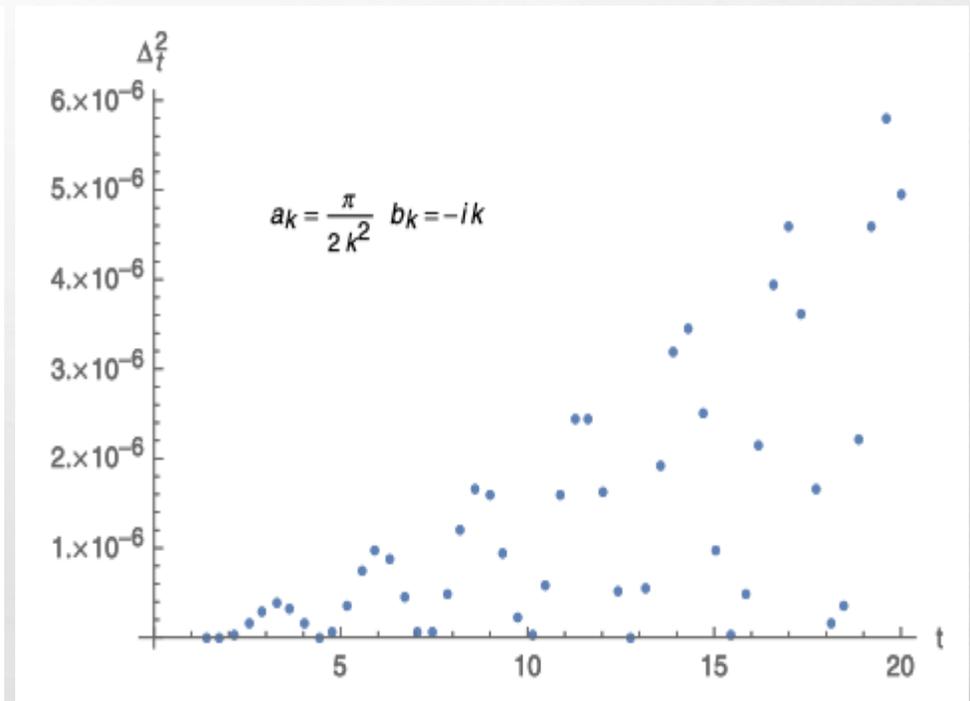
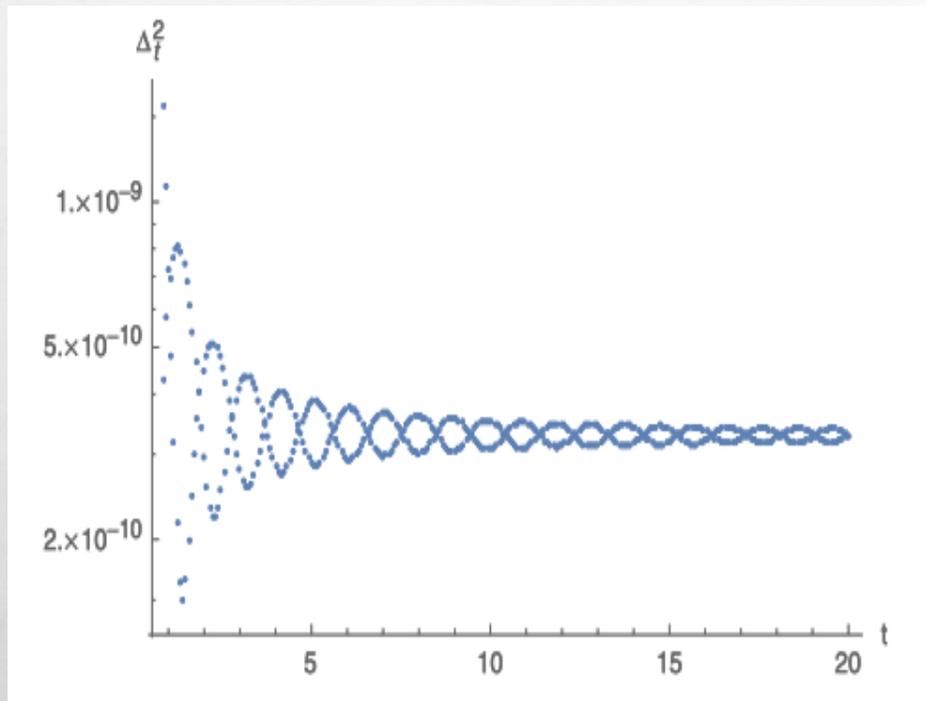
- With the same initial conditions,

$$a_k = \sqrt{\frac{\pi}{2k}}, \quad b_k = -i\sqrt{\frac{\pi}{2k}},$$



3. Applications to LQC (Cont.)

an observationally consistent power spectrum can be obtained:



3. Applications to LQC (Cont.)

- It should be noted that the above conclusions can be obtained only after [the general analytical solutions are known](#).
- The solutions are only up to the first-order of the UAA method. It can be easily generalised to high-orders [[T. Zhu, et al., 2016](#)].
- The physics of the initial conditions identified above at the silent point is unclear (if there is any).

3. Applications to LQC (Cont.)

- Now, we are ready to apply the UAA method to the dressed metric and hybrid approaches.
- We shall pay particular attention to:
 - ✓ the initial conditions, which are not only consistent with observations but also solve some anomalies of CMB [I. Agullo, J. Olmedo, V. Sreenath, 2020; A. Ashtekar, B. Gupta, V. Sreenath, 2021];
 - ✓ the non-Gaussianities, including the region near the quantum bounce [I. Agullo, B. Bolliet, V. Sreenath, 2018];
 - ✓ particle creations in the contracting phase [I. Agullo, A. Ashtekar, W. Nelson, 2013]
 - ✓ ...

4. Conclusions

- We have developed the UAA method systematically, and successfully applied it to various problems in several fields of physics, including:
 - ✓ the accurate calculations of power spectra of cosmological perturbations when quantum effects are taken into account, which were done mainly numerically previously;
 - ✓ gravitational waveforms in parity-violating theories of gravity;
 - ✓ QNMs of black holes.
- We expect that such analytical analysis will provide deeper and thorough understanding of the physics involved.



4. Conclusions (Cont.)

- One advantage of the UAA method is to allow us to estimate the upper bound of errors, and more important allow us to minimize the errors by,

$$\frac{\partial F(\xi, a_n, b_n)}{\partial a_n} = 0, \quad \frac{\partial F(\xi, a_n, b_n)}{\partial b_n} = 0 \quad \Leftrightarrow g = g(\xi, a_n), \quad q = q(\xi, b_n),$$

- provided that

$$\mathcal{V}_{\alpha, \beta}(F) < \infty, \quad \forall \alpha, \beta \in (\alpha_1, \alpha_2)$$

- Now, we are ready to apply it to the dressed metric and hybrid approaches in LQC and mLQCs [B.-F. Li, P. Singh, AW, 2021].



Thank you !!!



Any question?

