LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

ARNOLD SOMMERFELD CENTER FOR THEORETICAL PHYSICS

Phase transitions in Group field theory:

Towards the phase structure of the complete Lorentzian Barrett-Crane model

Andreas Pithis (LMU, ASC, MCQST)
in collaboration with
D. Oriti, L. Marchetti and J. Thürigen

mostly based on arXiv:1904.00598 (Phys. Rev. D 98, 126006 (2018)), 2110.15336 (J. High Energ. Phys. 2021, 201 (2021)) & wip

MCQST

July 18, 2022

Loops 2022 at Ecole Normale Superieure de Lyon (ENS Lyon)

Funded by
DFG Deutsche Forschungsgemeinschaft
German Research Foundation

Outline

- General motivation: Landau-Ginzburg mean-field method
- LG theory applied to Group field theory
- Conclusions

What is LG theory? How does it relate to the RG?

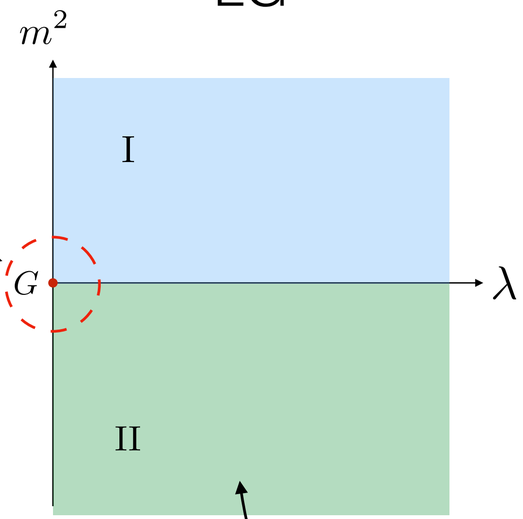
- statistical field theory method to describe **1st and 2nd order phase transitions**

- LG mean-field analysis **clarifies phase structure** of local field theories (coarse account)

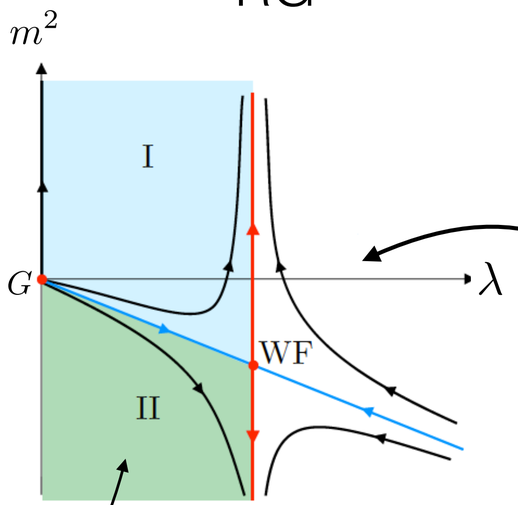
- transition to condensate phase with **nontrivial VEV** (non-perturbative vacuum) $\langle\varphi\rangle \neq 0$

e.g. $$S[\varphi] = \frac{1}{2}\int \mathrm{d}^3x\varphi(x)(-\Delta + m^2)\varphi(x) + \frac{\lambda}{4!}\int \mathrm{d}^3x\varphi(x)^4$$

LG



RG



LG theory studies fluctuations around the Gaussian fixed point (free theory)

involved RG studies go beyond small fluctuations

→ coarse account of the phase diagram

→ detailed account of the phase diagram

$\langle\varphi\rangle \neq 0$

What is LG theory? How does it relate to the RG?

- start with **free energy functional** as an expansion in terms of even + odd powers of the local field (order parameter) and its gradient

- consider **truncation** of this functional assumed to be valid from mesoscale to macroscale

- details on microphysics encoded in couplings and order parameter

- **order parameter** features only **universal properties** of the system
  (dimension of space, symmetries of order parameter)

dimension of underlying space

e.g. $S[\varphi] = \frac{1}{2}\int \mathrm{d}^3x\varphi(x)(-\Delta + m^2)\varphi(x) + \frac{\lambda}{4!}\int \mathrm{d}^3x\varphi(x)^4$

$$Z = \int \mathcal{D}\varphi \mathrm{e}^{-S[\phi]}$$

partition function (all configs)

free energy functional

order parameter

here: global symmetry $\mathbb{Z}_2$

- allows to **control thermodynamic phases** of the system by studying long-range correlations of order parameter fluctuations over the distance (**correlation length**) ξ

- beyond ξ correlations decay exponentially; it diverges at criticality

LG theory more concretely

(work in dimension d)

1) determine **uniform field configurations** which are minimizers of the free energy functional

$$\varphi_0 = 0 \text{ if } m^2 > 0 \text{ and } \varphi_0 = \pm\sqrt{-\frac{m^2}{\lambda/3!}} \text{ if } m^2 < 0$$

2) study **correlations of fluctuations** around this uniform background (aka Gaussian approximation)

2. a) **linearize classical equations of motion** using fluctuations over the background $\varphi(\vec{x}) \rightarrow \varphi_0 + \delta\varphi(\vec{x})$

$$\left(-\Delta + m^2\right)\delta\varphi(\vec{x}) + \frac{\lambda\varphi_0^2}{2}\delta\varphi(\vec{x}) = 0$$

2. b) solve for **correlation function** $\left(-\Delta + m^2 + \frac{\lambda\varphi_0^2}{2}\right) C(\vec{x}) = \delta(\vec{x})$ (go to Fourier representation)

2. c) correlator is exponentially decaying function ⟶ determine **correlation length** $\xi^2 = \frac{1}{-2m^2}, \quad m^2 < 0$

3) determine **domain of validity** by quantifying the strength of the fluctuations:
should remain small as well as the coupling then mean-field theory self-consistent

Ginzburg parameter $Q = \frac{\int_\xi \mathrm{d}^d x C(\vec{x})}{\int_\xi \mathrm{d}^d x \varphi_0^2}$

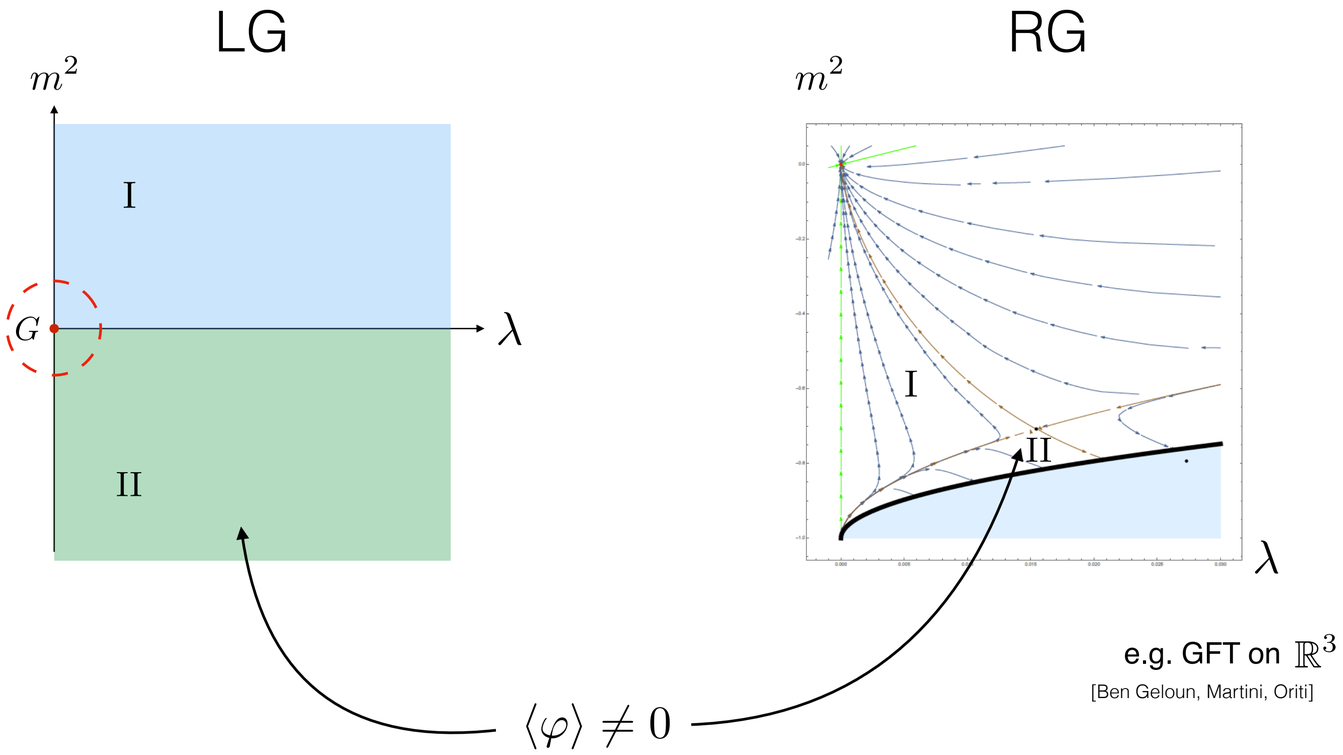
(measures strength of fluctuations; *should be small*)

$$Q \sim \lambda\xi^{4-d} \longrightarrow d_c = 4$$

critical dimension below which MFT seizes to be accurate; accounts for coarse picture of phase diagram (good enough)

Why bother in GFT? Applicable?

- transition to condensate phase in GFT with non-trivial VEV?!



e.g. GFT on $\mathbb{R}^3$
[Ben Geloun, Martini, Oriti]

- problem of the continuum limit in GFT/spin foam models
- mapping phases/phase structure of such models
- to this aim: exploit field theory character of GFTs

- condensate remains **hypothesis** for realistic models (but getting there); **test with LG theory applied to GFT**
- important for group field theory condensate cosmology: condensate phase is important pillar
- upshot: LG MFT applicable in GFT in spite of non-locality of its interactions, gauge invariance and simplicity

Landau-Ginzburg mean-field theory of GFTs

(goal: determine ingredients that allow for condensate phase)

[Thürigen, Pithis; Marchetti, Oriti, Thürigen,Pithis]

local scalar field theory:

LG theory gives coarse picture of phase structure thus sufficient to point to the formation of a condensate phase; fully accurate only above critical dimension

→ **method works also for GFTs** (non-local); and of course also when local dof are added

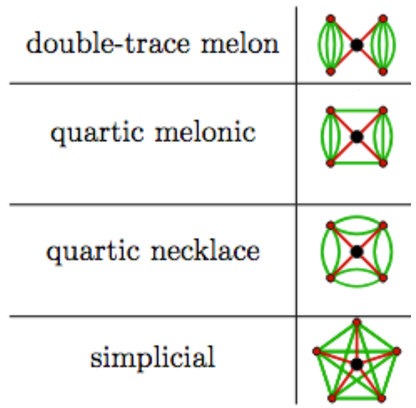
(shown for simplified models on Abelian compact/non-compact group with/out closure constraint with/out additional local dof; **wip** on Lorentz group and simplicity constraints imposed)

mean-field analysis for $\varphi(\boldsymbol{\phi};\boldsymbol{g}) : \mathbb{R}^{d_l} \times G^r \to \mathbb{R}, \quad G = \mathrm{U}(1) \text{ or } \mathbb{R}$

- extract critical dimension via Ginzburg quantity $Q \sim \lambda_\gamma \xi^{4-d_l-(r-s_0)}$

coupling, rank of group field, correlation length, quartic interaction, non-local contribution

specifies combinatorics



- various interactions (power and combinatorics) checked and result generalized

$$Q \sim \lambda_\gamma^{\frac{2}{V_\gamma - 2}} \xi^{\frac{2V_\gamma}{V_\gamma - 2} - d_l - (r - s_0)}$$

- impose closure constraint $r \to r - 1$

More realistic scenario - kinematics

[wip: Marchetti, Oriti, Thürigen, Pithis]

- work within extended formulation developed in the context of the Barrett-Crane model (based on projected SNs)

[Livine, Alexandrov; Baratin, Oriti; Jercher, Oriti, Pithis]

- **add non-dynamical timelike, spacelike or light like normal vectors X to domain of field**

⟶ allows to impose closure and simplicity covariantly and commutatively

→ GFT quantization of first order Palatini gravity

Here:

- **couple also free massless scalar fields (local dof) to lattice**
- **caveat: restrict to spacelike tetrahedra/timelike normals**

$$\Phi(\boldsymbol{\phi}, \boldsymbol{g}, X) = \Phi(\phi_1, ..., \phi_{d_{\text{loc}}}, g_1, ..., g_4, X) : \mathbb{R}^{d_{\text{loc}}} \times \text{SL}(2,\mathbb{C})^4 \times \mathbb{H}^3_+ \to \mathbb{R} \text{ or } \mathbb{C}$$

symmetries:

$$\Phi(\boldsymbol{\phi}, g_1, g_2, g_3, g_4, X) = \Phi(\boldsymbol{\phi}, g_1 u_1, g_2 u_2, g_3 u_3, g_4 u_4, X), \quad \forall u_i \in \text{SU}(2)_X, \qquad \textbf{(simplicity)}$$

$$\Phi(\boldsymbol{\phi}, g_1, g_2, g_3, g_4, X) = \Phi(\boldsymbol{\phi}, g_1 h^{-1}, g_2 h^{-1}, g_3 h^{-1}, g_4 h^{-1}, h \cdot X), \quad \forall h \in \text{SL}(2,\mathbb{C}) \qquad \textbf{(closure)}$$

More realistic scenario - kinematics

- decomposition of the field with the aforementioned symmetries:

$$\Phi(\boldsymbol{\phi}, \boldsymbol{g}, X) = \prod_{i=1}^{4} \left(\int \mathrm{d}\rho_i 4\rho_i^2 \sum_{j_i, m_i} D^{(\rho_i, 0)}_{j_i m_i 00}(g_i X) \right) \Phi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}(\boldsymbol{\phi})$$

$$\Phi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}(\boldsymbol{\phi}) \equiv \Phi^{\rho_1 0 \rho_2 0 \rho_3 0 \rho_4 0}_{j_1 m_1 00 j_2 m_2 00 j_3 m_3 00 j_4 m_4 00}(\boldsymbol{\phi})$$

Wigner matrices of $\mathrm{SL}(2, \mathbb{C})$ in the so-called unitary principal series

- integration over normal to get rid of irrelevant information on embedding

$$\Phi(\boldsymbol{\phi}, \boldsymbol{g}) = \int_{\mathbb{H}^3} \mathrm{d}X \Phi(\boldsymbol{\phi}, \boldsymbol{g}, X)$$

$$= \prod_{i=1}^{4} \left(\int \mathrm{d}\rho_i 4\rho_i^2 \sum_{\substack{j_i, m_i; \\ l_i, n_i}} D^{(\rho_i, 0)}_{j_i m_i l_i n_i}(g_i) \right) B^{\rho_1\rho_2\rho_3\rho_4}_{l_1 n_1 l_2 n_2 l_3 n_3 l_4 n_4} \Phi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}(\boldsymbol{\phi})$$

Barrett-Crane intertwiner $$B^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4} \equiv \int \mathrm{d}X \prod_{i=1}^{4} D^{(\rho_i, 0)}_{j_i m_i 00}(X)$$

- Fourier decomposition wrt local dof

$$\Phi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}(\boldsymbol{\phi}) = \int_{\mathbb{R}^{d_l}} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^{d_l}} \Phi^{\rho_1\rho_2\rho_3\rho_4}_{j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4}(\boldsymbol{k}) \mathrm{e}^{i\boldsymbol{\phi}\cdot\boldsymbol{k}}$$

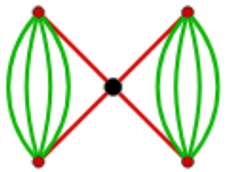
More realistic scenario - dynamics

$$S[\Phi] = S_0[\Phi] + S_{\mathrm{IA}}[\Phi]$$

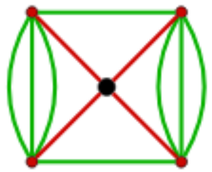
GFT action — kinetic term — interaction

$$S_0[\Phi] = \int_{\mathbb{R}^{d_{\mathrm{loc}}}} \mathrm{d}\boldsymbol{\phi} \int_{\mathrm{SL}(2,\mathbb{C})^4} \mathrm{d}\boldsymbol{g} \int_{\mathbb{H}^3_+} \mathrm{d}X \Phi(\boldsymbol{\phi}, \boldsymbol{g}, X) \left(-\sum_{i=1}^{d_{\mathrm{loc}}} \alpha_i \partial^2_{\phi_i} - \sum_{c=1}^{4} \Delta_c + \mu \right) \Phi(\boldsymbol{\phi}, \boldsymbol{g}, X)$$

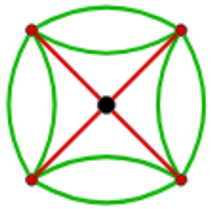
consider interactions of type:



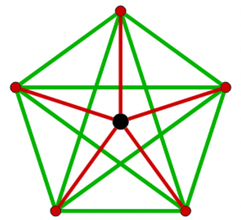
double-trace melon



simple melon



necklace



simplicial

e.g.

$$S_{\mathrm{IA,simplex}}[\Phi] = \frac{\lambda}{5!} \int_{\mathbb{R}^{d_{\mathrm{loc}}}} \mathrm{d}\boldsymbol{\phi} \int_{\mathrm{SL}(2,\mathbb{C})^{10}} [\mathrm{d}g]^{10} \int_{\mathbb{H}^{3\cdot 5}_+} [\mathrm{d}X]^5 \; \Phi_{1234}(\boldsymbol{\phi}, X_1)\Phi_{4567}(\boldsymbol{\phi}, X_2)\Phi_{7389}(\boldsymbol{\phi}, X_3)\Phi_{9620}(\boldsymbol{\phi}, X_4)\Phi_{0851}(\boldsymbol{\phi}, X_5)$$

More realistic scenario - regularization

- due to closure constraint together with projection onto uniform fields Φ_0 one has infinite volume factors as $\mathrm{SL}(2,\mathbb{C})$ is non-compact

→ have to regularize models: done by analytic continuation and compactification of $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{Spin}(4)$ [Dona, Gozzini, Nicotra]

concretely:

- at local level it amounts to **map** between corresponding **Lie algebras** $\mathfrak{spin}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \leftrightarrow \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$

- at global level it amounts to map between corresponding Lie groups via **mapping respective Cartan decompositions** into each other:

$\mathrm{SL}(2,\mathbb{C})$

$$\mathrm{SU}(2) \times \mathrm{A}^+ \times \mathrm{SU}(2) \to \mathrm{SL}(2,\mathbb{C})$$
$$(u, \mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3}, v) \mapsto u\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3}v^{-1}$$

$\mathrm{Spin}(4)$

$$\mathrm{SU}(2) \times \mathrm{T}^+ \times \mathrm{SU}(2) \to \mathrm{Spin}(4)$$
$$(u, \mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3}, v) \mapsto (u\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3}v^{-1}, u\mathrm{e}^{i\frac{1}{2}\frac{t}{a}\sigma_3}v^{-1})$$

$$\mathrm{A}^+ = \{\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3} | \eta \in \mathbb{R}_+\} \xrightarrow[\Lambda]{\text{introduce regulator}} \mathrm{A}^+_\Lambda = \{\mathrm{e}^{\frac{1}{2}\frac{\eta}{a}\sigma_3} | \eta \in [0,\Lambda)\} \xrightarrow[\eta \to -it]{\text{Wick rotate}} \mathrm{T}^+_\Lambda = \{\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3} | t \in [0,\Lambda)\} \xrightarrow[\Lambda \to 2\pi a]{\text{compactify}} \mathrm{T}^+ = \{\mathrm{e}^{-i\frac{1}{2}\frac{t}{a}\sigma_3} | t \in [0, 2\pi a)\}$$

- essentially amounts to mapping of respective homogeneous spaces into each other

$$\mathbb{H}^3 \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$$
$$\mathrm{d}H^2 = a^2\left(\left(\frac{\mathrm{d}\eta}{a}\right)^2 + \sinh^2\left(\frac{\eta}{a}\right)\mathrm{d}\Omega_2\right)$$

$$S^3 \cong \mathrm{Spin}(4)/\mathrm{SU}(2)$$
$$\mathrm{d}S^2 = a^2\left(\left(\frac{\mathrm{d}t}{a}\right)^2 + \sin^2\left(\frac{t}{a}\right)\mathrm{d}\Omega_2\right)$$

- map representation labels $\rho \to -ip$

→ work with $\mathrm{Spin}(4)$-representation theory instead

More realistic scenario - correlation function and length

Starting from regularized action:

→ **linearize equations of motion** over non-trivial background Φ_0

→ solve for regularized correlation function:

$$C(\boldsymbol{\phi},\boldsymbol{g}) = \int_{\mathbb{R}^{d_{\mathrm{loc}}}} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^{d_{\mathrm{loc}}}} \mathrm{e}^{i\boldsymbol{\phi}\cdot\boldsymbol{k}} \prod_{i=1}^{4} \left(\sum_{p_i} \frac{p_i^2}{\mathrm{vol}(\mathrm{T}^+)} \sum_{\substack{j_i,m_i;\\ l_i,n_i}} D^{(p_i,0)}_{j_i m_i l_i n_i}(g_i) \right) B^{p_1p_2p_3p_4}_{l_1n_1l_2n_2l_3n_3l_4n_4} \hat{C}^{p_1p_2p_3p_4}_{j_1m_1j_2m_2j_3m_3j_4m_4}(\boldsymbol{k})$$

$$\hat{C}^{p_1p_2p_3p_4}_{j_1m_1j_2m_2j_3m_3j_4m_4}(\boldsymbol{k}) = \frac{1}{\alpha_{\boldsymbol{p},\boldsymbol{j},\boldsymbol{m}} \sum_i k_i^2 + \frac{1}{a^2} \sum_c \left(-\mathrm{Cas}_{1,p_c}\right) + b_{\boldsymbol{p},\boldsymbol{j},\boldsymbol{m}}}$$

encapsulates remaining non-locality of interactions after projection onto Φ_0

→ analyze correlation function **mode-by-mode**

→ turns out that only the zero-mode behaviour of the correlator is important for us; there we can Wick rotate back and decompactify to $\mathrm{SL}(2,\mathbb{C})$

→ **only** these **zero-modes contribute to** the **correlation length** and **determine** the behaviour of the **Ginzburg Q**-parameter

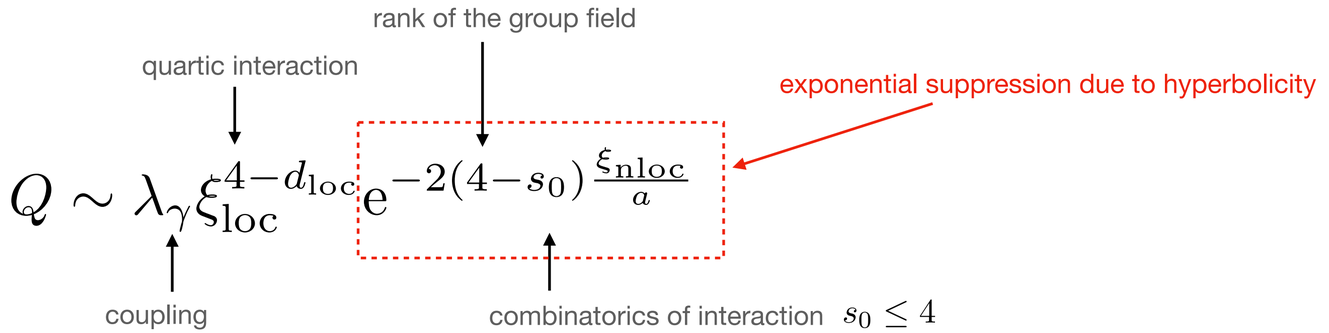
→ **result for correlation lengths** (via asymptotic analysis or second-moment-method): $\xi^2_{\mathrm{nloc}} \sim \frac{1}{a^2\mu^2} + \frac{1}{\mu}$ & $\xi^2_{\mathrm{loc}} \sim \frac{1}{\mu}$

modification due to hyperbolicity of domain

standard form for local degrees of freedom

More realistic scenario - Ginzburg Q

for finite skirt radius a :



flat limit:
$a \to \infty$

dimension of 3-hyperboloid

$$Q \sim \lambda_\gamma \xi_{\mathrm{loc}}^{4-d_{\mathrm{loc}}} \xi_{\mathrm{nloc}}^{4-3(4-s_0)}$$

impact of closure constraint
via the BC entertainer:

$$s_0 \to s_0 + 1$$ (i.e. one more zero-mode)

- can be generalized to arbitrary interactions
- agrees with results on local scalar field theory on hyperbolic domains (Benedetti, 1403.6712)

→ **Ginzburg Q (almost) always very small**

→ **LG mean-field theory can self-consistently describe phase transition** $(\Phi_0 = 0 \leftrightarrow \Phi_0 \neq 0)$

Conclusions

- LG theory is also applicable to GFT models in spite of their non-local interactions
- it informs us about the coarse phase structure of different models
- it informs us about the coarse phase structure of different models
- from there we can extract the rescaling relations of couplings needed for RG studies

Extensions

- consider all the bare causal structure (spacelike, timelike and lightlike tetrahedra)
- extension to other relevant models such as the EPRL-model
- conduct full-fledged (functional) RG and 1/N analyses of these models
- devise observables and tools to characterize different phases wrt their geometric properties

Thank you for your attention!