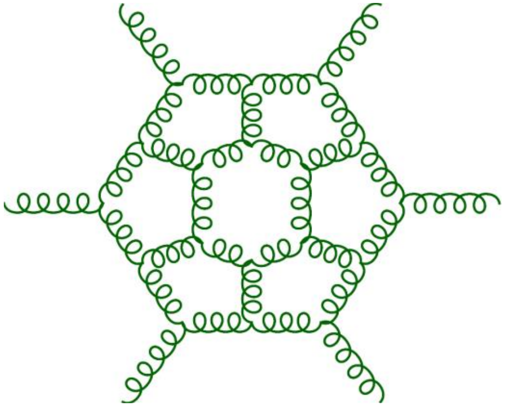
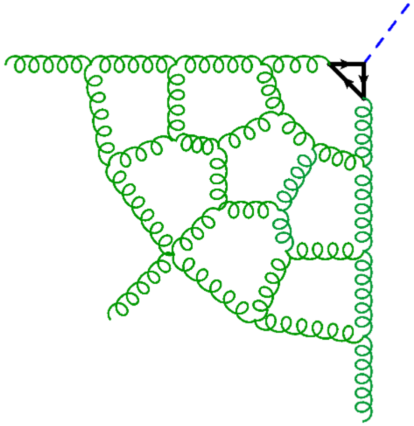
An Antipodal Duality Between Amplitudes and Form Factors





Lance Dixon (SLAC)

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm

2112.06243 and 2204.11901

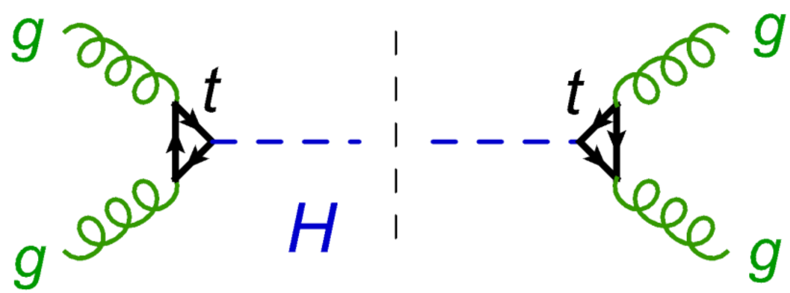
SAGEX Closing Meeting
Queen Mary University of London
23 June 2022

SAGEX
Scattering Amplitudes: from Geometry to Experiment

SLAC NATIONAL ACCELERATOR LABORATORY

Total cross section for producing Higgs boson at LHC via gluon fusion

Leading Order (LO)



- Higgs production at LHC is dominantly via **gluon fusion**, mediated by **top quark** loop.
- Since $\boldsymbol{2m_{top}} = 350\ \mathrm{GeV}$ $\gg \boldsymbol{m_{Higgs}} = 125\ \mathrm{GeV}$,

integrate out top quark to get leading local operator $H G_{\mu\nu}^{a} G^{\mu\nu\, a}$

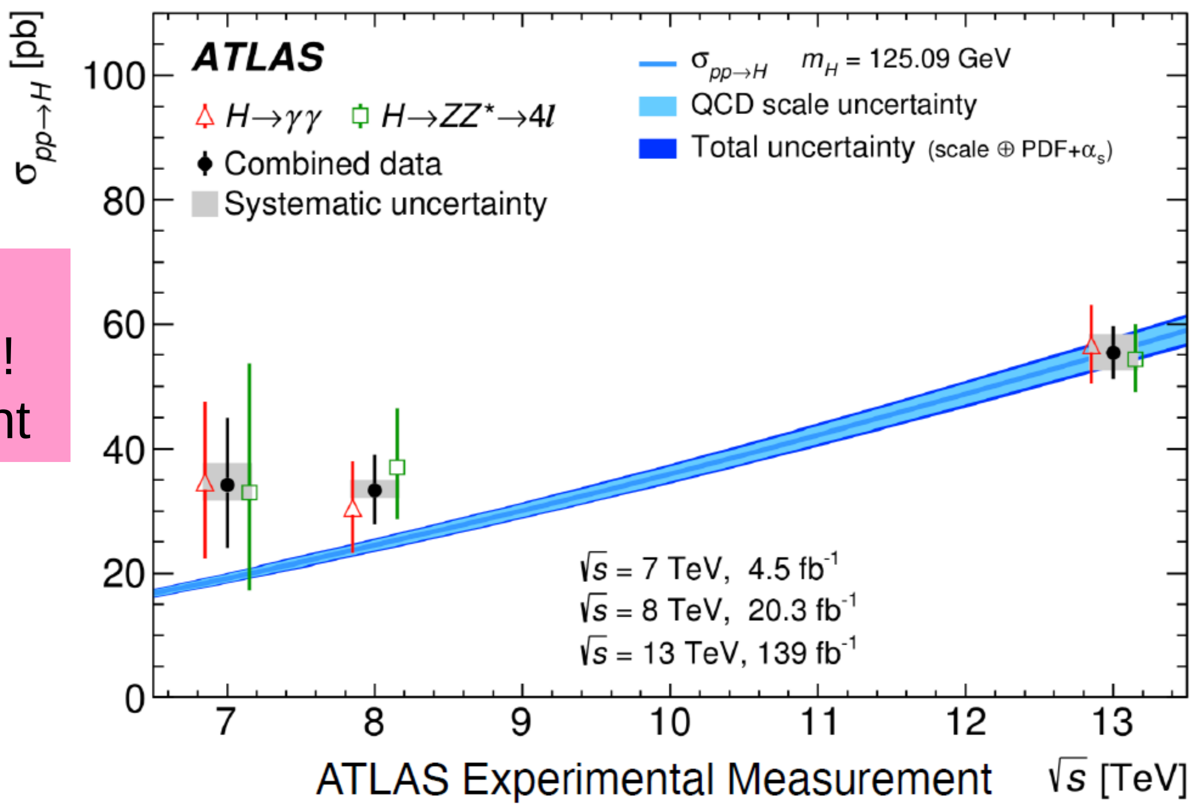
Perturbative Short-Distance Cross Section

$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R) \;=\; [\alpha_s(\mu_R)]^{n_\alpha}\left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi}\hat{\sigma}^{(1)}(\mu_F,\mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F,\mu_R) + \cdots\right]$$

LO NLO NNLO

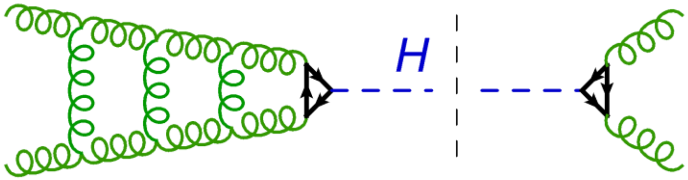
Higgs gluon fusion cross section at LHC vs. CM energy $\sqrt{s}$

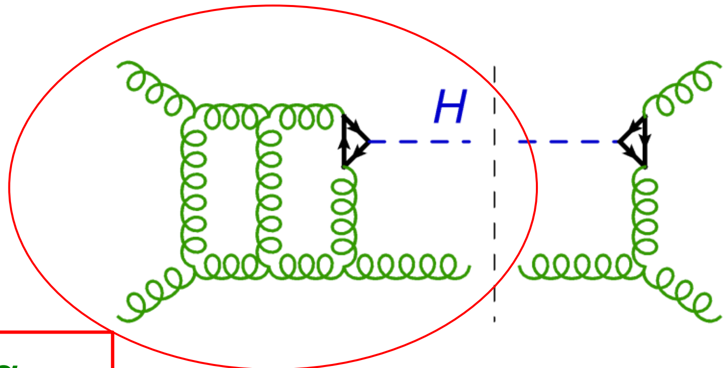
LO → NNNLO
→ factor of 2 or 3 increase!
Critical to match experiment

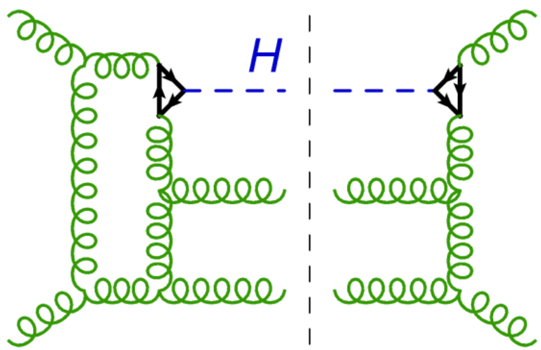


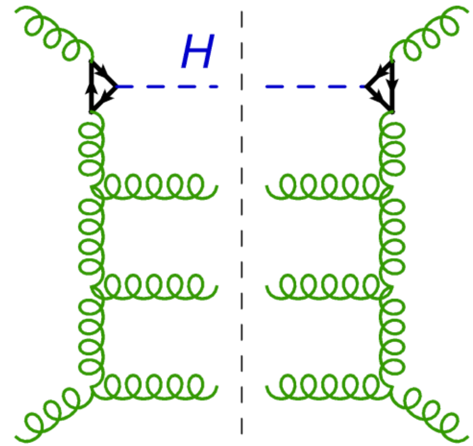
ATLAS Experimental Measurement

Some NNNLO QCD topologies









gg → *Hg*
@ 2 loops,
state of art
in QCD

+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

Scattering amplitudes are the underlying building blocks

Planar N=4 SYM, "hydrogen atom" of amplitudes

- QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $\mathrm{SU}(N_c)$, in large N_c (planar) limit
- Structure very rigid:

$$\text{Amplitudes} = \sum_i \mathit{rational}_i \times \mathit{transcendental}_i$$

- For planar N=4 SYM, rational structure well understood, focus on transcendental functions.
- Furthermore, at least three dualities hold:

1. AdS/CFT
2. Amplitudes dual to Wilson loops
3. New "antipodal" duality between amplitudes and form factors

Transcendental Structure

- N=4 SYM amplitudes have "uniform weight" (transcendentality) $2L$ at loop order L
- Weight ~ number of integrations, e.g.

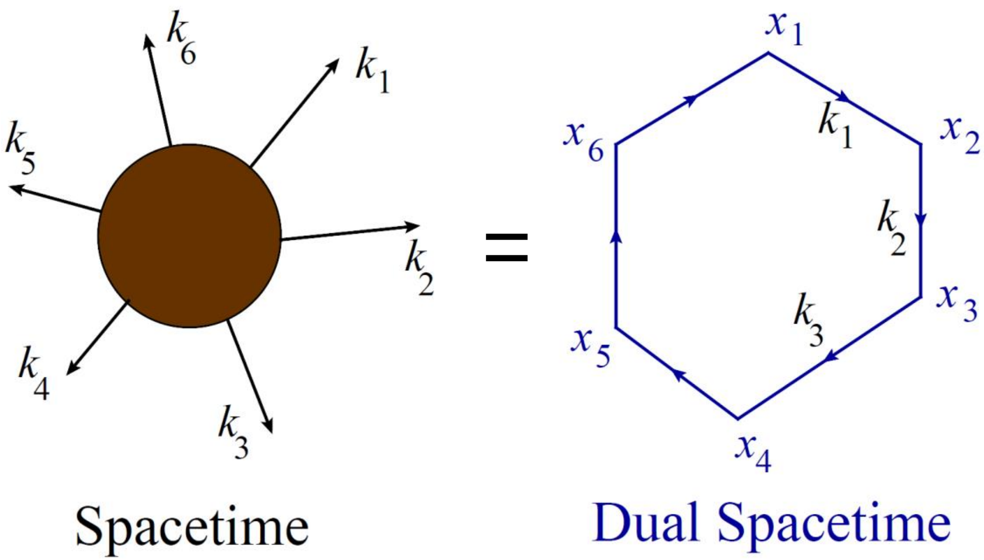
$$\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \qquad 1$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)] \qquad 2$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad n$$

- QCD amps typically all weights from 0 to $2L$

Amplitudes = Wilson loops



- Polygon vertices x_i are not positions but dual momenta,
  $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality holds at both strong and weak coupling

weak-weak duality, holds order-by-order

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

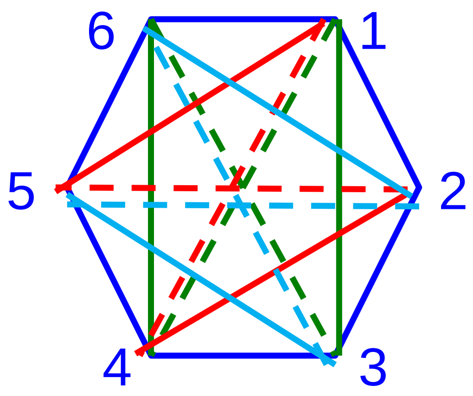
- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for n = 4,5

n = 6 → precisely 3 ratios:

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$v = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$w = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



n = 7 → 6 ratios.
In general, $3n$-15 ratios.

Different routes to perturbative amplitudes

Draw all Feynman graphs G_i

↓

Evaluate all Feynman rules: I_i

↓

Perform all loop integrations: A_i

↓

$$A = \sum_i A_i$$

Evaluate all unitarity cuts C_α

↓

Construct **local**: integrand I

↓

Perform all loop integrations: A_α

↓

$$A = \sum_\alpha A_\alpha$$

Bootstrap: Guess $A = \sum_m c_m F_m$

F_m known functions

$c_m \in \mathbb{Q}$ unknown constants

↓

Solve constraints, linear equations for $c_m \to r_m$

↓

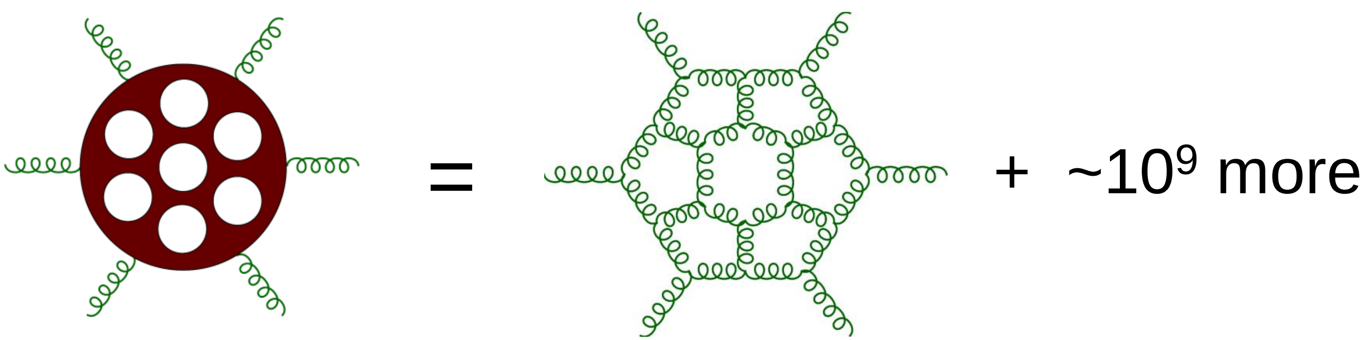
$$A = \sum_m r_m F_m$$

Hexagon function bootstrap

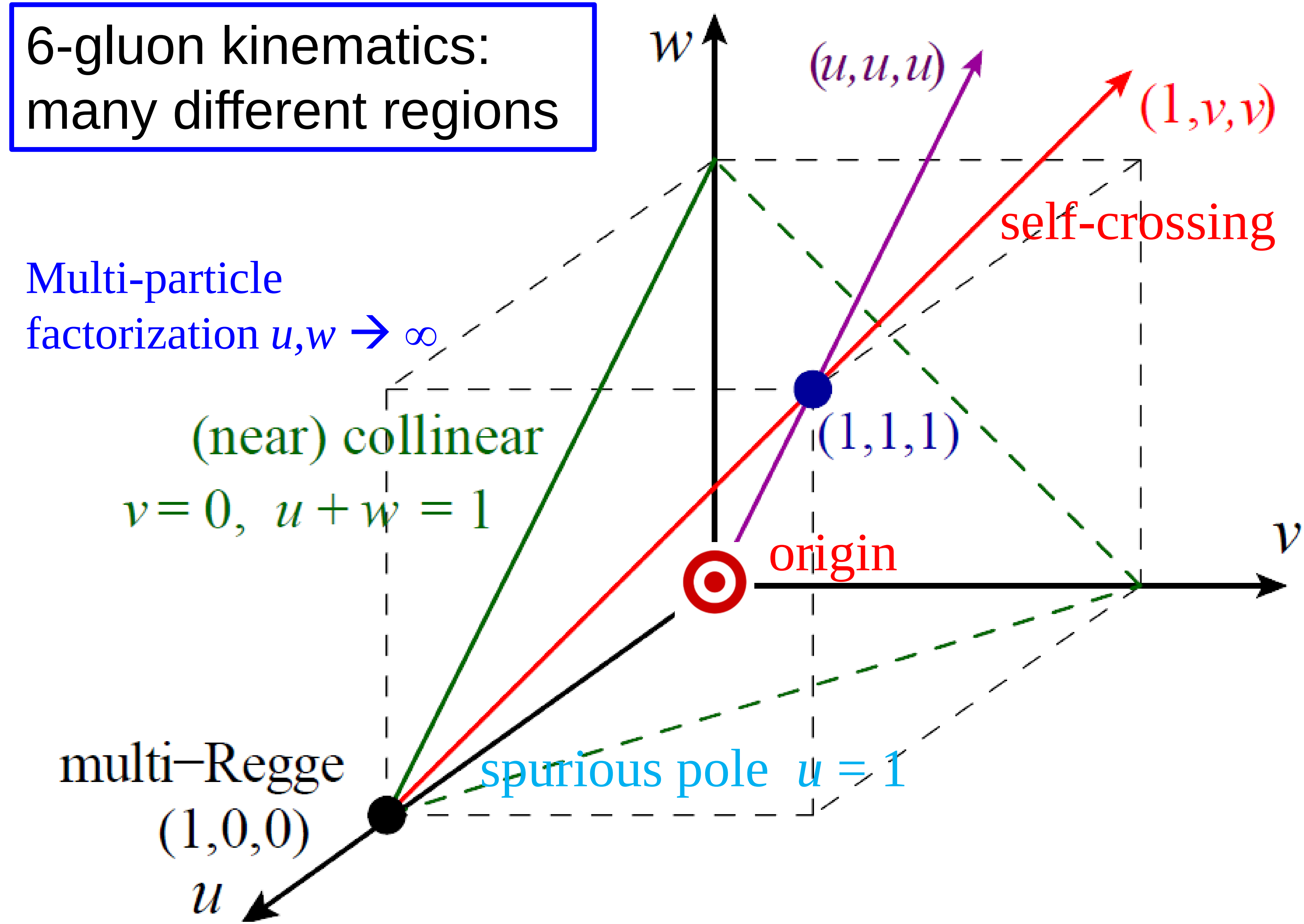
Loops

Loops	
3	LD, Drummond, Henn, 1108.4461, 1111.1704;
4,5	Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
6,7	Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890, 1906.07116; LD, Dulat, 22mm.nnnnn (NMHV 7 loop)

- Use analytical properties of perturbative (six point) amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops
- Step toward doing this nonperturbatively (no loops to peek inside) for general kinematics
- Same method used for “Higgs” form factor; see below

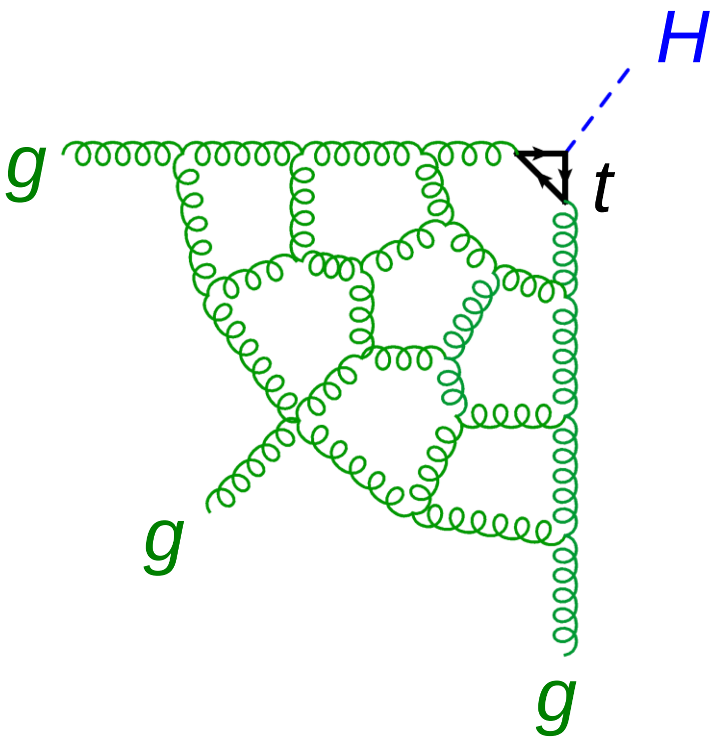


6-gluon kinematics: many different regions



Bootstrap Goldilocks "Higgs" amplitude [planar N=4 form factor] to 8 loops

	Loops
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2012.12286,	3,4,5
2204.11901	6,7,8



- Matrix elements of operator $G_{\mu\nu}^{a} G^{\mu\nu\,a}$ with n gluons in planar N=4 SYM
- *Hgg* form factor ($n = 2$) "too simple", no kinematic dependence beyond overall $(-s_{12})^{-L\epsilon}$
- *Hggg* ($n = 3$) is "just right", depends on only 2 dimensionless ratios
- 8 loop results for function of 2 variables are a "data mine" for discovering e.g. antipodal duality

$Hggg$ kinematics is two-dimensional

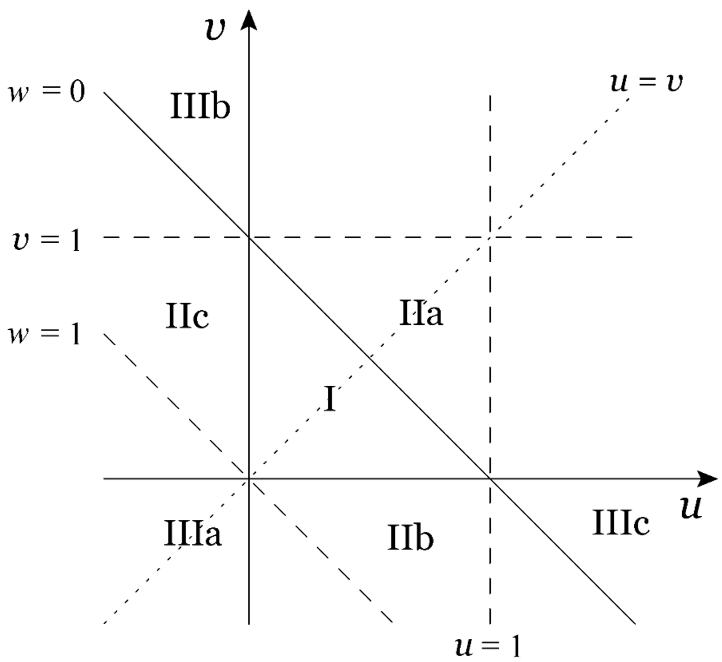
$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

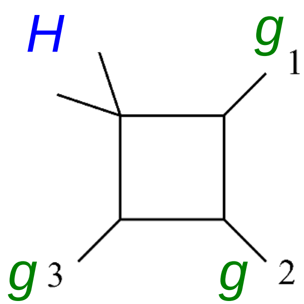
$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or

$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

One loop integrals/amplitudes



$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right)+\mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right)+\cdots$$

$$= \mathrm{Li}_2\left(1-\frac{1}{u}\right)+\mathrm{Li}_2\left(1-\frac{1}{v}\right)+\frac{1}{2}\ln^2\left(\frac{u}{v}\right)+\cdots$$

Two-loop story

- *Hggg* computed in QCD at 2 loops

  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Stress tensor 3-point form factor $\mathcal{F}_3$ in N=4 SYM computed next (QMUL, a decade ago)

  Brandhuber, Travaglini, Yang, 1201.4170
- Highest weight part of QCD result was **same as N=4 result!!**
- "Principle of maximal transcendentality"

  Kotikov, Lipatov, Velizhanin, hep-ph/0301021, hep-ph/0611204
- Does it hold here beyond two loops?
- Other operators: Ahmed et al., 1905.12770; Guo et al., 2205.12969

2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight n. Every function F obeys:

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w}$$

$$w = 1 - u - v$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To **bootstrap** *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight

Generalized polylogarithms

Chen, Goncharov, Brown,…

- Define as iterated integrals, e.g.

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

- Or define differentially: $dF = \sum_{s_k \in \mathcal{L}} F^{s_k} \, d \ln s_k$

- A Hopf algebra "co-acts" on space of polylogarithms,

$$\Delta: F \rightarrow F \otimes F$$

- Derivative dF is one piece of Δ: $\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{L}} F^{s_k} \otimes \ln s_k$
- so we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F
- $\boldsymbol{s_k}$ **are letters in the symbol alphabet** $\mathcal{L}$

Generalized polylogarithms (cont.)

- $\{n\text{-}1,1\}$ coaction can be applied iteratively
- Define $\{n\text{-}2,1,1\}$ double coproducts, F^{s_k,s_j},

via derivatives of $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$dF^{s_j} \equiv \sum_{s_k \in \mathcal{L}} F^{s_k,s_j}\, d\ln s_k$$

- And so on for $\{n\text{-}m,1,\ldots,1\}$ m^{th} coproducts of F.
- Maximal iteration, n times for weight n function, is

the symbol,

$$\mathcal{S}[F] = \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}}\, d\ln s_{i_1} \ldots d\ln s_{i_n} \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}}\, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example: Classical polylogarithms

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

- Regular at $x = 0$, branch cut starts at $x = 1$.
- Iterated differentiation gives the symbol $\mathcal{S}$:

$$\mathcal{S}[\mathrm{Li}_n(x)] = \mathcal{S}[\mathrm{Li}_{n-1}(x)] \otimes x$$
$$= \cdots = -(1-x) \otimes x \otimes \ \ldots \otimes x$$

- Branch cut discontinuities displayed in first entry of symbol, e.g. clip off leading $(1-x)$ to compute discontinuity at $x = 1$.
- Derivatives computed from symbol by clipping last entry, multiplying by that $d\ln(\ldots)$.

Example: Harmonic Polylogarithms in one variable (HPL{0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize classical polylogs
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x) d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

- Symbol alphabet: $\mathcal{L} = \{x, 1-x\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$ is number of binary strings: 2^{2L}
- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol

Symbol alphabet $\mathcal{L}$ for *Hggg*

Gehrmann, Remiddi, hep-ph/0008287

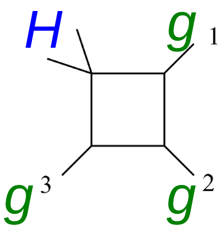
- Comparing
$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$
$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w}$$

$$w = 1 - u - v$$

with $\quad dF = \sum_{s_k \in \mathcal{S}} F^{s_k} \, d\ln s_k$

we see that the alphabet is $\quad \mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$

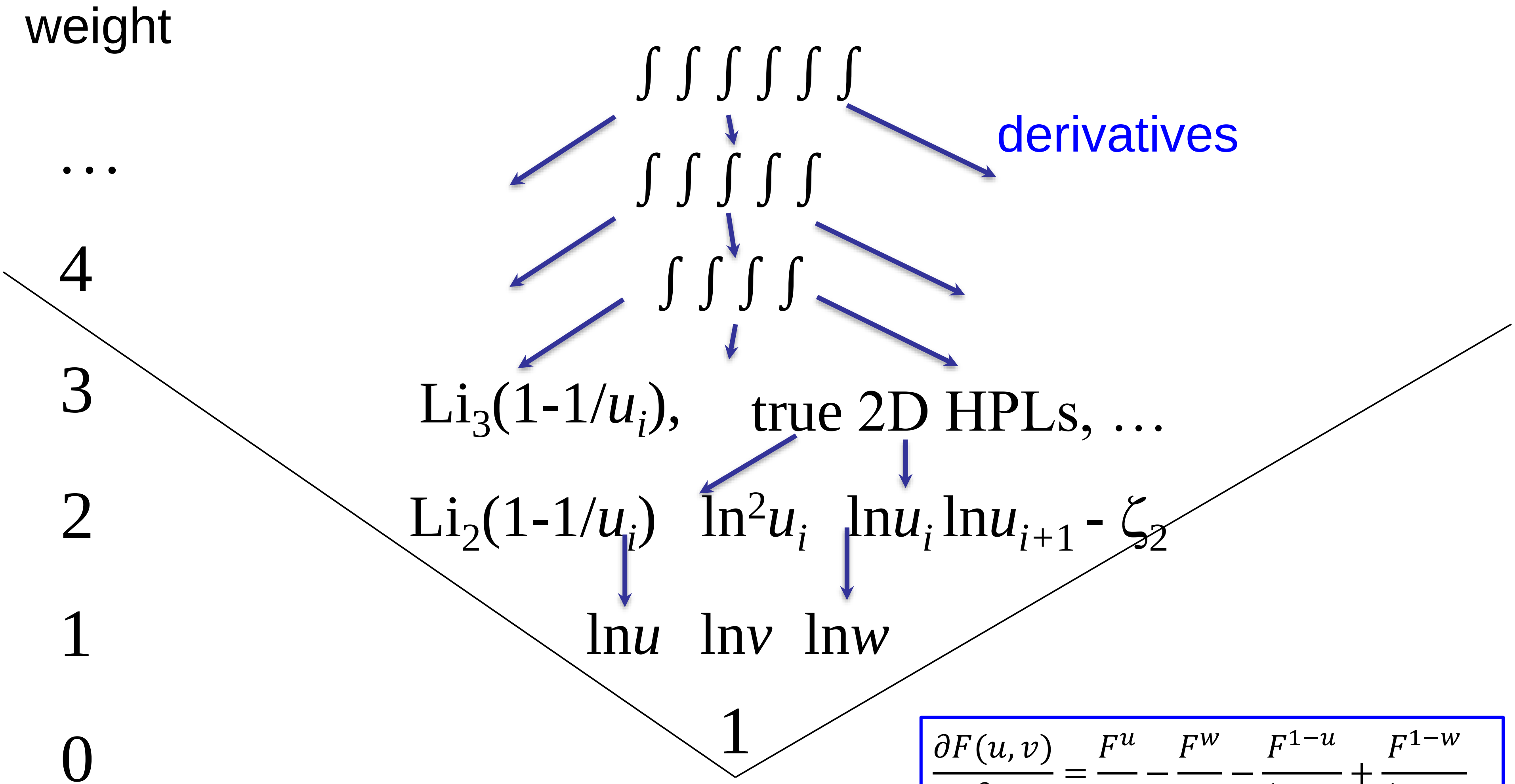
For example,



$$F = \mathrm{Li}_2\left(1 - \frac{1}{u}\right) + \mathrm{Li}_2\left(1 - \frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

→ symbol $\mathcal{S}[F] = u \otimes (1-u) + v \otimes (1-v) - u \otimes v - v \otimes u$

Heuristic view of function space



$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

Symbol alphabets for n-gluon amplitudes

parity-odd letters, algebraic in $\hat{u}, \hat{v}, \hat{w}$

$n = 6$ has 9 letters: $\mathcal{L}_6 = \{\hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w\}$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703; LD, Drummond, Henn, 1108.4461; Caron-Huot, LD, von Hippel, McLeod, 1609.00669

$n = 7$ has 42 letters

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Papathanasiou, Spradlin 1412.3763

$n = 8$ has at least 222 letters, could even be infinite as $L \to \infty$

Arkani-Hamed, Lam, Spradlin, 1912.08222;
Drummond, Foster, Gürdoğan, Kalousios, 1912.08217, 2002.04624;
Henke, Papathanasiou 1912.08254, 2106.01392;
Z. Li, C. Zhang, 2110.00350

Back to 3-gluon form factor

- Motivated by 6 gluon case, switch to equivalent alphabet

$$\mathcal{L}' = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

- Symbols of form factor $F_3^{(L)}$ at 1 and 2 loops: just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)\, b \otimes d + \text{dihedral}$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 4\, b \otimes d \otimes d \otimes d + 2\, b \otimes b \otimes b \otimes d + \text{dihedral}$$

dihedral cycle: $a \to b \to c \to a, \quad d \to e \to f \to d$
dihedral flip: $a \leftrightarrow b, \quad d \leftrightarrow \mathrm{e}$

Simplest analytic form is for $v \to \infty$

$$\mathcal{L}' = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

$$\mathcal{L}' \to \{ \frac{1}{u},\ 1 - \frac{1}{u} \}$$

→ Harmonic polylogarithms $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{u})$

$$F_3^{(1)}(v \to \infty) = 2H_{0,1} + 6\zeta_2$$

$$F_3^{(2)}(v \to \infty) = -8H_{0,0,0,1} - 4H_{0,1,1,1} + 12\zeta_2 H_{0,1} + 13\zeta_4$$

$$\begin{aligned} F_3^{(3)}(v \to \infty) = {} & 96H_{0,0,0,0,0,1} + 16H_{0,0,0,1,0,1} + 16H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 8H_{0,0,1,0,1,1} \\ & + 8H_{0,0,1,1,0,1} + 16H_{0,1,0,0,0,1} + 8H_{0,1,0,0,1,1} + 12H_{0,1,0,1,0,1} + 4H_{0,1,0,1,1,1} \\ & + 8H_{0,1,1,0,0,1} + 4H_{0,1,1,0,1,1} + 4H_{0,1,1,1,0,1} + 24H_{0,1,1,1,1,1} \\ & - \zeta_2(32H_{0,0,0,1} + 8H_{0,0,1,1} + 4H_{0,1,0,1} + 52H_{0,1,1,1}) \\ & - \zeta_3(8H_{0,0,1} - 4H_{0,1,1}) - 53\zeta_4 H_{0,1} - \frac{167}{4}\zeta_6 + 2(\zeta_3)^2 \end{aligned}$$

8 loop result has $\sim 2^{2\times 8-2} = 16{,}384$ terms

6-gluon amplitude is simplest for $(\hat{u},\hat{v},\hat{w}) = (1,\hat{v},\hat{v})$

- Let $H_{\vec{w}} \equiv H_{\vec{w}}(1-\frac{1}{\hat{v}})$

$$
\begin{aligned}
A_6^{(1)}(1,\hat{v},\hat{v}) &= 2H_{0,1} \\
A_6^{(2)}(1,\hat{v},\hat{v}) &= -8H_{0,1,1,1} - 4H_{0,0,0,1} - 4\zeta_2 H_{0,1} - 9\zeta_4 \\
A_6^{(3)}(1,\hat{v},\hat{v}) &= 96H_{0,1,1,1,1,1} + 16H_{0,1,0,1,1,1} + 16H_{0,0,0,1,1,1} + 16H_{0,1,1,0,1,1} + 8H_{0,0,1,0,1,1} \\
&\quad + 8H_{0,1,0,0,1,1} + 16H_{0,1,1,1,0,1} + 8H_{0,0,1,1,0,1} + 12H_{0,1,0,1,0,1} + 4H_{0,0,0,1,0,1} \\
&\quad + 8H_{0,1,1,0,0,1} + 4H_{0,0,1,0,0,1} + 4H_{0,1,0,0,0,1} + 24H_{0,0,0,0,0,1} \\
&\quad + \zeta_2(8H_{0,0,0,1} + 8H_{0,1,0,1} + 48H_{0,1,1,1}) \\
&\quad + 42\zeta_4 H_{0,1} + 121\zeta_6
\end{aligned}
$$

Exact map at symbol level, with $\frac{1}{\hat{v}} = 1 - \frac{1}{u}$, $0 \leftrightarrow 1$,
if you also reverse order of symbol entries / HPL indices!!!
Works to 7 loops, where $\sim 2^{2\times 7-2} = 4{,}096$ terms agree

Antipodal duality

weak-weak duality

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)$$

Antipode map S, at symbol level, reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

Kinematic map is

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u + v + w = 1$ to parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

also corresponds to "twisted forward scattering":

$$\hat{k}_{i+n}^{\mu} = -\hat{k}_i^{\mu}, \qquad i = 1,2,\ldots,n \qquad (n = 3 \; here)$$

6-gluon alphabet and symbol map

- $\mathcal{L}_6 = \{ \hat{u}, \hat{v}, \widehat{w}, 1-\hat{u}, 1-\hat{v}, 1-\widehat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w \}$ → 1 for $\Delta = 0$

→ $\mathcal{L}'_6 = \{ \hat{a} = \frac{\hat{u}}{\hat{v}\widehat{w}},\ \hat{b} = \frac{\hat{v}}{\widehat{w}u},\ \hat{c} = \frac{\widehat{w}}{\hat{u}\hat{v}},\ \hat{d} = \frac{1-\hat{u}}{\hat{u}},\ \hat{e} = \frac{1-\hat{v}}{\hat{v}},\ \hat{f} = \frac{1-\widehat{w}}{\widehat{w}} \}$

- Kinematic map on letters:

$\sqrt{\hat{a}} = d, \quad \hat{d} = a, \qquad$ *plus cyclic relations*

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\frac{1}{2})\hat{b}\otimes\hat{d} + \text{dihedral}$$

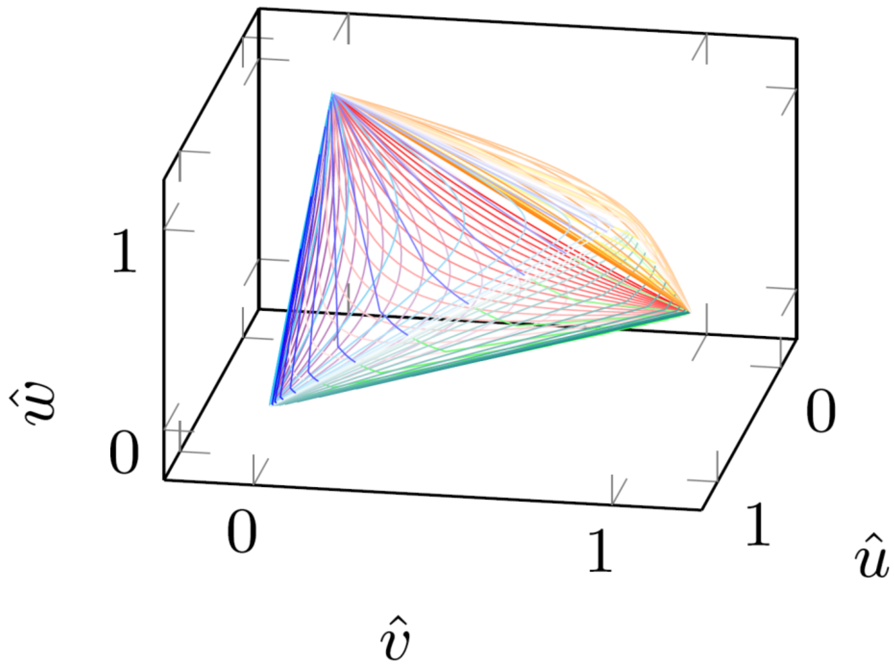
$$\mathcal{S}\left[A_6^{(2)}\right] = \hat{b}\otimes\hat{d}\otimes\hat{d}\otimes\hat{d} + \frac{1}{2}\hat{b}\otimes\hat{b}\otimes\hat{b}\otimes\hat{d} + \text{dihedral}$$

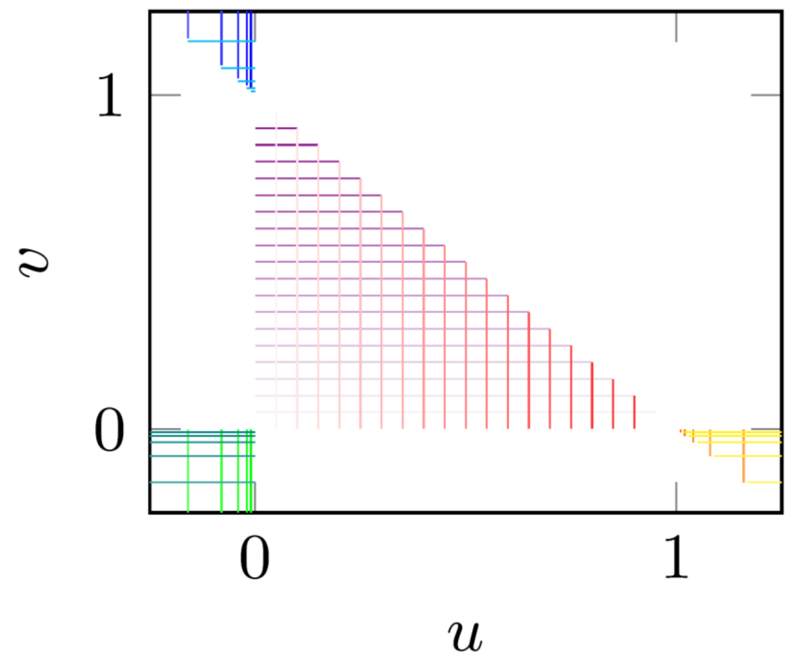
$$\ldots$$

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

- Works through 7 loops!

Map covers entire phase space for 3-gluon form factor



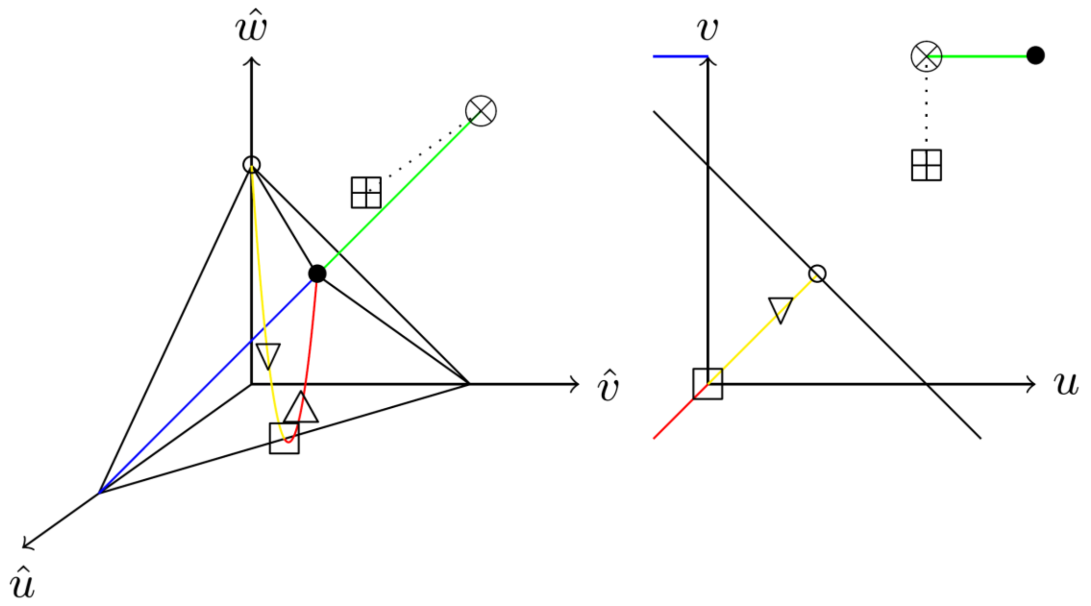


- Soft is dual to collinear; collinear is dual to soft
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \hat{w} > 1$

Many special dual points

There is an "f" alphabet at all these points: a way of writing multiple zeta values (MZV's) so that coaction is manifest.

F. Brown, 1102.1310; O. Schnetz, HyperlogProcedures



	$(\hat{u},\hat{v},\hat{w})$	(u,v,w)	functions
▽	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\sqrt[6]{1}$
□	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,1)$	$\mathrm{Li}_2(\frac{1}{2})$ + logs
●	$(1,1,1)$	$\lim_{u\to\infty}(u,u,1-2u)$	MZVs
○	$(0,0,1)$	$(\frac{1}{2},\frac{1}{2},0)$	MZVs + logs
△	$(\frac{3}{4},\frac{3}{4},\frac{1}{4})$	$(-1,-1,3)$	$\sqrt[6]{1}$
⊞	(∞,∞,∞)	$(1,1,-1)$	alternating sums
⊗	$\lim_{\hat{v}\to\infty}(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(1,v,-v)$	MZVs
green–blue line	$(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(u,v,1-u-v)$	HPL$\{0,1\}$
red–yellow line	$(\hat{u},\hat{u},(1-2\hat{u})^2)$	$(u,u,1-2u)$	HPL$\{-1,0,1\}$

Simplest point

- $(\hat{u},\hat{v},\hat{w}) = (1,1,1) \Leftrightarrow u,v \to \infty$
- At this point,

$$A_6^{(1)}(\cdot) = 0 \qquad F_3^{(1)}(\cdot) = 8\zeta_2$$

$$A_6^{(2)}(\cdot) = -9\zeta_4 \qquad F_3^{(2)}(\cdot) = 31\zeta_4$$

$$A_6^{(3)}(\cdot) = 121\zeta_6 \qquad F_3^{(3)}(\cdot) = -145\zeta_6$$

$$A_6^{(4)}(\cdot) = 120 f_{3,5} - 48\zeta_2 f_{3,3} - \frac{6381}{4}\zeta_8 \qquad F_3^{(4)}(\cdot) = 120 f_{5,3} + \frac{11363}{4}\zeta_8$$

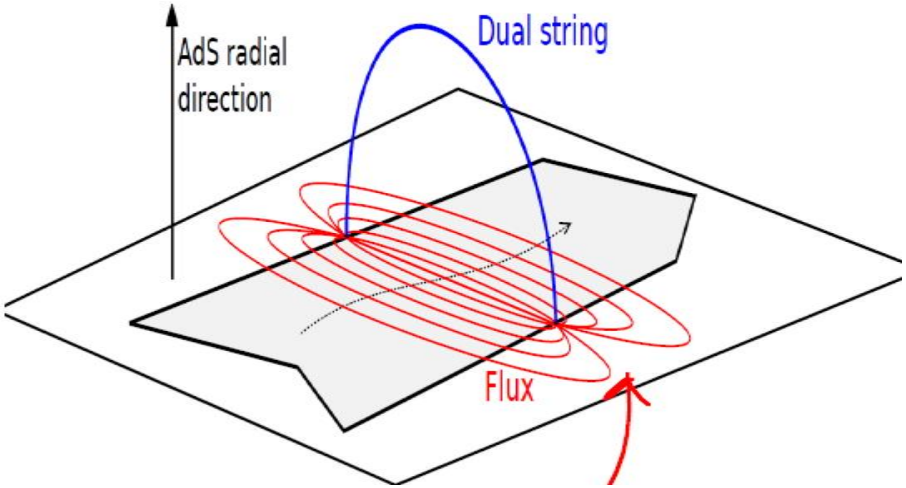
$$A_6^{(5)}(\cdot) = -2688 f_{3,7} - 1560 f_{5,5} + \mathcal{O}(\pi^2) \qquad F_3^{(5)}(\cdot) = -2688 f_{7,3} - 1560 f_{5,5} + \mathcal{O}(\pi^2)$$

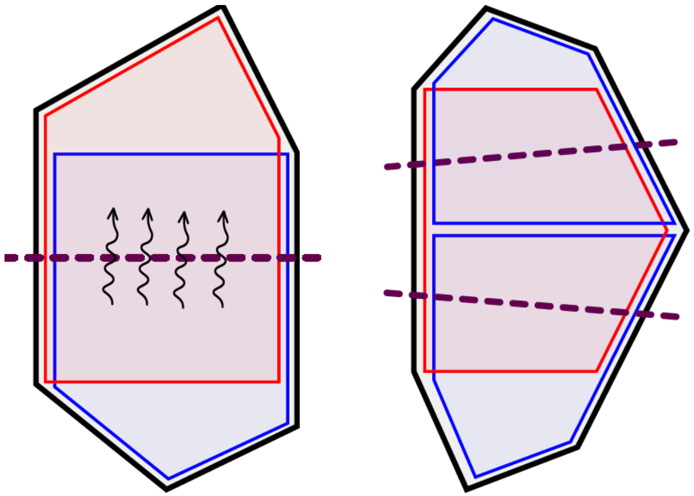
$$A_6^{(6)}(\cdot) = 48528 f_{3,9} + 37296 f_{5,7} + 21120 f_{7,5} + \mathcal{O}(\pi^2) \qquad F_3^{(6)}(\cdot) = 48528 f_{9,3} + 37296 f_{7,5} + 21120 f_{5,7} + \mathcal{O}(\pi^2)$$

- Reversing ordering of letters in f-alphabet, blue values show that antipodal duality holds beyond symbol level, modulo $i\pi$
- modulo $i\pi$ is best we can get from antipode map

Bootstrap boundary data:
Flux tubes at finite coupling

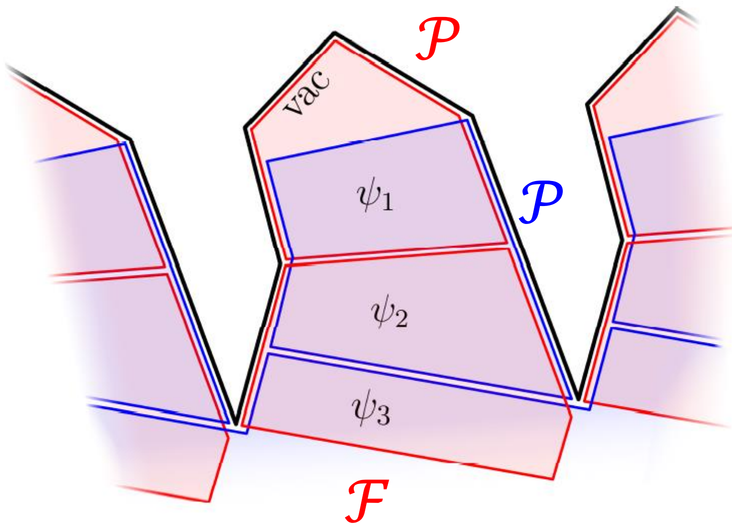
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

A New Form Factor OPE



- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$

Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

OPE representation

- 6-gluon amplitude:

$$\mathcal{W}_{\text{hex}} = \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0)\, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi}$$

$T = e^{-\tau}, S = e^{-\sigma}, F = e^{i\phi}$. $\quad v = \frac{T^2}{1+T^2} \rightarrow 0,$

weak-coupling, $E = k + \mathcal{O}(g^2)$ → expansion in T^k

- 3-gluon form factor: $\psi = helicity\ 0\ pairs\ of\ states$

$$\mathcal{W}_3 = \sum\!\!\!\!\!\!\int_{\psi} e^{-E_\psi \tau + ip_\psi \sigma} \mathcal{P}(0|\psi) \mathcal{F}(\psi)$$

weak-coupling → expansion in T^{2k} (no azimuthal angle ϕ)

OPE parametrizations

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})},$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \qquad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2}$$

$(\hat{F} = 1 \text{ for } \Delta = 0)$

- Form factor:

$$u = \frac{1}{1 + S^2 + T^2}, \qquad v = \frac{T^2}{1 + T^2},$$

$$w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))},$$

- Apply kinematic map → $\hat{T} = \frac{T}{S}, \qquad \hat{S} = \frac{1}{TS}$
- Apparently some correspondence between single flux tube excitations for the amplitude (T^1) and double (or bound state) excitations for the form factor (T^2)

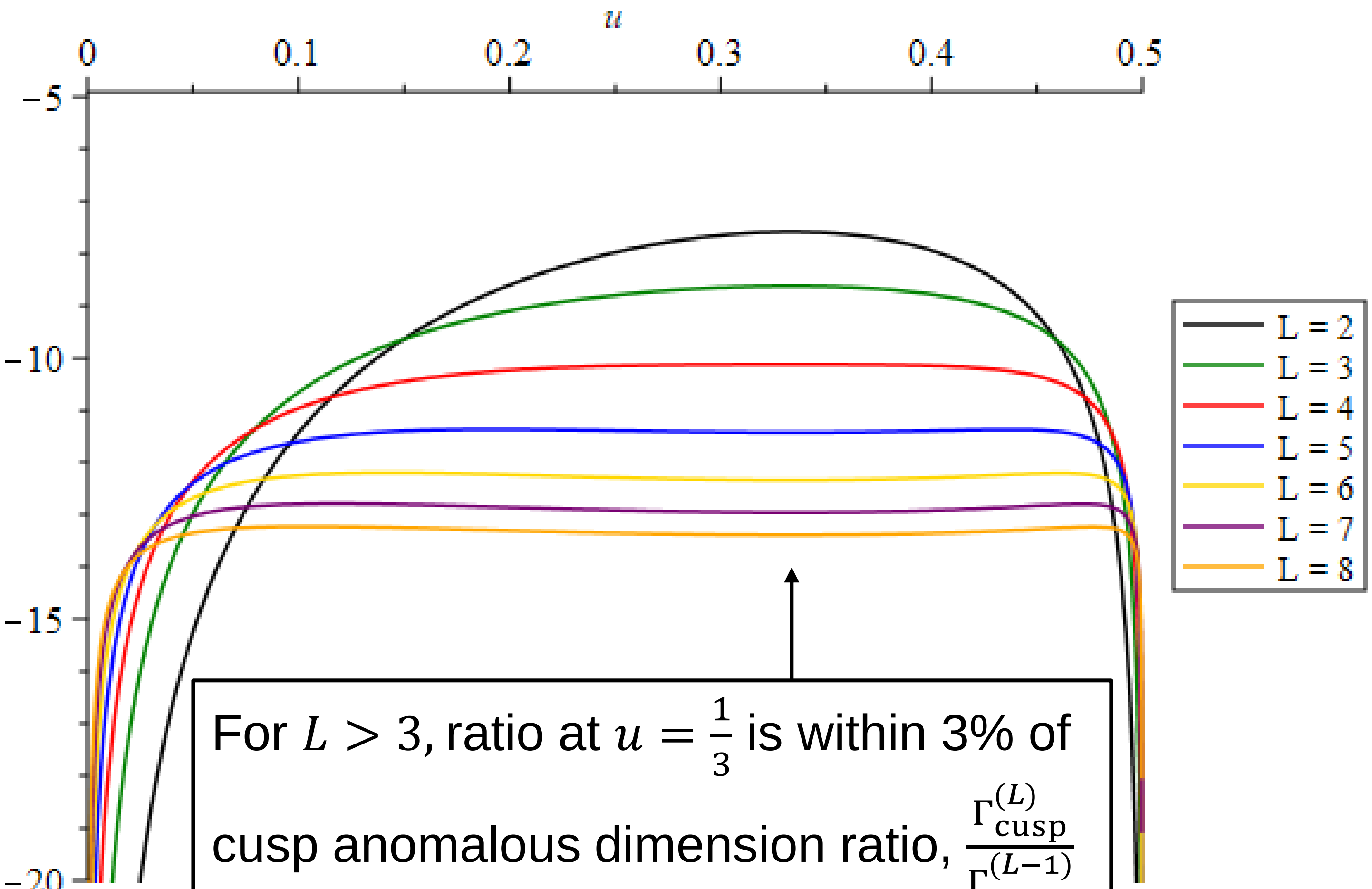
Finite radius of convergence

- Planar N=4 SYM has no renormalons ($\beta(g) = 0$) and no instantons ($e^{-1/g_{\rm YM}^2} = e^{-N_c/\lambda}$)
- Perturbative expansion can have finite radius of convergence, unlike QCD, QED, whose perturbative series are asymptotic.
- For cusp anomalous dimension, using coupling $g^2 \equiv \frac{N_c g_{\rm YM}^2}{16\pi^2} = \frac{\lambda}{16\,\pi^2}$, radius is $\frac{1}{16}$

Beisert, Eden, Staudacher (BES), 0610251

- Ratio of successive loop orders $\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}} \to -16$
- Find same radius of convergence in high-loop-order behavior of amplitudes and form factors, in most kinematic regions.

Euclidean Region numerics



$$\frac{\mathcal{E}^{(L)}(u,u,1-2u)}{\mathcal{E}^{(L-1)}(u,u,1-2u)}$$

$\mathcal{E} \equiv F_3$

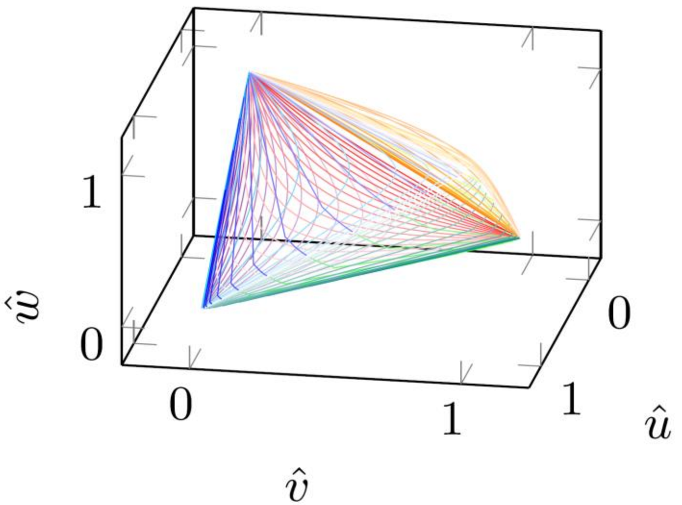
For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}}$

→ same finite radius of convergence…

Exploit/test antipodal duality at 8 loops

LD, Y.-T. Liu, to appear

- Given form factor, antipodal duality determines symbol of MHV 6 gluon amplitude at 8 loops on $\Delta = 0$ surface.
- Lift symbol into bulk. Only 3 free parameters!
- 2 killed at origin, $(\hat{u}, \hat{v}, \hat{w}) \to (0,0,0)$
- last killed in process of lifting to full function level
- Need one OPE data point to kill one beyond-symbol ambiguity $\propto \zeta_8$



8 loop MHV 6-gluon amplitude at $(\hat{u}, \hat{v}, \hat{w}) = (1,1,1)$

LD, Y.-T. Liu, to appear

$$
\begin{aligned}
A_6^{(8)}(1,1,1) = {} & 9122624\, f_{9,7} + 11543472\, f_{7,9} + 5153280\, f_{11,5} + 19603536\, f_{5,11} + 23915376\, f_{3,13} \\
& + 371520\, f_{5,3,3,5} + 400320\, f_{3,3,5,5} + 400320\, f_{3,5,3,5} + 825216\, f_{3,3,3,7} \\
& - \zeta_2\,(701856\, f_{7,7} + 1303232\, f_{9,5} + 430656\, f_{5,9} + 2061312\, f_{11,3} - 309696\, f_{3,11} \\
& \qquad + 160128\, f_{3,5,3,3} + 160128\, f_{3,3,5,3} + 117888\, f_{3,3,3,5} + 148608\, f_{5,3,3,3}) \\
& - \zeta_4\,(3243888\, f_{5,7} + 3475296\, f_{7,5} + 3909696\, f_{9,3} + 3215472\, f_{3,9} + 353664\, f_{3,3,3,3}) \\
& - \zeta_6\,(3612804\, f_{5,5} + 3791520\, f_{7,3} + 3409152\, f_{3,7}) - \zeta_8\,(3720664\, f_{5,3} + 3456614\, f_{3,5}) \\
& - \frac{19560489}{5}\,\zeta_{10}\, f_{3,3} - \frac{512193667550809}{7639104}\,\zeta_{16}
\end{aligned}
$$

- Blue values successfully predicted by antipodal duality
- Result consistent with coaction principle at weight 16.

Summary & Open Questions

- Form factors as well as scattering amplitudes in planar N=4 SYM can now be **bootstrapped** to high loop order
- Comparing the 6-gluon amplitude to the 3-gluon form factor, a strange new antipodal duality emerges, swapping the role of branch cuts and derivatives

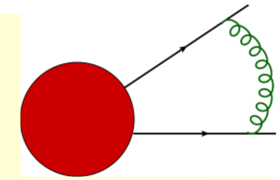
- Underlying physical reason for this duality? Relation to flux tube representation?
- (How) does it hold at strong coupling?
- Does it hold at 8g-4gFF level? Anywhere else?
- How much more can we exploit it to learn more about both amplitudes and form factors?

The End

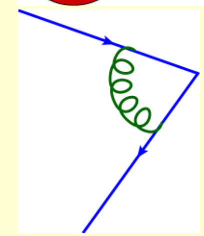
SAGEX

Scattering Amplitudes:
from Geometry to Experiment

Removing Amplitude (or Form Factor) Infrared Divergences



- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension Γ_{cusp}

  – known to all orders in planar N=4 SYM:
  Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\mathrm{BDS-like}}(s_{i,i+1},\epsilon)} = \exp\left[\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{E}^{(1)} + R_6\right]$$

remainder function

BDS & BDS-like normalization for $\mathcal{F}_3$

$$\frac{\mathcal{F}_3}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\mathrm{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon)\right)M^{1-\mathrm{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u,v,w)\right]\right\}$$

BDS ansatz

remainder function only a function of u, v, w; vanishes in all collinear limits, but no adjac... y constraints

split 1-loop amplitude judiciously:

$$\frac{\mathcal{F}_3^{1-\mathrm{loop}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} \equiv M^{1-\mathrm{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$M(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{3}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum^{3} \ldots$$

$$\mathcal{E}^{(1)}(u,v \ldots \ldots\left(1-\frac{1}{v}\right) + \mathrm{Li}_2\left(1-\frac{1}{w}\right)\Big] \qquad \mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$$

$\mathcal{E}$ obeys "adjacency constraints"

Now div... ...y:

$$\frac{\mathcal{F}_3^{\mathrm{BDS-like}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\mathrm{cusp}}}{4} + \mathcal{O}(\epsilon)\right)M(L\epsilon) + C^{(L)}\right]\right\} \Rightarrow$$

$$\mathcal{E} = \exp\left[\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{E}^{(1)} + R\right]$$

8-4 Kinematic Map in OPE Parametrization

- 8-point amplitude has D_8 dihedral symmetry; change it to D_4 of the form factor by requiring

$$\hat{T}_3 = \hat{T}_1, \qquad \hat{S}_3 = \hat{S}_1, \qquad \hat{F}_3 = \hat{F}_1$$

- To get $\mathcal{S}[R_8^{(2)}]$ to have only 8 final

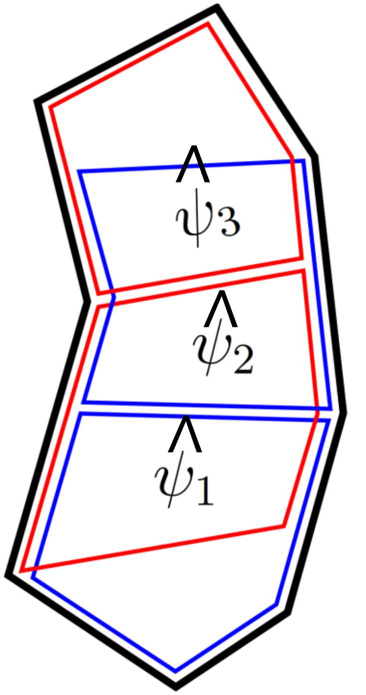
entries, we also fix $\hat{F}_1 = \hat{F}_2 = 1$.

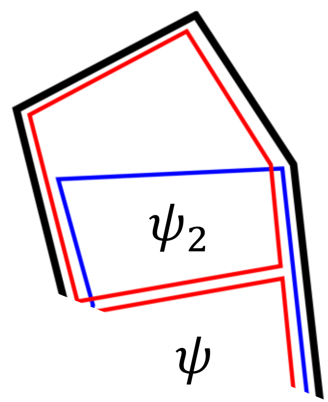
- The kinematic map becomes

$$\hat{T}_1 = \frac{T}{S}, \hat{S}_1 = \frac{1}{TS},$$

$$\hat{T}_2 = \frac{T_2}{S_2}, \hat{S}_2 = \frac{1}{T_2 S_2}$$

and requires $F_2 = i$





8-gluon Amp ←→ 4-gluon FF

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, in progress

- We have a candidate kinematic map for a 4-dimensional surface (4-gluon FF is 5d).
- $\mathcal{S}[R_8^{(2)}]$ is known S. Caron-Huot, 1105.5606
- The kinematic+antipodal maps take it to a symbol with 40 letters, the first 8 of which are "right": $u_i = \frac{s_{i,i+1}}{s_{1234}}, \quad v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$
- Still have to check this candidate 2-loop 4-gluon form factor vs. FFOPE

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.

  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Many “empirical” adjacency constraints

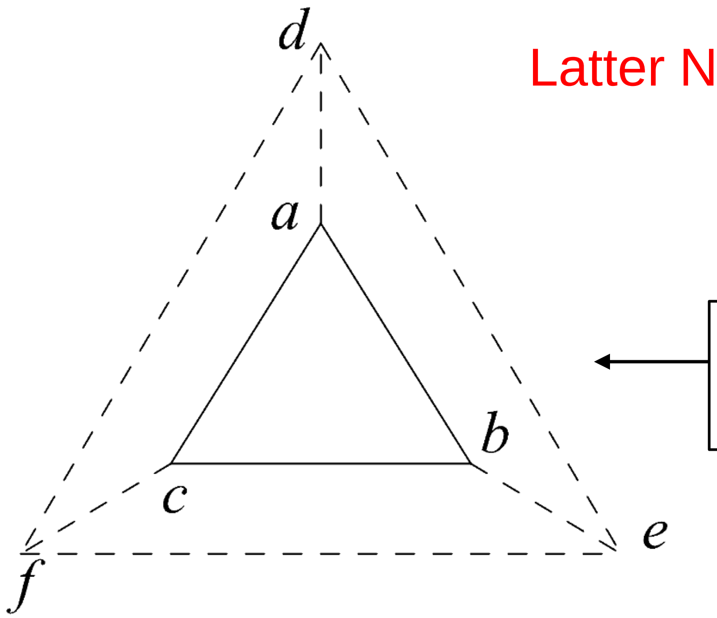
$$F^{d,e} = F^{e,d} = F^{e,f} = F^{f,e} = F^{f,d} = F^{d,f} = 0$$

Hold for 2 loop QCD amplitudes too, planar and nonplanar!

LD, Mcleod, Wilhelm, 2012.12286

$$F^{a,d} = F^{d,a} = F^{b,e} = F^{e,b} = F^{c,f} = F^{f,c} = 0$$

Latter NEW: Hold for planar N=4 SYM to 8 loops!



Mnemonic for dihedral symmetry;
6 dashed lines indicate 12 forbidden pairs.

Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L = 1$	1	3	1														
$L = 2$	1	3	6	3	1												
$L = 3$	1	3	9	12	6	3	1										
$L = 4$	1	3	9	21	24	12	6	3	1								
$L = 5$	1	3	9	21	46	45	24	12	6	3	1						
$L = 6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L = 7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L = 8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right

Multi-final entry relations

1. $\mathcal{E}^{a} = 0$ (plus dihedral images)

2. $\mathcal{E}^{a,e} = \mathcal{E}^{a,f}$ (plus …)

3. $\mathcal{E}^{a,b,d} = 0, \quad \mathcal{E}^{a,e,e} = -\,\mathcal{E}^{a,f,f},$

$$\mathcal{E}^{e,a,f} = \mathcal{E}^{f,a,f} - \mathcal{E}^{a,f,f}$$

4. …

Number of remaining parameters in form-factor ansatz after imposing constraints

L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	6654	34219	????	????
dihedral symmetry	11	51	247	1219	????	????	????
$(L-1)$ final entries	5	9	20	44	86	191	191
L^{th} discontinuity	2	5	17	38	75	171	164
collinear limit	0	1	2	8	19	70	6
OPE $T^2 \ln^{L-1} T$	0	0	0	4	12	56	0
OPE $T^2 \ln^{L-2} T$	0	0	0	0	0	36	0
OPE $T^2 \ln^{L-3} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-4} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-5} T$	0	0	0	0	0	0	0

The [Dual] Conformal Group

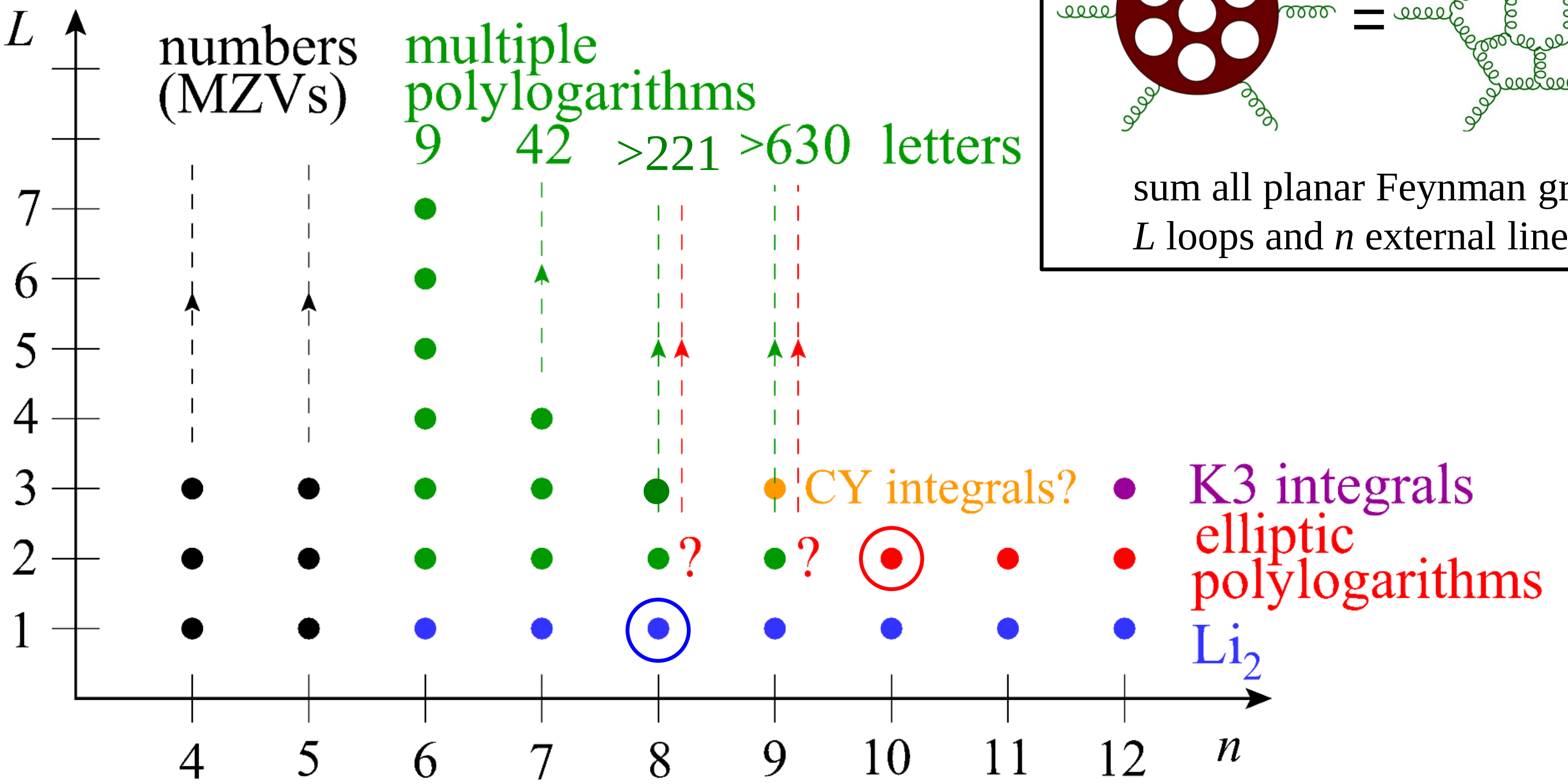
SO(4,2) $\supset$ SO(3,1) [rotations+boosts] + translations+dilatations + special-conformal

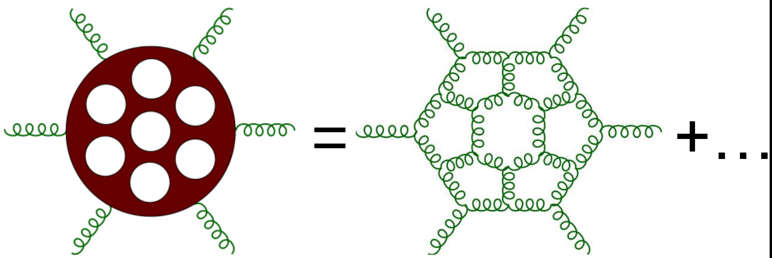
15 = 3 + 3 + 4 + 1 + 4

- The nontrivial generators are special conformal K^μ
- Correspond to inversion · translation · inversion
- To obtain a [dual] conformally invariant function $f(x_{ij}^2)$ just have to check invariance under inversion,

$$x_i^\mu \to x_i^\mu / x_i^2$$

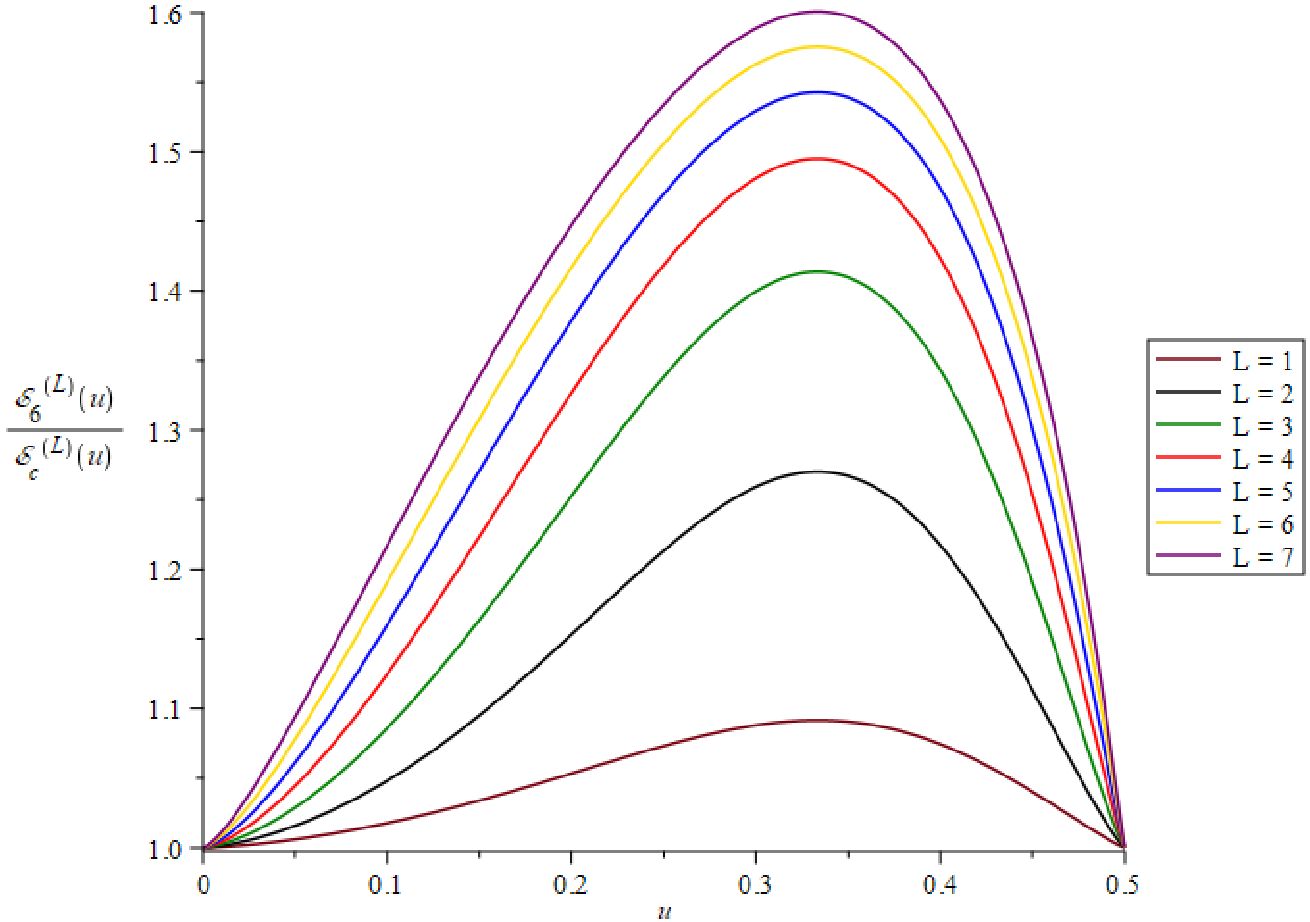
Beyond $n = 8$





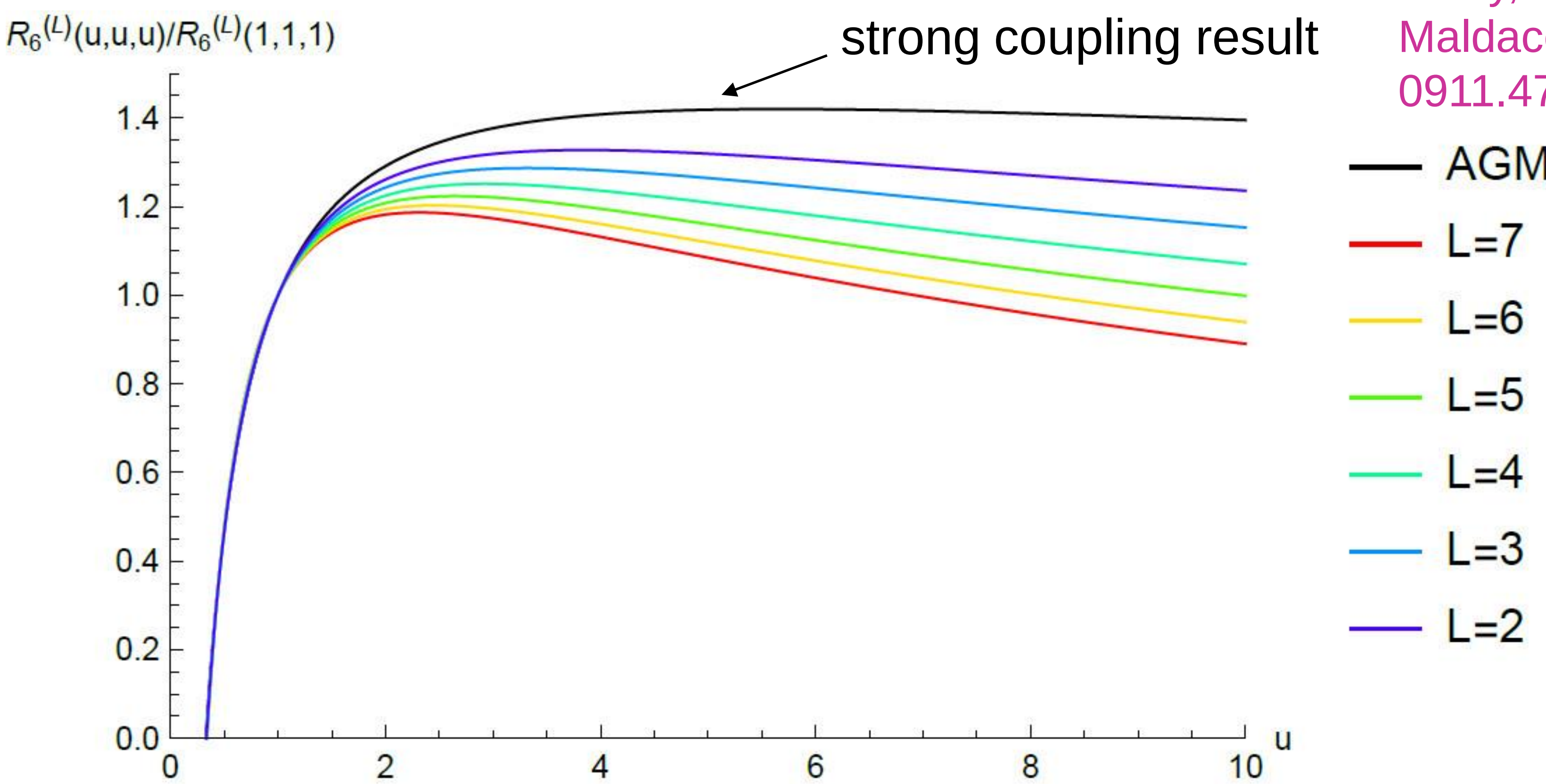
sum all planar Feynman graphs with
L loops and n external lines

Numerical implications of antipodal duality?



Example: MHV finite remainder $R_6^{(L)}$ on (u,u,u)

Alday, Gaiotto, Maldacena, 0911.4708



- **Amazing proportionality** of **each** perturbative coefficient at small u, and also with the strong coupling result

Origin at weak coupling

- Remarkably, MHV remainder R_6 and closely−related quantity $\ln\mathcal{E}$ are quadratic in logarithms through 7 loops CDDvHMP, 1903.10890

$$\ln\mathcal{E}(u_i) \approx -\frac{\Gamma_{\rm oct}}{24}\ln^2(u_1u_2u_3) - \frac{\Gamma_{\rm hex}}{24}\sum_{i=1}^{3}\ln^2\frac{u_i}{u_{i+1}} + C_0$$

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
$\Gamma_{\rm oct}$	4	$-16\zeta_2$	$256\zeta_4$	$-3264\zeta_6$	$\frac{126976}{3}\zeta_8$
$\Gamma_{\rm cusp}$	4	$-8\zeta_2$	$88\zeta_4$	$-876\zeta_6 - 32\zeta_3^2$	$\frac{28384}{3}\zeta_8 + 128\zeta_2\zeta_3^2 + 640\zeta_3\zeta_5$
$\Gamma_{\rm hex}$	4	$-4\zeta_2$	$34\zeta_4$	$-\frac{603}{2}\zeta_6 - 24\zeta_3^2$	$\frac{18287}{6}\zeta_8 + 48\zeta_2\zeta_3^2 + 480\zeta_3\zeta_5$
C_0	$-3\zeta_2$	$\frac{77}{4}\zeta_4$	$-\frac{4463}{24}\zeta_6 + 2\zeta_3^2$	$\frac{67645}{32}\zeta_8 + 6\zeta_2\zeta_3^2 - 40\zeta_3\zeta_5$	$-\frac{4184281}{160}\zeta_{10} - 65\zeta_4\zeta_3^2 - 120\zeta_2\zeta_3\zeta_5 + 228\zeta_5^2 + 420\zeta_3\zeta_7$

- Coefficients involve same BES kernel as for cusp, but “tilted” by angle α,

$$\Gamma_{\rm cusp} = \Gamma_{\alpha=\pi/4} \qquad \Gamma_{\rm oct} = \Gamma_{\alpha=0} \qquad \Gamma_{\rm hex} = \Gamma_{\alpha=\pi/3}$$

B. Basso, LD, G. Papathanasiou, 2001.05460