From conformal symmetries and integrability to the Electron-Ion Collider

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Based on work done in collaboration with:

- *Resummation of small-x double logarithms in QCD: inclusive deep-inelastic scattering*
  J. Davies, C.-H. Kom, S. M., and A. Vogt `arXiv:2202.10362`
- *Low moments of the four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:2111.15561`
- *Renormalization of non-singlet quark operator matrix elements for off-forward hard scattering*
  S. M. and S. Van Thurenhout, `arXiv:2107.02470`
- *Approximate four-loop QCD corrections to the Higgs-boson production cross section*
  G. Das, S. M., and A. Vogt `arXiv:2004.00563`
- *Soft corrections to inclusive deep-inelastic scattering at four loops and beyond*
  G. Das, S. M., and A. Vogt `arXiv:1912.12920`
- *Five-loop contributions to low-N non-singlet anomalous dimensions in QCD*
  F. Herzog, S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt
  `arXiv:1812.11818`
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1805.09638`
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`

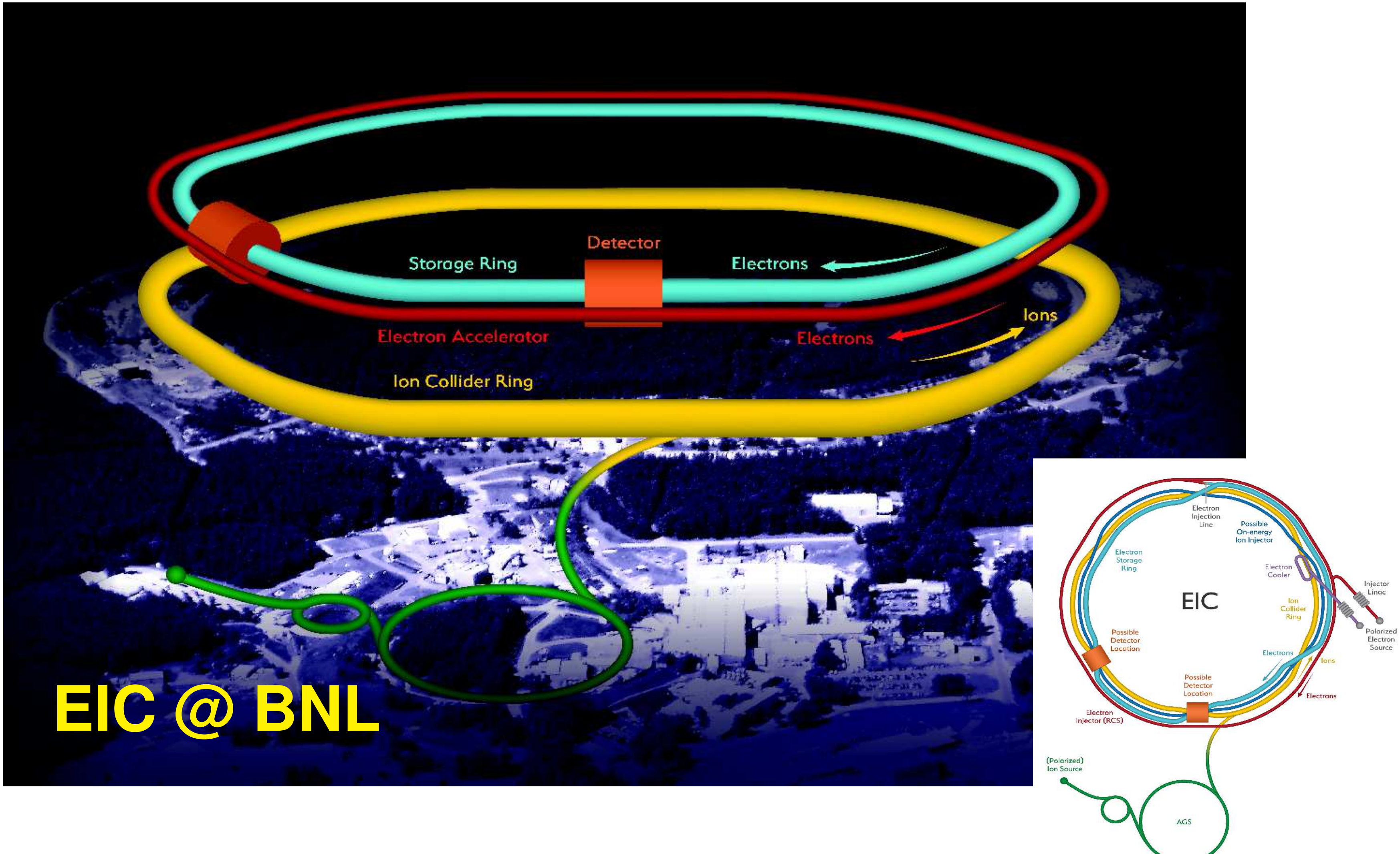
Deep-inelastic scattering

Once upon a time ...

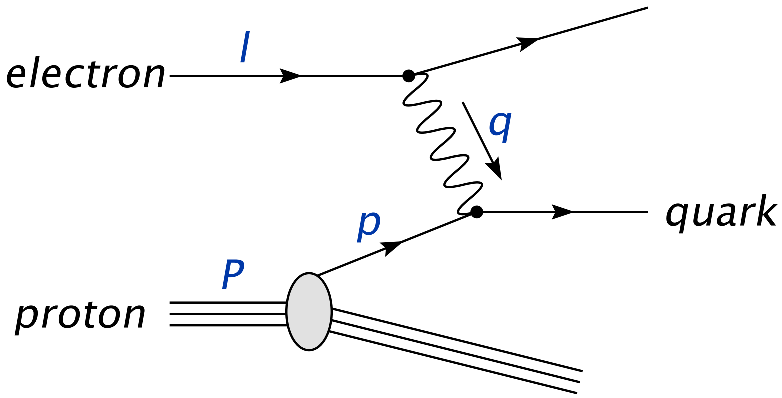
- HERA: deep structure of proton at highest Q^2 and smallest x

Bright future for precision hadron physics

- Electron-Ion Collider
  A machine that will unlock the secrets of the strongest force in Nature



Deep-inelastic scattering



Kinematic variables

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure functions (up to order $\mathcal{O}(1/Q^2)$)

$$F_a\left(x, Q^2\right) = \sum_i \left[C_{a,i}\left(\alpha_s(\mu^2), \mu^2/Q^2\right) \otimes PDF\left(\mu^2\right)\right](x)$$

- Coefficient functions up to N^{4}LO (work in progress)

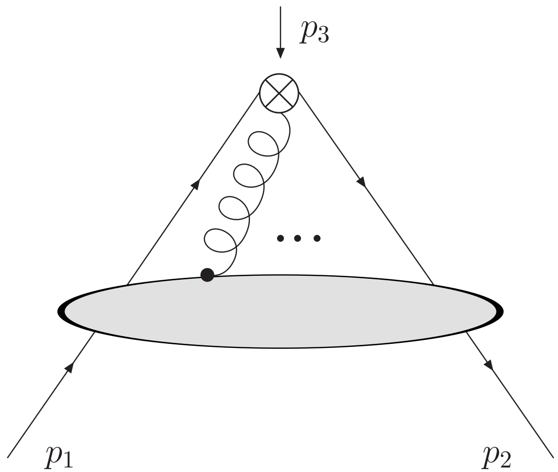
$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \alpha_s^3 c_{a,i}^{(4)} + \ldots\right)$$

- Evolution equations up to N^{3}LO (work in progress)
  - non-singlet ($2n_f - 1$ scalar) and singlet (2×2 matrix) equations

$$\frac{d}{d \ln \mu^2} PDF(x, \mu^2) = \left[P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)\right](x)$$

  - splitting functions $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \ldots$

Operator matrix elements



- Quark operator of spin-N and twist two
  $O^{\psi}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\, \gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\, \psi$
- N covariant derivatives
  $D_{\mu,ij} = \partial_\mu \delta_{ij} + \mathrm{i} g_s \,(t_a)_{ij}\, A^a_\mu$
  sandwiched between quark fields ψ, $\overline{\psi}$
- Evaluation of operators in matrix elements $A^{\psi\psi}$ with external quark states

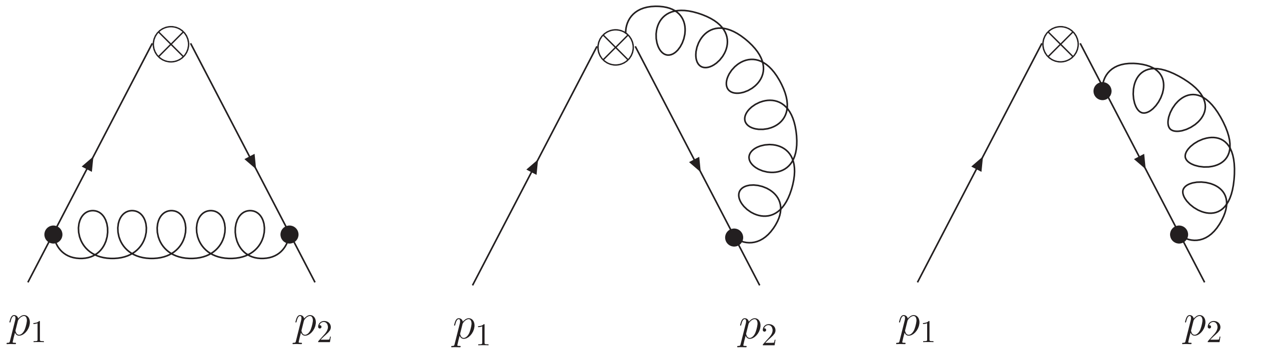
$$A^{\psi\psi}_{\{\mu_1,\dots,\mu_N\}} = \langle \psi(p_1) | O^{\psi}_{\{\mu_1,\dots,\mu_N\}}(p_3) | \overline{\psi}(p_2) \rangle$$

- Anomalous dimensions $\gamma(\alpha_s, N)$ govern scale dependence of renormalized operators

$$\frac{d}{d \ln \mu^2} O^{\rm ren} = -\gamma\, O^{\rm ren} \qquad\qquad \gamma(N) = -\int_0^1 dx\, x^{N-1} P(x)$$

- Zero-momentum transfer through operator reduces problem to computation of propagator-type diagrams

One-loop computation



- Computation of loop integral in $D = 4 - 2\epsilon$ dimensions and expansion in ϵ
  - anomalous dimension $\gamma(N)$ from ultraviolet divergence

$$\Delta^{\mu_1} \ldots \Delta^{\mu_N} \langle \psi(p_1) | O^{\psi}_{\{\mu_1,\ldots,\mu_N\}}(0) | \overline{\psi}(-p_1) \rangle =$$

$$= \quad 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \gamma^{(0)}(N) + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2)$$

$$= \quad 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \left\{ C_F \left(4S_1(N) + \frac{2}{N+1} - \frac{2}{N} - 3 \right) \right\} + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2)$$

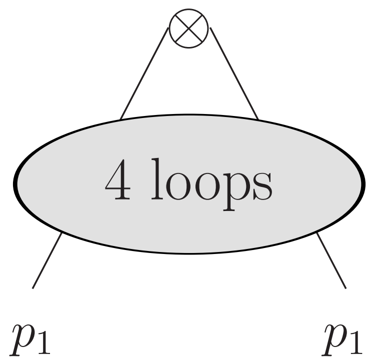
- One-loop result with harmonic sum $S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$
- Details in *The Theory of Quark and Gluon Interactions*
  F.J. Yndurain

Four-loop computation

- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira ‘91
- Parametric reduction of four-loop massless propagator diagrams with integration-by-parts identities encoded in **Forcer** Ruijl, Ueda, Vermaseren ‘17
- Symbolic manipulations with **Form** Vermaseren ‘00; Kuipers, Ueda, Vermaseren, Vollinga ‘12 and multi-threaded version **TForm** Tentyukov, Vermaseren ‘07
- Diagrams of same topology and color factor combined to meta diagrams
- Non-singlet anomalous dimension
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for $\gamma_{\rm ns}^{\pm}$
  - 1 three- and 29 four-loop meta diagrams for $\gamma_{\rm ns}^{\rm s}$

Fixed Mellin moments

- Computation of anomalous dimensions $\gamma(N)$ for Mellin moments mostly up to $N = 18$
  - sometimes higher for complicated topologies ($N = 19$, $N = 20$, . . .)
  - much higher for “easy” topologies, e.g., n_f-dependent ($N \simeq 80$, . . .)



Analytic reconstruction

- Sufficiently many Mellin moments allow for reconstruction of analytic all-N expressions through solution of Diophantine equations
- Anomalous dimensions $\gamma(N)$ given by harmonic sums up to weight 7

$$S_{\pm m_1,\dots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\dots,m_k}(i)$$

  - $2 \cdot 3^{w-1}$ sums at weight w
- Reciprocity relation (RR) $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(\alpha_s)\right)$ reduces number of 2^{w-1} sums at weight w for $\gamma_{\rm u}$
  - additional denominators with powers $1/(N+1)$ give $2^{w+1} - 1$ objects (255 at weight 7)
- Large-n_c limit only needs harmonic sums with positive index
  - weight w RR sums given by Fibonacci number $F(w)$
  - total number of unknowns (including powers $1/(N+1)$) amount to $F(w+4) - 2$ (87 at $w = 7$)
- Additional 46 constraints from large-x/small-x ($N \to \infty$/$N \to 0$) limit
- Solution becomes feasible with 18 Mellin moments for $\gamma_{\rm ns}^{\pm}$

Large-x behavior

The large x-limit: $x \to 1$

- Structure of diagonal splitting functions $P_{\rm ii}$ (for $\rm i = q, g$) at large x

$$P_{ii}^{(n-1)}(x) \;=\; \frac{A_{n,\rm i}}{(1-x)_+} + B_{n,\rm i}\,\delta(1-x) + C_{n,\rm i}\,\ln(1-x) + D_{n,\rm i}$$

- Cusp anomalous dimension $A_{n,\rm i}$ (known from $1/\epsilon^2$-poles of QCD form factor)
  Large-n_c (Henn, Lee, Smirnov, Smirnov, Steinhauser '16; S. M., Ruijl, Ueda, Vermaseren, Vogt '17);
  n_f terms (Grozin '18; Henn, Peraro, Stahlhofen, Wasser '19); n_f^2 terms (Davies, Ruijl, Ueda, Vermaseren, Vogt '16; Lee, Smirnov, Smirnov, Steinhauser '17); n_f^3 terms (Gracey '94; Beneke, Braun, '95);
  quartic colour factors (Lee, Smirnov, Smirnov, Steinhauser '19; Henn, Korchemsky, Mistlberger '19)
- virtual anomalous dimension $B_{n,\rm i}$ (parts related to $1/\epsilon$-poles of QCD form factor)
- subleading coefficients $C_{n,\rm i}$, $D_{n,\rm i}$ known from lower order cusp anomalous dimension (S.M., Vermaseren, Vogt '04, Dokshitzer, Marchesini, Salam '05)

Small-x behavior (I)

The small x-limit: $x \to 0$

- Structure of non-singlet splitting functions $P_{\rm ns}^{\pm}$ at small x
  - double-logarithmic contributions with very large coefficients
  - resummation for $P_{\rm ns}^{+}$ to leading logarithmic (LL) accuracy in Mellin-N space Kirschner, Lipatov '83

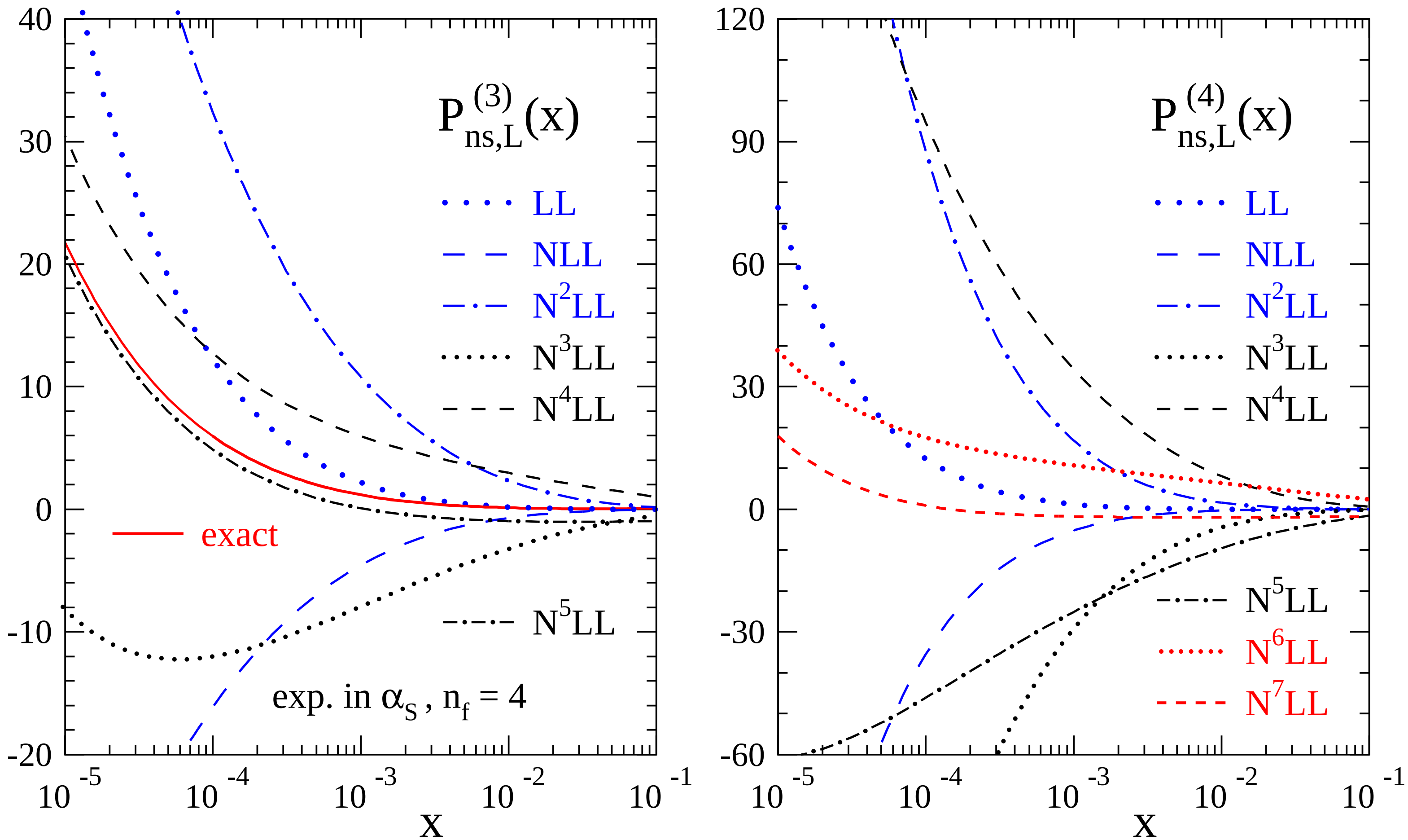
$$P_{\rm ns,LL}^{+}(N,\alpha_s) \;=\; \frac{N}{2}\left\{1-\left(1-\frac{2\alpha_s C_F}{\pi N^2}\right)^{1/2}\right\}$$

- Large-n_c limit with intriguing structure Velizhanin '14

$$P_{\rm ns}^{+}(N,\alpha_s)\,\left(P_{\rm ns}^{+}(N,\alpha_s)-N+\beta(\alpha_s)/\alpha_s\right) \;=\; O(1)$$

  - Laurent expansion about $N=0$
- Exploit structure of the (unfactorized) structure functions in dimensional regularization
- Resummation in terms of modified Bessel functions to N^{7}LL accuracy Davies, Kom, S.M., Vogt '22

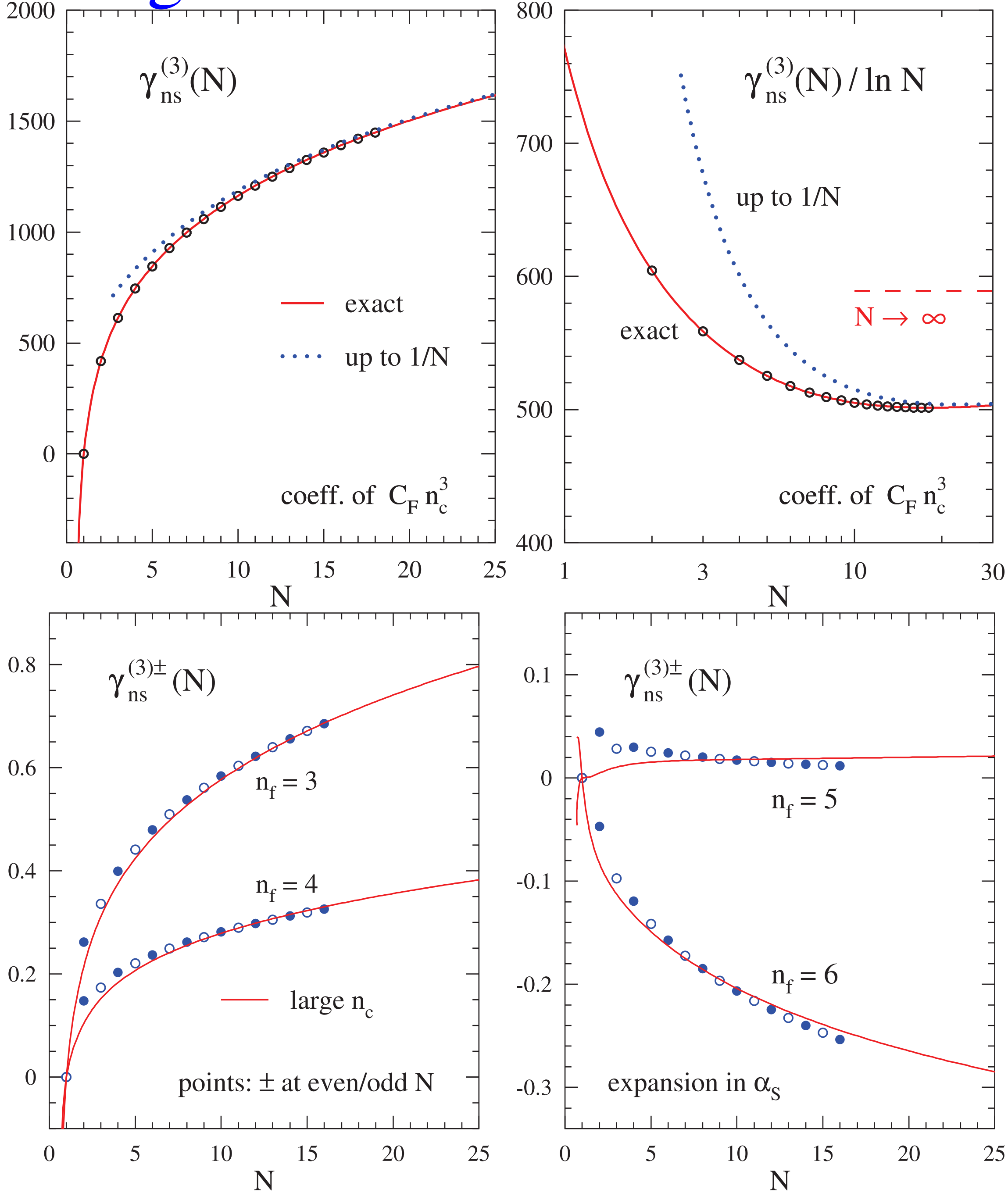
Small-x behavior (II)



- Splitting functions $P_{\rm ns}^{(3),+}$ (left) and $P_{\rm ns}^{(4),+}$ (right) Davies, Kom, S.M., Vogt '22
  - small-x approximations to the four-flavour splitting functions $P^{(n)}_{\rm ns,\,L}(x)$ in the large-n_c limit
  - predictions up to N^{7}LL

Four-loop non-singlet Mellin moments

- Top:
  n_f^0 part of anomalous dimensions $\gamma_{\rm ns}^{(3)\pm}(N)$ in large-n_c limit and large-N expansion

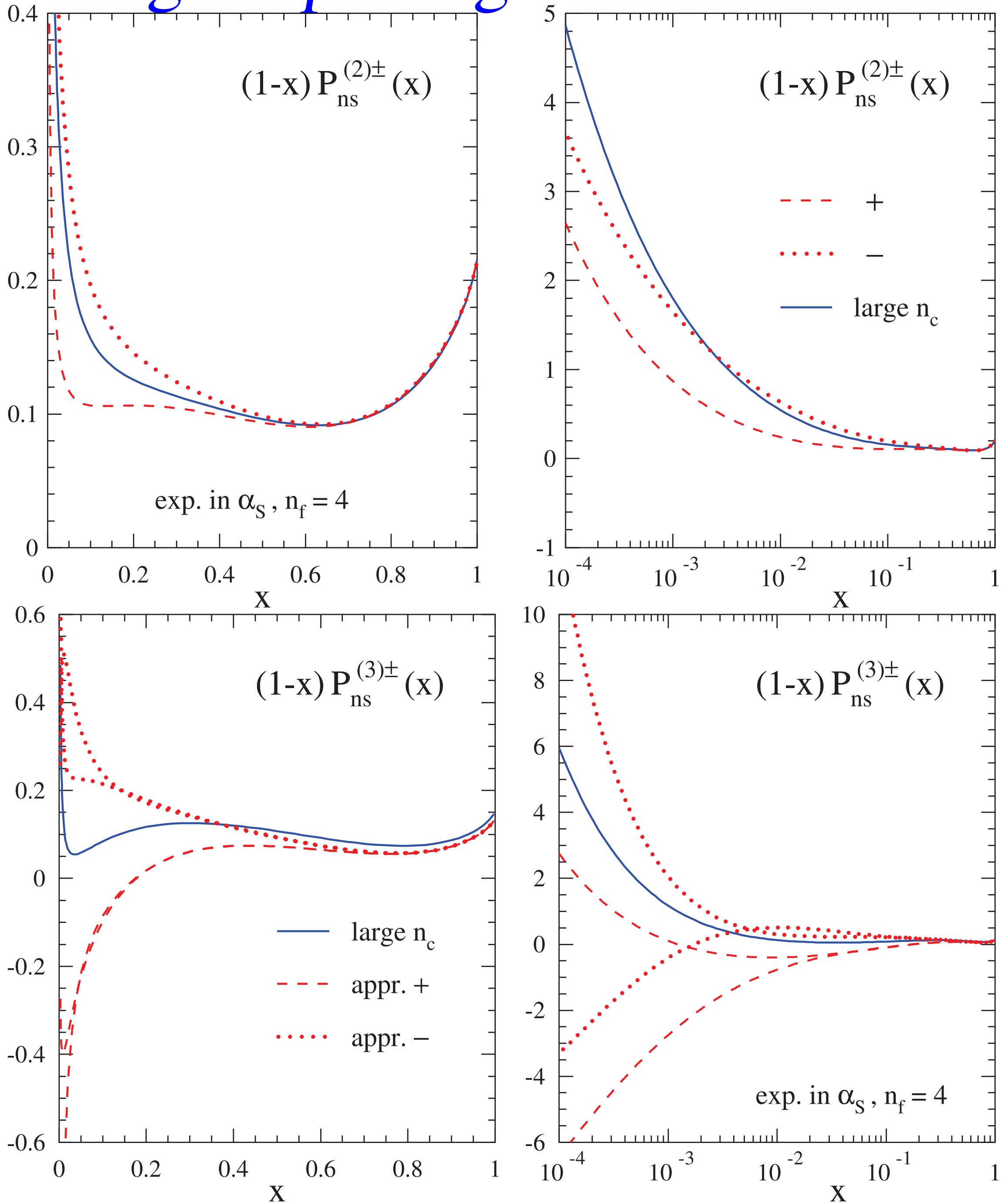


- Bottom: results for even-N ($\gamma_{\rm ns}^{(3)+}(N)$) and odd-N ($\gamma_{\rm ns}^{(3)-}(N)$) in large-n_c limit for $n_f = 3, \ldots 6$

Four-loop non-singlet splitting functions

- Top:
  three-loop $P_{\rm ns}^{(2)\pm}(x)$
  and large-n_c limit
  with $n_f = 4$

- Bottom:
  four-loop $P_{\rm ns}^{(3)\pm}(x)$
  and uncertainty bands
  beyond large-n_c limit
  with $n_f = 4$



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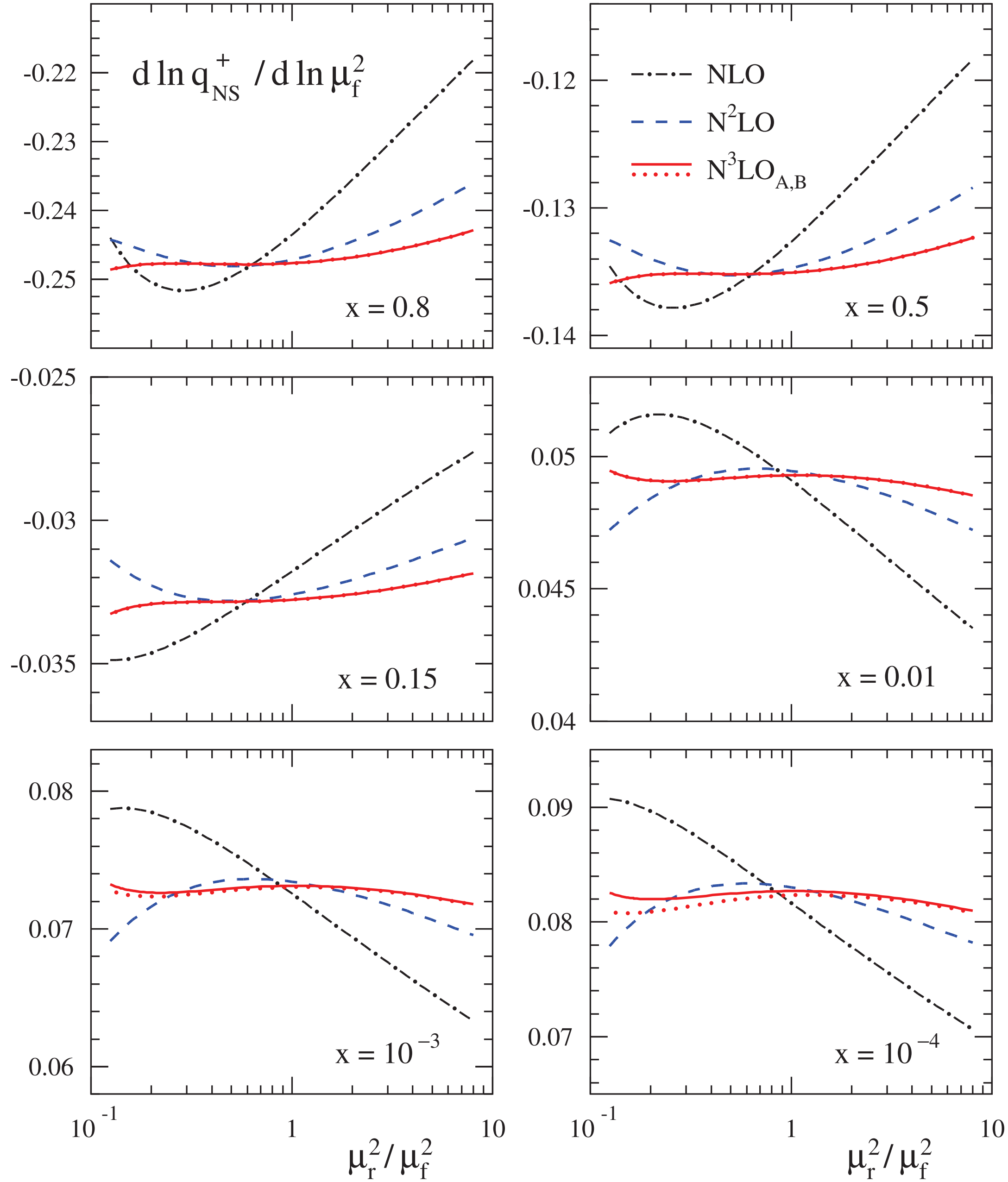
Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\rm ns}^{+} / d \ln \mu_f^2$
  - non-singlet shape

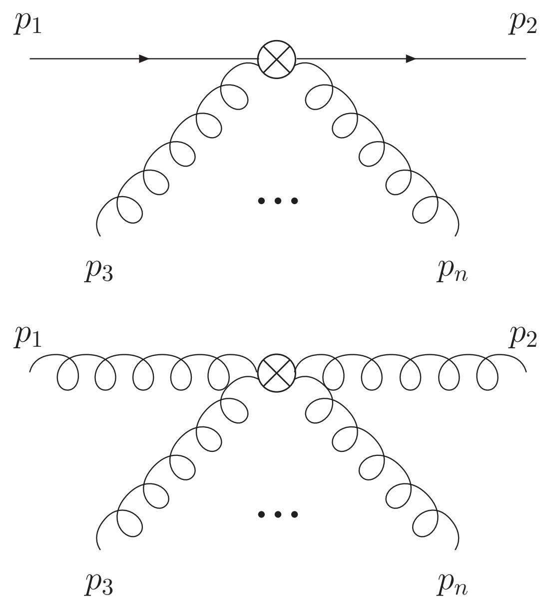
$$x q_{\rm ns}^{+}(x, \mu_0^2) \;=\; x^{\,0.5}(1-x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



Singlet

Operator matrix elements



- Singlet operators of spin-N and twist two

$$O^{\rm q}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi\ ,$$

$$O^{\rm g}_{\{\mu_1,\dots,\mu_N\}} = F_{\nu\{\mu_1} D_{\mu_2} \cdots D_{\mu_{N-1}} F_{\mu_N\}}{}^{\nu}$$

- Quartic Casimir terms at four loops are effectively 'leading-order'
  - $d^{(4)}_{xy} \equiv d_x^{abcd} d_y^{abcd}$ for representations labels x, y with generators T_r^a

$$d_r^{abcd} = \frac{1}{6}\,{\rm Tr}\,(\,T_r^a T_r^b T_r^c T_r^d + \text{ five } bcd \text{ permutations}\,)$$

  - anomalous dimensions fulfil relation for $\mathcal{N} = 1$ supersymmetry

$$\gamma^{(3)}_{\rm qq}(N) + \gamma^{(3)}_{\rm gq}(N) - \gamma^{(3)}_{\rm qg}(N) - \gamma^{(3)}_{\rm gg}(N) \stackrel{Q}{=} 0$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums
  - quartic Casimir terms fulfil stronger condition Belitsky, Müller, Schäfer '99

$$\gamma^{(0)}_{\rm qg}(N)\,\gamma^{(3)}_{\rm gq}(N) \stackrel{Q}{=} \gamma^{(0)}_{\rm gq}(N)\,\gamma^{(3)}_{\rm qg}(N)$$

Analytic results

- Reconstruction of analytic all-N expressions for ζ_5 terms from solution of Diophantine equations
  - example for $\gamma_{\rm gg}^{(3)}$ with $\eta = \frac{1}{N} - \frac{1}{N+1}$ and $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$\gamma_{\rm gg}^{(3)}(N)\Big|_{\zeta_5\, d_{AA}^{(4)}/n_A} = \frac{64}{3}\Big(30\,(12\,\eta^2 - 4\,\nu^2 - S_1(4\,S_1 + 8\,\eta - 8\,\nu - 11) - 7\nu) + 188\,\eta - \frac{751}{3} - \frac{1}{6}\,N\,(N+1)\Big)$$

- Recall large-N limit of anomalous dimensions

$$\gamma_{\rm ii}^{(k)}(N)\Big|_{N\to\infty} = A_{n,\rm i}\,\ln(N) + \mathcal{O}(\text{const}_N)$$

- Terms $S_1(N)^2 \sim \ln(N)^2$ and $N(N+1)$ proportional to ζ_5 must be compensated in large-N limit

Universal anomalous dimension

- Universal anomalous dimension $\gamma_{\rm uni}$ in $N = 4$ SYM to three loops
  Kotikov, Lipatov, Onishchenko, Velizhanin '04
  - One-loop example: $\gamma^{(0)}_{\rm uni}(N) = 4 n_c S_1$ emerges from

$$\gamma^{(0)}_{\rm qq}(N) = C_F \left(-3 + 2\frac{1}{N+1} - 2\frac{1}{N} + 4S_1 \right) \quad \text{or}$$

$$\gamma^{(0)}_{\rm gg}(N) = C_A \left(-\frac{11}{3} - \frac{4}{N-1} - \frac{4}{N+1} + \frac{4}{N+2} + \frac{4}{N} + 4S_1 \right) + \frac{2}{3} n_f$$

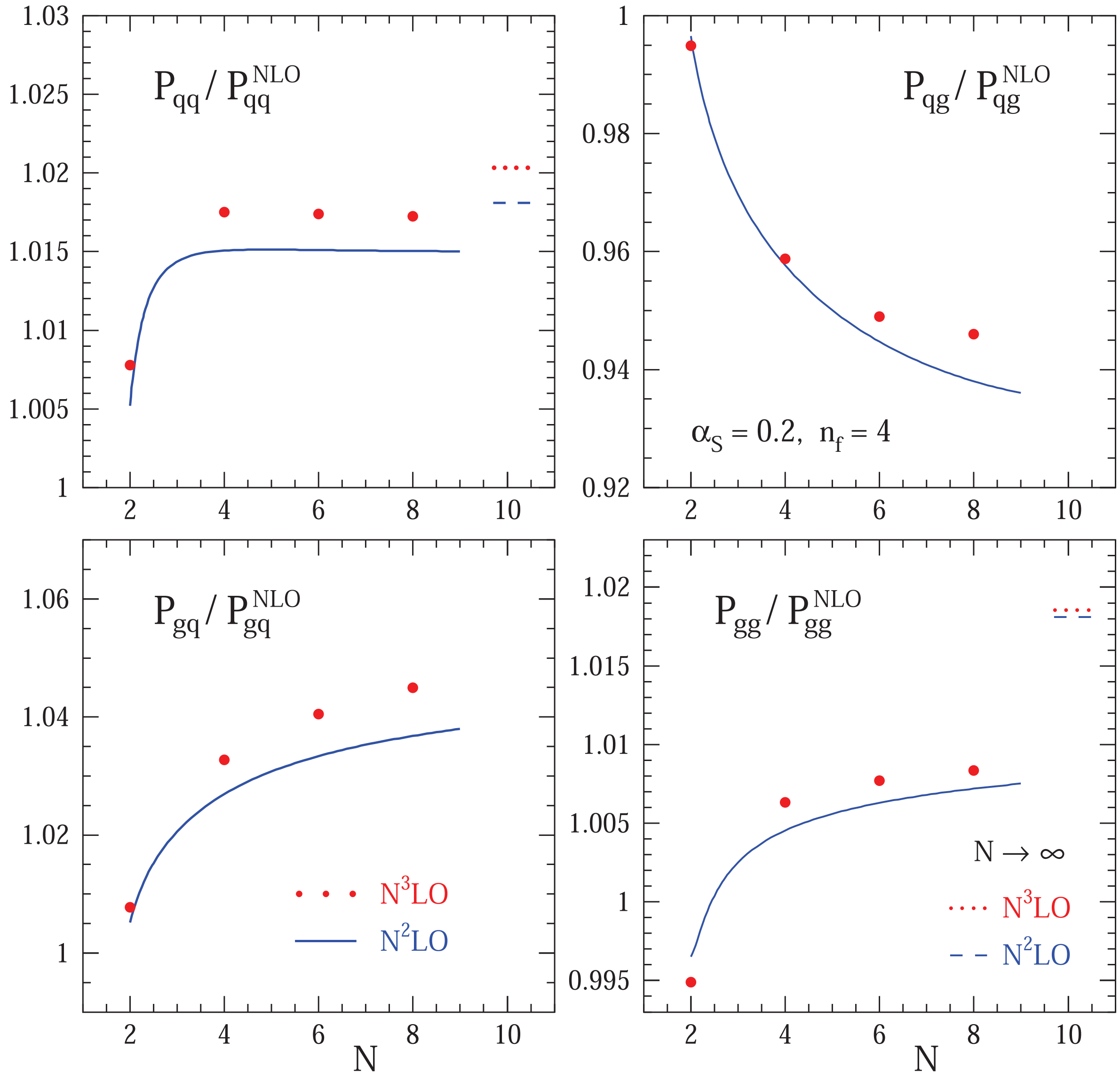
- Starting at four loops wrapping corrections to complement asymptotic Bethe ansatz
  - four-loop Bajnok, Janik, Lukowski '08, five-loop Lukowski, Rej, Velizhanin '09, six-loop [. . .], . . .
- $\gamma(N)^{\rm wrap,(4)} \simeq S_1(N)^2\, f^{\rm wrap}(N)$

$$f^{\rm wrap}(N) \;=\; 5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5$$

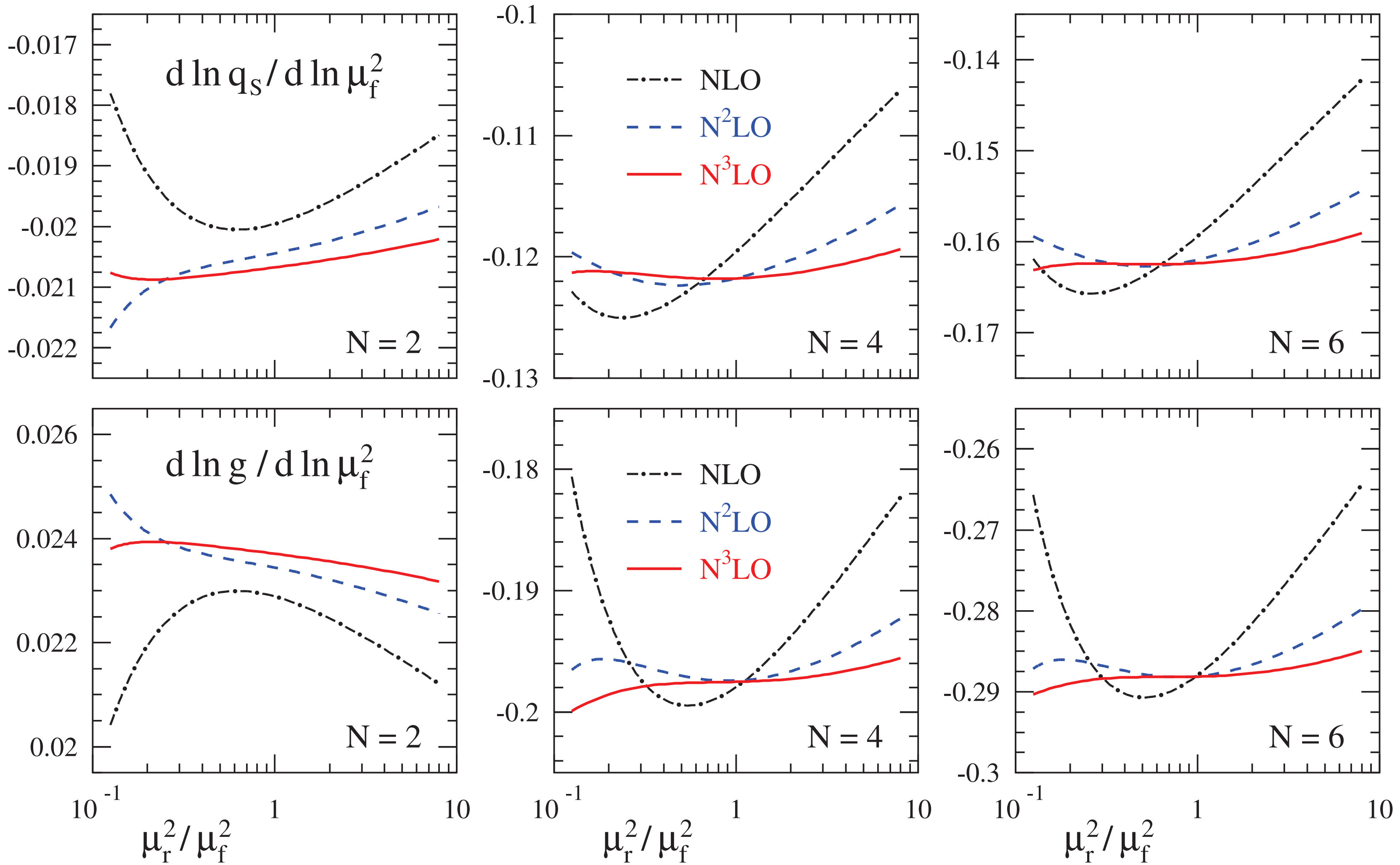
- Three-loop Wilson coefficient $c^{(3)}_{\rm ns}(N)$ S.M., Vermaseren, Vogt '05
  - $c^{(3)}_{\rm ns}(N) \simeq C_F \left(C_F - \frac{C_A}{2} \right)^2 \left\{ N(N+1)\, f^{\rm wrap}(N) \right\}$
- Non-planar part of $\gamma_{\rm uni}$ in $N = 4$ SYM at four loops Kniehl, Velizhanin '21

Four-loop singlet Mellin moments

- Singlet moments at NNLO (lines) and N^3LO (even-N points) normalized to NLO results
  - $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$
- Large-N limits in qq- and gg-channel



Scale stability of singlet evolution



- Renormalization-scale dependence of singlet PDFs $d \ln q_s^{\pm} / d \ln \mu_f^2$ and $d \ln g^{\pm} / d \ln \mu_f^2$ at $N = 2,\ 4,$ and 6 using NLO, NNLO and N^{3}LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

Five-loop Mellin moments

Five-loop Mellin moments

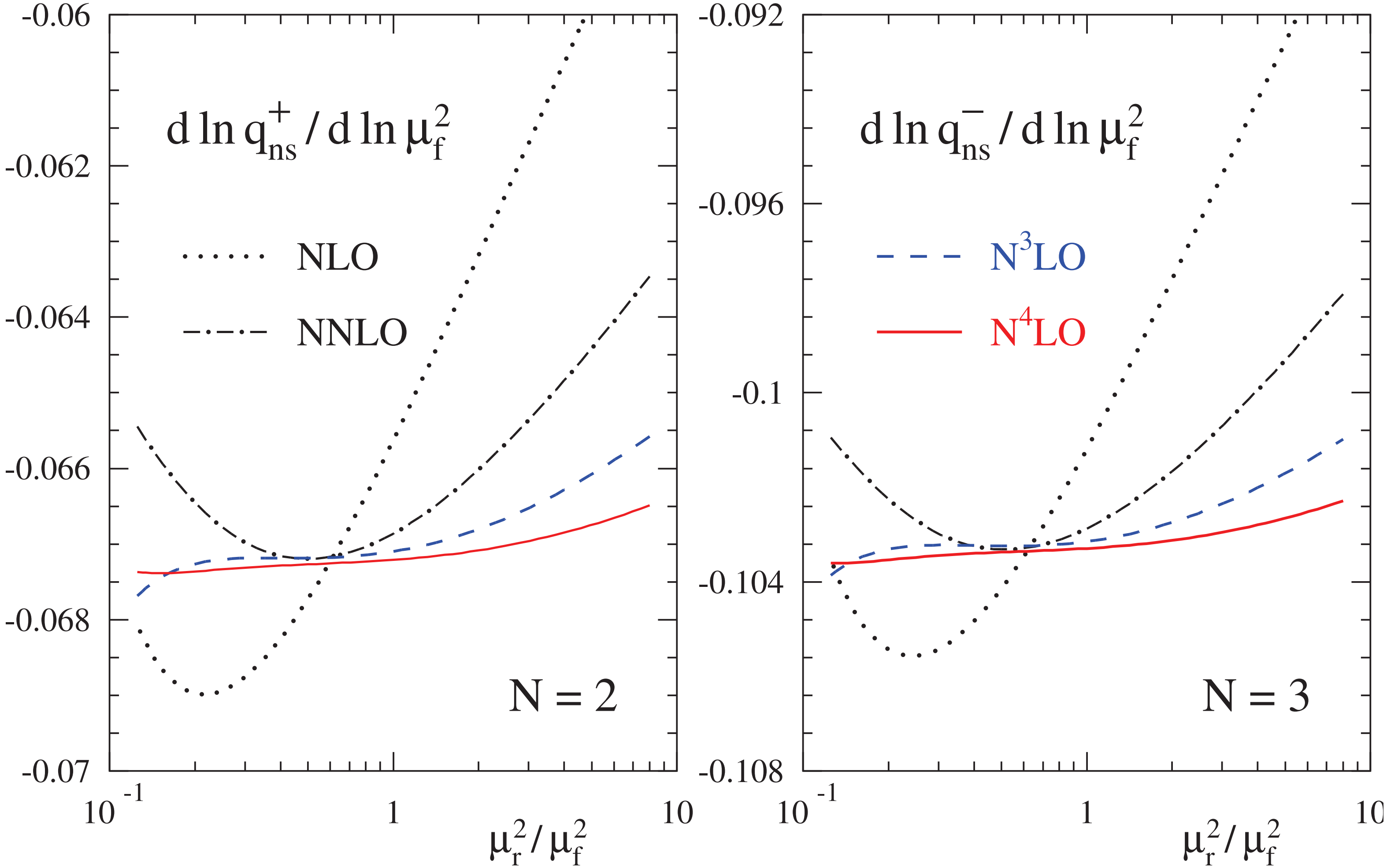
- Moments $N = 2$ and $N = 3$ for nonsinglet anomalous dimensions $\gamma_{\rm ns}^{\pm}$
- Implementation by Herzog, Ruijl '17 of local R^* operation Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '84 for reduction of five-loop self-energy diagrams to four-loop ones computed with **Forcer** Ruijl, Ueda, Vermaseren '17

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)+}(N=2) \;=\;
& C_F^5 \left[\frac{9306376}{19683} - \frac{802784}{729}\zeta_3 - \frac{557440}{81}\zeta_5 + \frac{12544}{9}\zeta_3^2 + 8512\zeta_7 \right] \\
& - C_A C_F^4 \left[\frac{81862744}{19683} - \frac{1600592}{243}\zeta_3 + \frac{59840}{81}\zeta_4 - \frac{142240}{27}\zeta_5 + 3072\zeta_3^2 - \frac{35200}{9}\zeta_6 + 19936\zeta_7 \right] \\
& + C_A^2 C_F^3 \left[\frac{63340406}{6561} - \frac{1003192}{243}\zeta_3 - \frac{229472}{81}\zeta_4 + \frac{61696}{27}\zeta_5 + \frac{30976}{9}\zeta_3^2 - \frac{35200}{9}\zeta_6 + 15680\zeta_7 \right] \\
& - C_A^3 C_F^2 \left[\frac{220224724}{19683} + \frac{4115536}{729}\zeta_3 - \frac{170968}{27}\zeta_4 - \frac{3640624}{243}\zeta_5 + \frac{70400}{27}\zeta_3^2 + \frac{123200}{27}\zeta_6 + \frac{331856}{27}\zeta_7 \right] \\
& + C_A^4 C_F \left[\frac{266532611}{39366} + \frac{2588144}{729}\zeta_3 - \frac{221920}{81}\zeta_4 - \frac{3102208}{243}\zeta_5 + \frac{74912}{81}\zeta_3^2 + \frac{334400}{81}\zeta_6 + \frac{178976}{27}\zeta_7 \right] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \left[\frac{15344}{81} - \frac{12064}{27}\zeta_3 - \frac{704}{3}\zeta_4 + \frac{58400}{27}\zeta_5 - \frac{6016}{3}\zeta_3^2 - \frac{19040}{9}\zeta_7 \right] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_F \left[\frac{23968}{81} - \frac{733504}{81}\zeta_3 + \frac{176320}{81}\zeta_5 + \frac{6400}{3}\zeta_3^2 + \frac{77056}{9}\zeta_7 \right] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_A \left[\frac{82768}{81} - \frac{555520}{81}\zeta_3 + \frac{10912}{9}\zeta_4 - \frac{1292960}{81}\zeta_5 + \frac{84352}{27}\zeta_3^2 + \frac{140800}{27}\zeta_6 + 12768\zeta_7 \right] \\
& + n_f C_F^4 \left[\frac{1824964}{19683} - \frac{463520}{243}\zeta_3 + \frac{21248}{81}\zeta_4 - \frac{16480}{81}\zeta_5 + \frac{6656}{9}\zeta_3^2 - \frac{6400}{9}\zeta_6 + \frac{8960}{3}\zeta_7 \right] \\
& - n_f C_A C_F^3 \left[\frac{3375082}{6561} - \frac{420068}{243}\zeta_3 - \frac{48256}{81}\zeta_4 + \frac{458032}{81}\zeta_5 + \frac{3968}{3}\zeta_3^2 - \frac{8000}{3}\zeta_6 + \frac{4480}{3}\zeta_7 \right] \\
& + n_f C_A^2 C_F^2 \left[\frac{15291499}{13122} + \frac{1561600}{243}\zeta_3 - \frac{114536}{27}\zeta_4 - \frac{252544}{243}\zeta_5 + \frac{24896}{27}\zeta_3^2 + \frac{13600}{27}\zeta_6 + \frac{11200}{27}\zeta_7 \right] \\
& - n_f C_A^3 C_F \left[\frac{48846580}{19683} + \frac{4314308}{729}\zeta_3 - \frac{274768}{81}\zeta_4 - \frac{1389080}{243}\zeta_5 + \frac{27808}{81}\zeta_3^2 + \frac{184000}{81}\zeta_6 + \frac{39088}{27}\zeta_7 \right] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \left[\frac{22096}{27} + \frac{43712}{81}\zeta_3 - \frac{512}{9}\zeta_4 - \frac{217280}{81}\zeta_5 - \frac{25088}{27}\zeta_3^2 + \frac{25600}{27}\zeta_6 - 2464\zeta_7 \right] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \left[\frac{170752}{81} - \frac{328832}{81}\zeta_3 + \frac{650240}{81}\zeta_5 - \frac{8192}{9}\zeta_3^2 - \frac{35840}{9}\zeta_7 \right] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \left[\frac{207824}{81} + \frac{251392}{81}\zeta_3 - \frac{5632}{9}\zeta_4 - \frac{522880}{81}\zeta_5 + \frac{15872}{27}\zeta_3^2 + \frac{70400}{27}\zeta_6 - \frac{29120}{9}\zeta_7 \right] \\
& + n_f^2 C_F^3 \left[\frac{1082297}{6561} - \frac{145792}{243}\zeta_3 + \frac{1072}{81}\zeta_4 + \frac{55552}{81}\zeta_5 + \frac{1792}{9}\zeta_3^2 - \frac{3200}{9}\zeta_6 \right] \\
& + n_f^2 C_A C_F^2 \left[\frac{332254}{2187} - \frac{85016}{243}\zeta_3 + \frac{20752}{27}\zeta_4 - \frac{28544}{81}\zeta_5 - \frac{13952}{27}\zeta_3^2 + \frac{1600}{27}\zeta_6 \right] \\
& + n_f^2 C_A^2 C_F \left[\frac{631400}{6561} + \frac{214268}{243}\zeta_3 - 784\zeta_4 - \frac{53344}{243}\zeta_5 + \frac{25472}{81}\zeta_3^2 + \frac{22400}{81}\zeta_6 \right] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[\frac{43744}{81} - \frac{35648}{81}\zeta_3 - \frac{1792}{9}\zeta_4 - \frac{52480}{81}\zeta_5 + \frac{2048}{27}\zeta_3^2 + \frac{12800}{27}\zeta_6 \right] \\
& + n_f^3 C_F^2 \left[\frac{265510}{19683} + \frac{11872}{729}\zeta_3 - \frac{128}{3}\zeta_4 + \frac{512}{27}\zeta_5 \right] \\
& + n_f^3 C_A C_F \left[\frac{168677}{19683} + \frac{11872}{729}\zeta_3 + \frac{2752}{81}\zeta_4 - \frac{4096}{81}\zeta_5 \right] - n_f^4 C_F \left[\frac{5504}{19683} + \frac{1024}{729}\zeta_3 - \frac{128}{81}\zeta_4 \right]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)-}(N=3) \;=\;
& C_F^5 \left[\frac{81472935625}{80621568} + \frac{99382175}{23328}\zeta_3 - \frac{3395975}{162}\zeta_5 - \frac{9650}{9}\zeta_3^2 + \frac{34685}{2}\zeta_7 \right] \\
& - C_A C_F^4 \left[\frac{286028134219}{80621568} - \frac{23916529}{7776}\zeta_3 - 4490\zeta_3^2 + \frac{134090}{81}\zeta_4 - \frac{2468075}{108}\zeta_5 - \frac{55000}{9}\zeta_6 + \frac{155155}{4}\zeta_7 \right] \\
& + C_A^2 C_F^3 \left[\frac{20173099267}{3359232} - \frac{15401281}{864}\zeta_3 + \frac{732787}{1296}\zeta_4 + \frac{1972075}{216}\zeta_5 - \frac{63830}{9}\zeta_3^2 - \frac{79750}{9}\zeta_6 + \frac{139895}{4}\zeta_7 \right] \\
& - C_A^3 C_F^2 \left[\frac{166662991819}{20155392} - \frac{36397493}{2916}\zeta_3 - \frac{103763}{54}\zeta_4 + \frac{30994565}{3888}\zeta_5 - \frac{133990}{27}\zeta_3^2 - \frac{72875}{54}\zeta_6 + \frac{2127335}{108}\zeta_7 \right] \\
& + C_A^4 C_F \left[\frac{75932079965}{10077696} - \frac{27693563}{23328}\zeta_3 - \frac{1791229}{1296}\zeta_4 - \frac{9417425}{1944}\zeta_5 - \frac{96700}{81}\zeta_3^2 + \frac{163625}{81}\zeta_6 + \frac{199640}{27}\zeta_7 \right] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \left[\frac{81725}{162} - \frac{33505}{18}\zeta_3 - \frac{1100}{3}\zeta_4 + \frac{52025}{18}\zeta_5 - \frac{7000}{3}\zeta_3^2 - \frac{48125}{36}\zeta_7 \right] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_F \left[\frac{231575}{36} + \frac{6351445}{324}\zeta_3 - \frac{2927225}{162}\zeta_5 + \frac{23210}{3}\zeta_3^2 - \frac{200410}{9}\zeta_7 \right] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_A \left[\frac{165871}{54} + \frac{1816625}{162}\zeta_3 - \frac{41800}{9}\zeta_4 - \frac{4456145}{162}\zeta_5 + \frac{196880}{27}\zeta_3^2 + \frac{200750}{27}\zeta_6 - \frac{7525}{4}\zeta_7 \right] \\
& + n_f C_F^4 \left[\frac{1776521549}{40310784} - \frac{1332919}{486}\zeta_3 + \frac{5000}{9}\zeta_3^2 + \frac{33290}{81}\zeta_4 - \frac{30325}{81}\zeta_5 - \frac{10000}{9}\zeta_6 + \frac{14000}{3}\zeta_7 \right] \\
& - n_f C_A C_F^3 \left[\frac{3737356319}{3359232} - \frac{2327111}{432}\zeta_3 + \frac{1280}{3}\zeta_3^2 + \frac{262069}{648}\zeta_4 + \frac{1693715}{162}\zeta_5 - \frac{14000}{3}\zeta_6 + \frac{7000}{3}\zeta_7 \right] \\
& + n_f C_A^2 C_F^2 \left[\frac{5637513931}{3359232} + \frac{2711207}{486}\zeta_3 - \frac{5020}{27}\zeta_3^2 - \frac{457499}{108}\zeta_4 + \frac{508820}{243}\zeta_5 - \frac{20375}{27}\zeta_6 + \frac{50155}{108}\zeta_7 \right] \\
& - n_f C_A^3 C_F \left[\frac{8766012215}{2519424} + \frac{45697231}{5832}\zeta_3 + \frac{1195}{81}\zeta_3^2 - \frac{2848403}{648}\zeta_4 - \frac{1808870}{243}\zeta_5 + \frac{222250}{81}\zeta_6 + \frac{250915}{108}\zeta_7 \right] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \left[\frac{24385}{27} - \frac{334010}{81}\zeta_3 - \frac{8480}{9}\zeta_3^2 + \frac{1622600}{81}\zeta_5 - \frac{135380}{9}\zeta_7 \right] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \left[\frac{297889}{162} + \frac{154970}{81}\zeta_3 - \frac{62600}{27}\zeta_3^2 + \frac{3700}{9}\zeta_4 - \frac{122780}{81}\zeta_5 - \frac{36500}{27}\zeta_6 - 910\zeta_7 \right] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \left[\frac{241835}{162} + \frac{333487}{81}\zeta_3 + \frac{30560}{27}\zeta_3^2 - 10780/9\,\zeta_4 - \frac{316900}{81}\zeta_5 + \frac{110000}{27}\zeta_6 - \frac{71960}{9}\zeta_7 \right] \\
& + n_f^2 C_F^3 \left[\frac{512848319}{1679616} - \frac{57109}{54}\zeta_3 + \frac{2800}{9}\zeta_3^2 + \frac{9118}{81}\zeta_4 + \frac{86440}{81}\zeta_5 - \frac{5000}{9}\zeta_6 \right] \\
& + n_f^2 C_A C_F^2 \left[\frac{1080083}{5832} - \frac{296729}{972}\zeta_3 - \frac{21800}{27}\zeta_3^2 + \frac{56327}{54}\zeta_4 - \frac{42860}{81}\zeta_5 + \frac{2500}{27}\zeta_6 \right] \\
& + n_f^2 C_A^2 C_F \left[\frac{61747877}{419904} + \frac{2496811}{1944}\zeta_3 + \frac{39800}{81}\zeta_3^2 - \frac{3503}{3}\zeta_4 - \frac{88990}{243}\zeta_5 + \frac{35000}{81}\zeta_6 \right] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[\frac{19435}{27} - \frac{53366}{81}\zeta_3 + \frac{3200}{27}\zeta_3^2 - \frac{3160}{9}\zeta_4 - \frac{70000}{81}\zeta_5 + \frac{20000}{27}\zeta_6 \right] \\
& + n_f^3 C_F^2 \left[\frac{28758139}{1259712} + \frac{21673}{729}\zeta_3 - \frac{610}{9}\zeta_4 + \frac{800}{27}\zeta_5 \right] \\
& + n_f^3 C_A C_F \left[\frac{13729181}{1259712} + \frac{14947}{729}\zeta_3 + \frac{4390}{81}\zeta_4 - \frac{6400}{81}\zeta_5 \right] - n_f^4 C_F \left[\frac{259993}{629856} + \frac{1660}{729}\zeta_3 - \frac{200}{81}\zeta_4 \right]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)\,\rm v}(N=3) \;=\; \gamma_{\rm ns}^{(4)-}(N=3)
& + n_f \frac{d_{abc}d^{abc}}{N_F} \Bigg\{ C_F^2 \left[\frac{79906955}{46656} + \frac{246955}{54}\zeta_3 - \frac{504550}{81}\zeta_5 \right] \\
& - C_A C_F \left[\frac{9797321}{3888} - \frac{475655}{54}\zeta_3 + \frac{17600}{9}\zeta_4 + \frac{516950}{81}\zeta_5 - \frac{500}{9}\zeta_3^2 + \frac{2800}{9}\zeta_7 \right] \\
& + C_A^2 \left[\frac{166535}{486} - \frac{1783913}{324}\zeta_3 + \frac{5555}{9}\zeta_4 + \frac{507515}{81}\zeta_5 - \frac{2035}{27}\zeta_3^2 - \frac{5500}{27}\zeta_6 - \frac{2765}{18}\zeta_7 \right] \\
& + n_f C_A \left[\frac{285985}{3888} + \frac{41954}{81}\zeta_3 + \frac{160}{27}\zeta_3^2 - \frac{1010}{9}\zeta_4 - \frac{56480}{81}\zeta_5 + \frac{1000}{27}\zeta_6 \right] \\
& + n_f C_F \left[\frac{1098323}{3888} - \frac{49720}{81}\zeta_3 + \frac{3200}{9}\zeta_4 \right] - n_f^2 \left[\frac{21823}{1944} \right] \Bigg\}
\end{aligned}
$$

Scale stability of evolution



- Renormalization-scale dependence of $d\ln q^{\pm}_{\rm ns}/d\ln\mu_f^2$ at $N = 2$ and $N = 3$ using NLO, NNLO, N^{3}LO and N^{4}LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

Coefficient functions at four loops

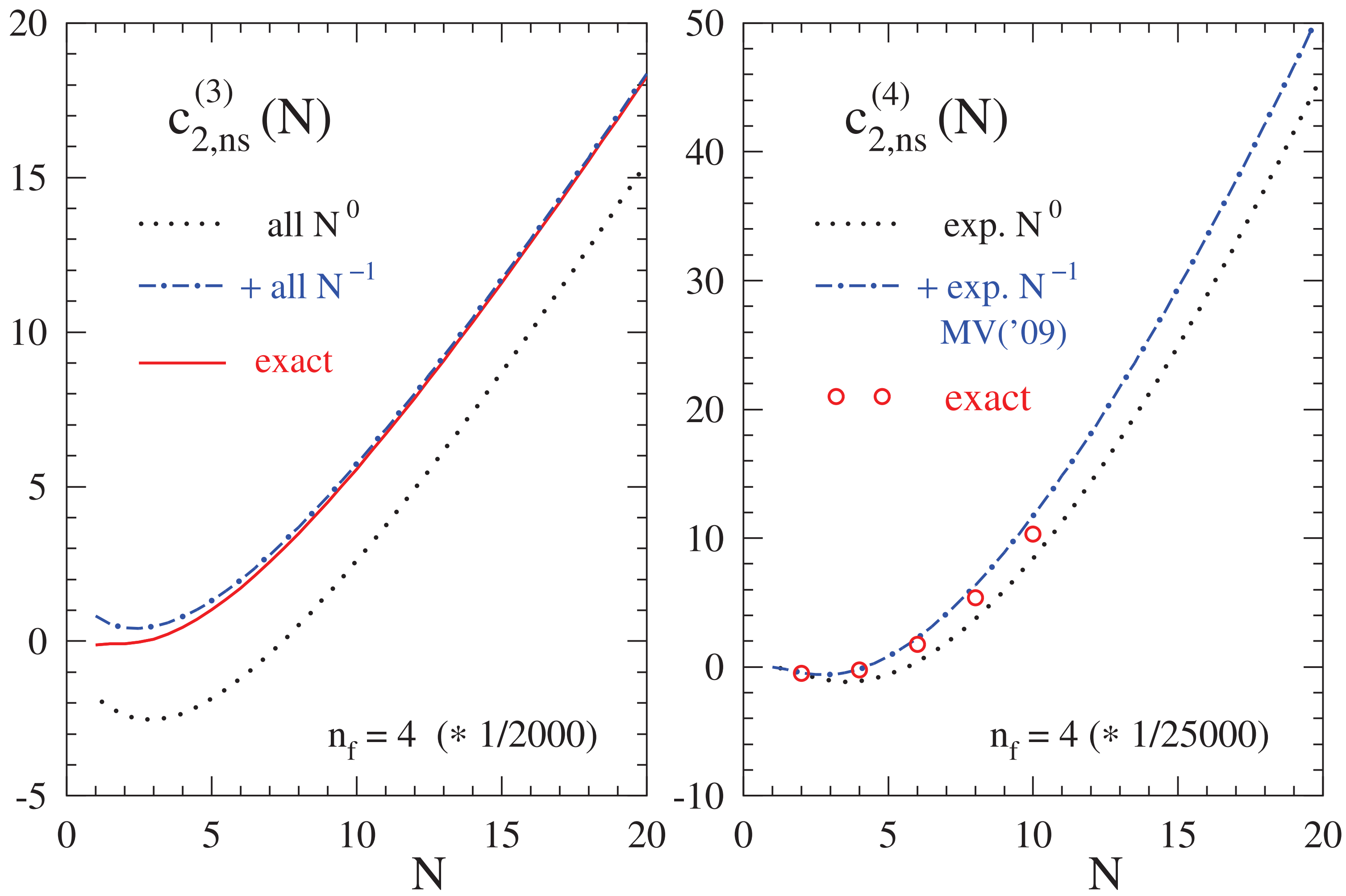
Four-loop non-singlet Mellin moments

- Perturbative expansion of non-singlet coefficient functions
  - Mellin moments $N = 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14$ of $C_{2,\mathrm{ns}}$ and $C_{L,\mathrm{ns}}$ (moments $N = 12,\ 14$ in limit of large n_c)
  - Mellin moments $N = 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15$ of $C_{3,-}$ (moments $N = 11,\ 13,\ 15$ in limit of large n_c)
- Numerical results for $C_{2,\mathrm{ns}}(N, n_f)$

S.M., Ruijl, Ueda, Vermaseren, Vogt *to appear*

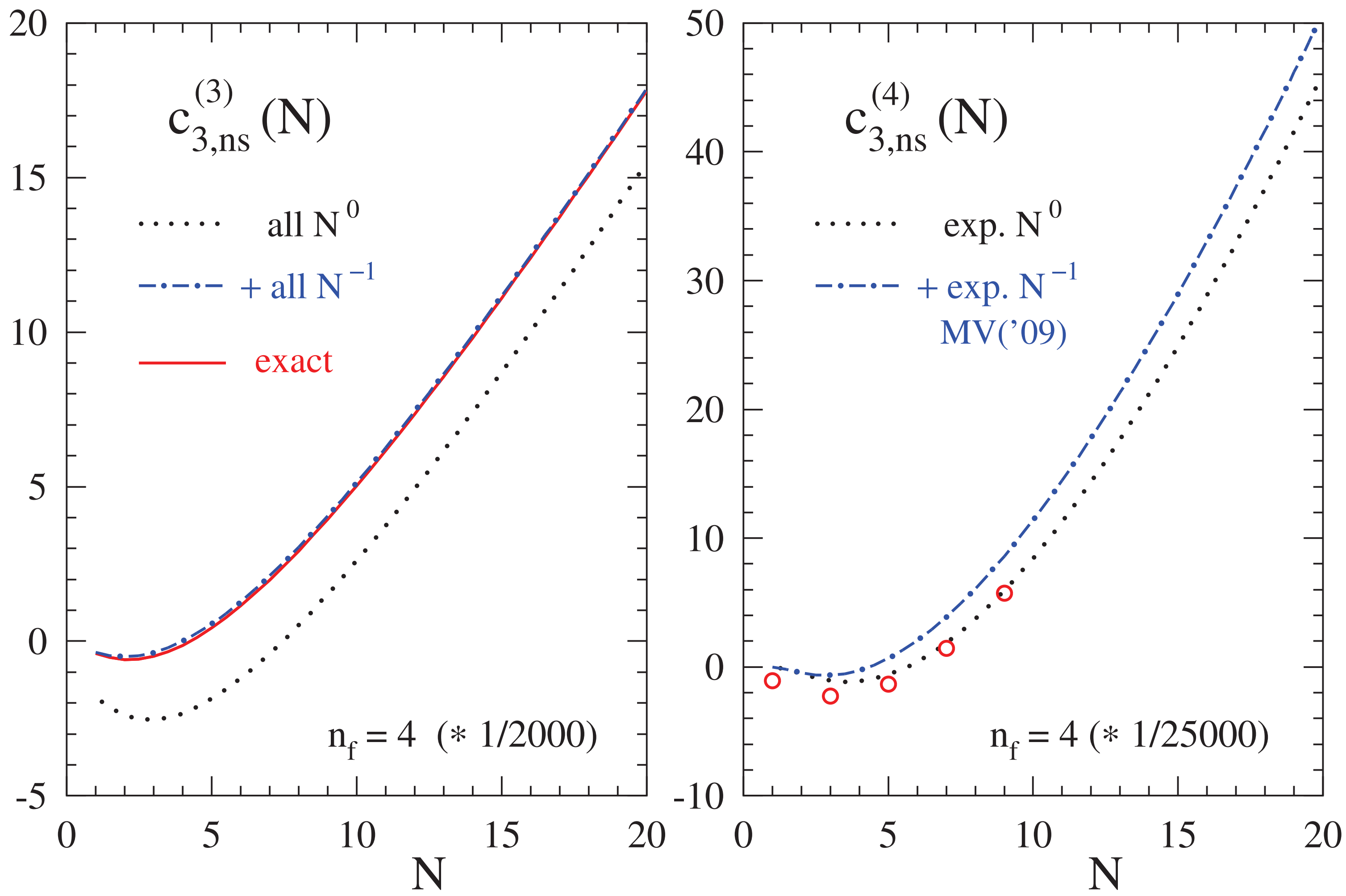
$$
\begin{aligned}
C_{2,\mathrm{ns}}(2,4) &= 1 + 0.0354\ \alpha_s - 0.0231\ \alpha_s^2 - 0.0613\ \alpha_s^3 - 0.4746\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(4,4) &= 1 + 0.4828\ \alpha_s + 0.4711\ \alpha_s^2 + 0.4727\ \alpha_s^3 - 0.2458\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(6,4) &= 1 + 0.8894\ \alpha_s + 1.2054\ \alpha_s^2 + 1.7572\ \alpha_s^3 + 1.7748\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(8,4) &= 1 + 1.2358\ \alpha_s + 2.0208\ \alpha_s^2 + 3.5294\ \alpha_s^3 + 5.3921\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(10,4) &= 1 + 1.5359\ \alpha_s + 2.8608\ \alpha_s^2 + 5.6244\ \alpha_s^3 + 10.324\ \alpha_s^4\ .
\end{aligned}
$$

Four-loop non-singlet Mellin moments



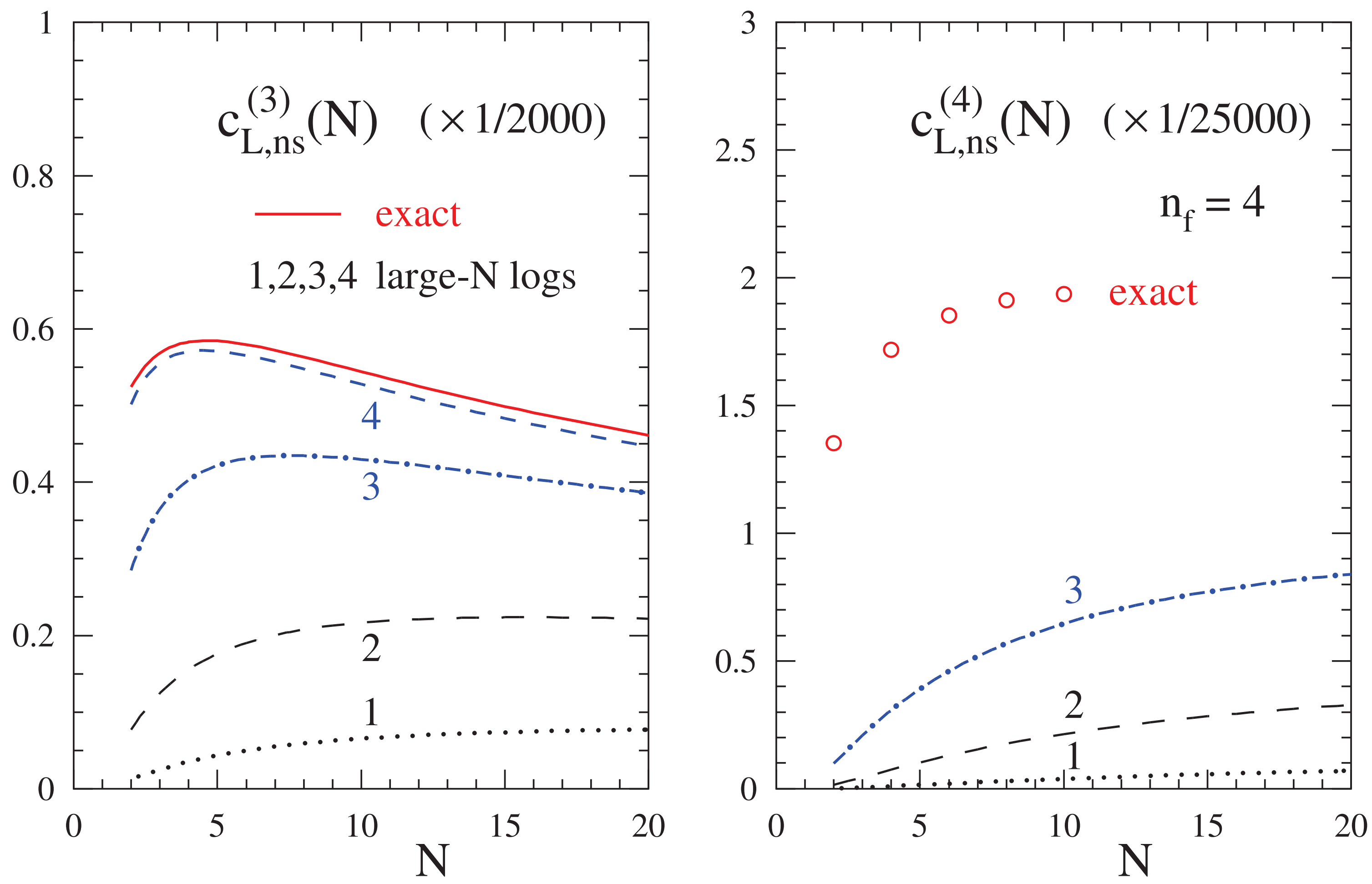
- Exact results for $c^{(3)}_{2,\rm ns}$ ($\mathrm{N^3LO}$) at $n_f = 4$ (rescaled by $2000 \simeq (4\pi)^3$)
- Moments for $c^{(4)}_{2,\rm ns}$ ($\mathrm{N^4LO}$) at $n_f = 4$ (rescaled by $25000 \simeq (4\pi)^4$)
- Comparison with contributions provided by large-N resummations

Four-loop non-singlet Mellin moments



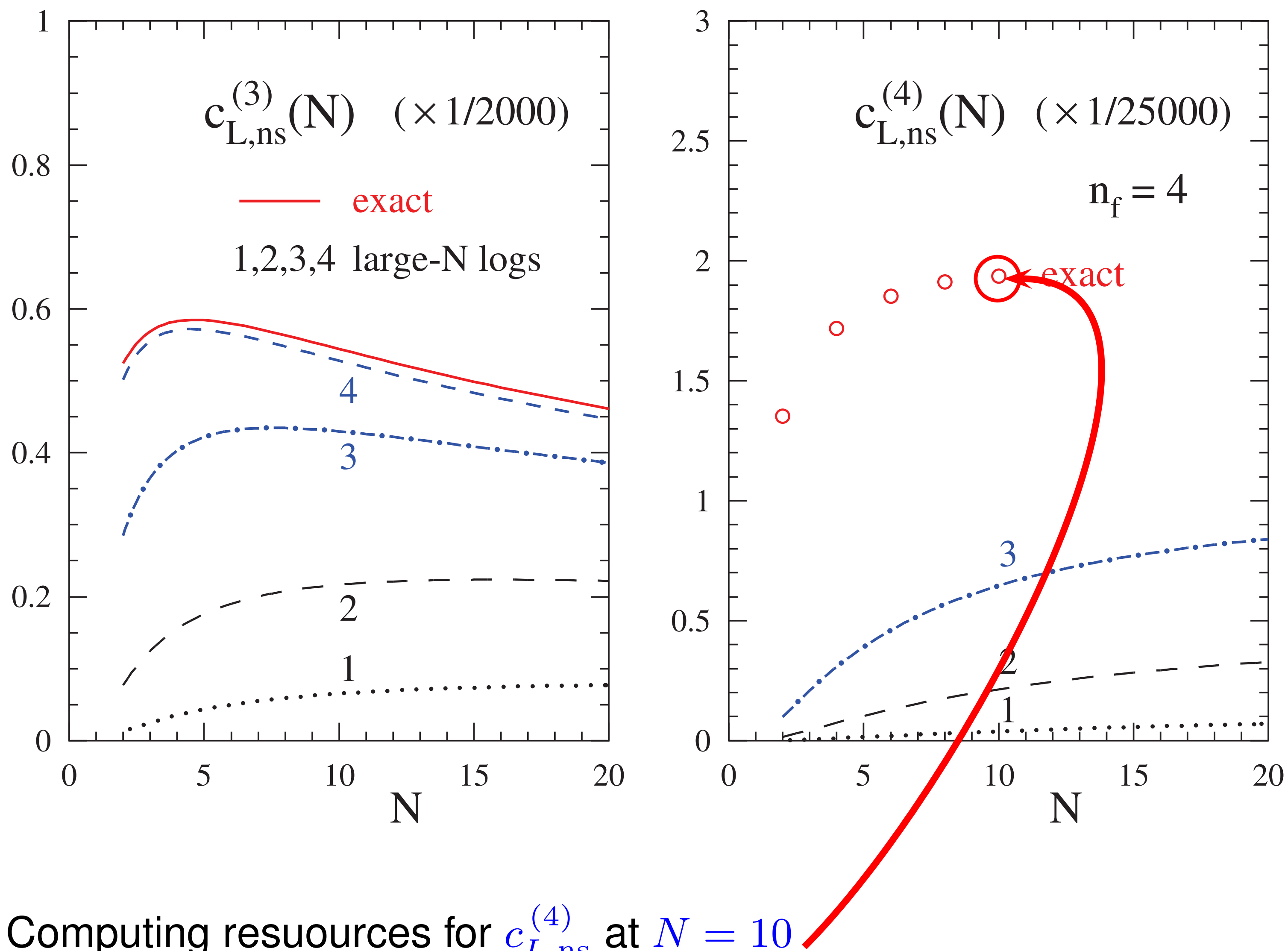
- Exact results for $c^{(3)}_{3,\mathrm{ns}}$ (N^{3}LO) at $n_f = 4$ (rescaled by $2000 \simeq (4\pi)^3$)
- Moments for $c^{(4)}_{3,\mathrm{ns}}$ (N^{4}LO) at $n_f = 4$ (rescaled by $25000 \simeq (4\pi)^4$)
- Comparison with contributions provided by large-N resummations

Four-loop non-singlet Mellin moments



- Exact results for $c^{(3)}_{L,\mathrm{ns}}$ (N^{3}LO) and moments for $c^{(4)}_{3,\mathrm{ns}}$ (N^{4}LO) at $n_f = 4$
- Tower of logarithms $\ln^4(N)/N$, ..., $\ln(N)/N$ at N^{3}LO
- Tower of logarithms $\ln^6(N)/N$, ..., $\ln^4(N)/N$ at N^{4}LO

Four-loop non-singlet Mellin moments



- Computing resuources for $c^{(4)}_{L,\mathrm{ns}}$ at $N = 10$
  - single core CPU time $\mathcal{O}(800.000)$h (**TForm** speed-up is $\mathcal{O}(10)$h)
  - $\mathcal{O}(20)$ TByte of disk space at intermediate stages of computation

Threshold resummation

- Coefficient function in large x-limit have large logarithms at n$^{\rm th}$-order

$$\alpha_s^n \, \frac{\ln^{2n-1}(1-x)}{(1-x)_+} \longleftrightarrow \alpha_s^n \, \ln^{2n}(N)$$

- Threshold resummation in Mellin space

$$C^N \; = \; (1 + \alpha_s \, g_{01} + \alpha_s^2 \, g_{02} + \ldots) \cdot \exp\left(G^N\right) \; + \; \mathcal{O}(N^{-1} \ln^n N)$$

- Control over logarithms $\ln(N)$ with $\lambda = \beta_0 \alpha_s \ln(N)$ to N^{k}LL accuracy

$$G^N \quad = \quad \ln(N) g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \ldots$$

  - $g_1(\lambda)$: LL Sterman ‘87; Appell, Mackenzie, Sterman ‘88
  - $g_2(\lambda)$: NLL Catani Trenatdue ‘89
  - $g_3(\lambda)$: NNLL or N^{2}LL Vogt ‘00; Catani, Grazzini, de Florian, Nason ‘03
  - $g_4(\lambda)$: N^{3}LL S.M., Vermaseren, Vogt ‘05
  - $g_5(\lambda)$: N^{4}LL Das, S.M., Vogt ‘19
- Resummed G^N predicts fixed orders in perturbation theory
  - generating functional for towers of large logarithms

DIS coefficient functions at four loops

Result

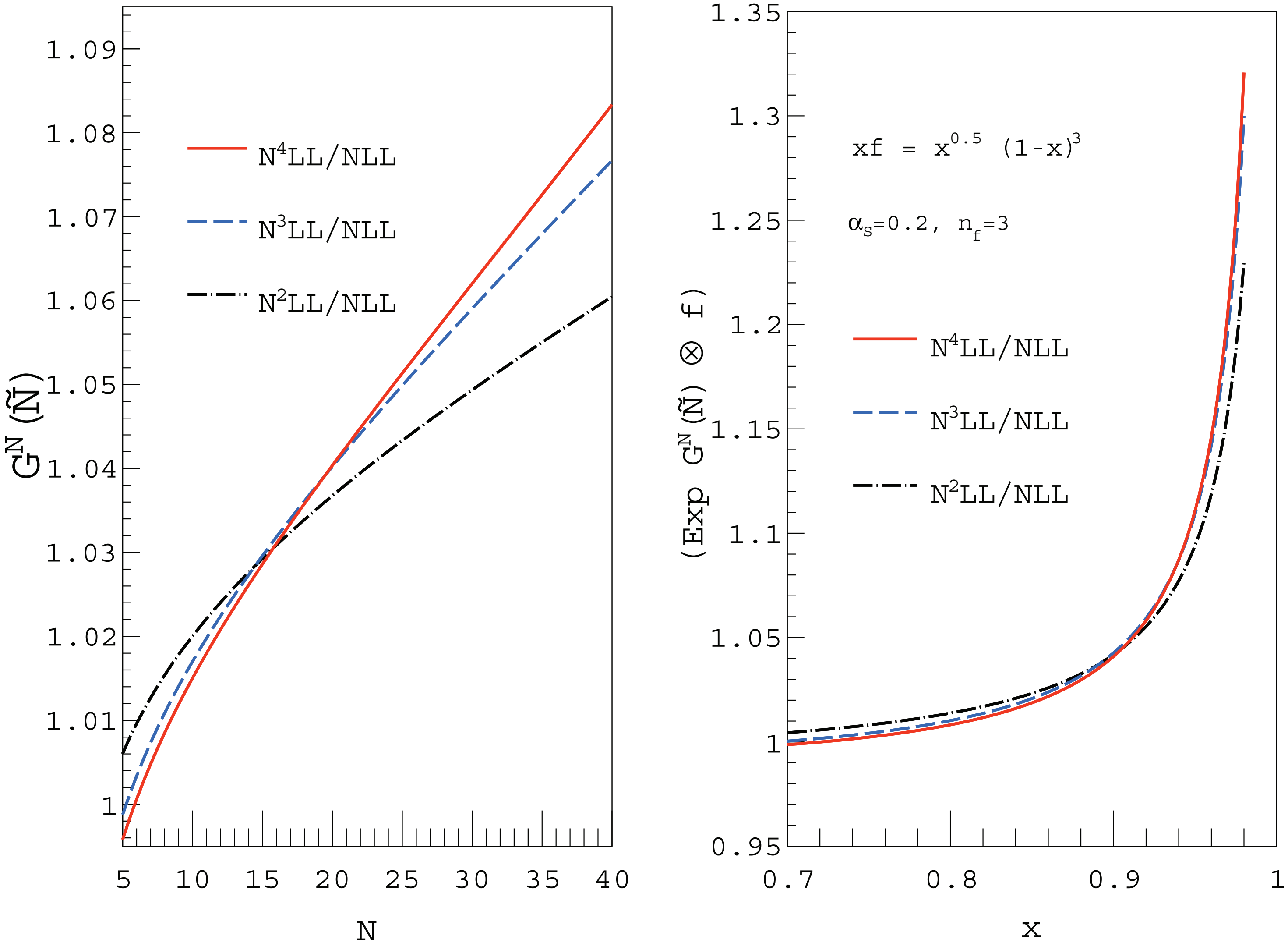
- Four-loop coefficient function $c_{2,\mathrm{q}}^{(4)}$ known $\frac{\ln^7(1-x)}{(1-x)_+}, \ldots, \frac{1}{(1-x)_+}$
- New result for $\frac{1}{(1-x)_+}$ term
  - best estimate (using partial large-n_c information)

$$c_{2,\mathrm{q}}^{(4)}\Big|_{\frac{1}{(1-x)_+}} = (3.874 \pm 0.010) \cdot 10^4 + (-3.496490 \pm 0.000003) \cdot 10^4\, n_f$$

$$+2062.715\, n_f^2 - 12.08488\, n_f^3 + 47.55183\, n_f\, fl_{11}$$

- Based on results for
  - Quark and gluon form factors at four loops in QCD
    Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser '22
  - eikonal anomalous dimensions Dixon, Magnea, Sterman '08
  - Mellin moments of DIS structure functions at four loops

Numerical results for DIS



- Left: Resummed exponent G^N normalized to NLL for DIS plotted successively up to N^{4}LL for $\alpha_s = 0.2$ and $n_f = 3$
- Right: Resummed series convoluted with typical shape for a quark distribution $xf = x^{0.5}(1-x)^3$ up to N^{4}LL

Off-forward kinematics

Operator matrix elements

- QCD applications to hard processes use nonlocal operators of partons at light-like separation

$$\mathcal{O}_\mu(x; z_1, z_2) = \bar{\psi}(x + z_1)\gamma_\mu[z_1, z_2]\psi(x + z_2)$$

  - quark and anti-quark fields joined by Wilson line along '$+$'-direction

$$[z_1, z_2] = \text{Pexp}\left[ig \int_{z_2}^{z_1} dt A_+(t)\right]$$

- Expansion of $\mathcal{O}_\mu(x; z_1, z_2)$ at short distances leads to local operators
  - (anti-)quark fields with covariant derivatives $D_\mu = \partial_+ - igA_+$

$$\bar{\psi}(x)(\overleftarrow{D}_+)^m \gamma_\mu (\overrightarrow{D}_+)^k \psi(x)$$

Applications

- (Generalized) parton distributions: PDFs and GPDs
- Hard exclusive reactions with identified hadrons $N(p)$ and $N(p')$ in initial and final state: $\gamma^* N(p) \to \gamma N(p')$ (DVCS)
- Meson-photon transition form factors $\gamma^* \to \gamma\pi$

Braun, Manashov, S.M., Schönleber '21, Gao, Huber, Ji, Wang '21

Light-ray operators

- Short-distance expansion yields light-ray operators $\mathcal{O}_\mu(x; z_1, z_2)$ with light-like direction n

$$[\mathcal{O}](x; z_1, z_2) \quad \equiv \quad \sum_{m,k} \frac{z_1^m z_2^k}{m!k!} \left[\bar{\psi}(x) (\overleftarrow{D} \cdot n)^m \not{n} (n \cdot \overrightarrow{D})^k \psi(x) \right]$$

- multiplicative renormalization $[\mathcal{O}] = Z\mathcal{O}$
- Light-ray operators satisfy renormalization group equation Balitsky, Braun '87

$$\Big(\mu^2 \partial_{\mu^2} + \beta(a_s) \partial_{a_s} + \mathcal{H}(a_s) \Big) [\mathcal{O}](x; z_1, z_2) = 0$$

- Integral operator $\mathcal{H}(a_s)$ acts on light-cone coordinates of fields $z_{12}^\alpha = z_1(1-\alpha) + z_2 \alpha$

$$\mathcal{H}(a_s)[\mathcal{O}](z_1, z_2) \; = \; \int_0^1 d\alpha \int_0^1 d\beta \, h(\alpha, \beta) [\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

- Evolution kernel $h(\alpha, \beta)$
- Mellin moments $\gamma_{N,N} = \int_0^1 d\alpha \int_0^1 d\beta \, (1 - \alpha - \beta)^{N-1} h(\alpha, \beta)$ are anomalous dimensions of leading-twist local operators with $N = m + k$ derivatives

Evolution equations

- Leading-order result for evolution kernel

$$\mathcal{H}^{(1)}\, f(z_1, z_2) \quad = 4C_F \Big\{ \int_0^1 d\alpha \, \frac{\bar{\alpha}}{\alpha} \Big[2f(z_1, z_2) - f(z_{12}^{\alpha}, z_2) - f(z_1, z_{21}^{\alpha}) \Big]$$

$$- \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \, f(z_{12}^{\alpha}, z_{21}^{\beta}) + \tfrac{1}{2} f(z_1, z_2) \Big\}$$

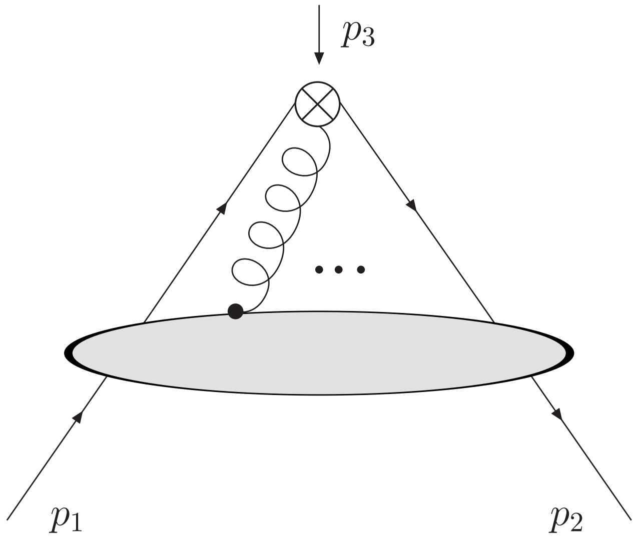
- Expression comprises all classical leading-order QCD evolution equations
  - PDFs Altarelli, Parisi '77; $\gamma_{N,N}^{(0)}$
  - meson light-cone distribution amplitudes
    Efremov, Radyushkin, Brodsky, Lepage
  - general evolution equation for GPDs Belitsky, Müller '99; $h^{(1)}(\alpha, \beta)$

Task

- Push accuracy of evolution equations to NNLO and beyond
- Computation of anomalous dimensions in forward and off-forward kinematics to three and four loops

Operator matrix elements

- Off-forward kinematics considers matrix elements with general momentum assignments $\langle \psi(p_1) | \mathcal{O}^{NS}_{\mu_1 \ldots \mu_N}(p_3) | \overline{\psi}(p_2) \rangle$
  - standard local non-singlet quark operator
    $\mathcal{O}^{NS}_{\mu_1 \ldots \mu_N} = \mathcal{S}\, \overline{\psi} \lambda^\alpha \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} \psi$



- Short-distance expansion of light-ray operators uses basis of local operators in terms of Gegenbauer polynomials

$$\mathcal{O}^{\mathcal{G}}_{N,k} = (\partial_{z_1} + \partial_{z_2})^k C^{3/2}_N \left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}} \right) \mathcal{O}(z_1, z_2) \Big|_{z_1 = z_2 = 0}$$

- Evolution equation for renormalized operators $[\mathcal{O}^{\mathcal{G}}_{N,k}]$

$$\left(\mu^2 \partial_{\mu^2} + \beta(a_s) \partial_{a_s} \right) [\mathcal{O}^{\mathcal{G}}_{N,k}] = \sum_{j=0}^{N} \gamma^{\mathcal{G}}_{N,j} [\mathcal{O}^{\mathcal{G}}_{j,k}] \qquad \text{with } \gamma^{\mathcal{G}}_{N,j} = 0 \text{ if } j > N$$

Conformal symmetry

- Full conformal algebra in 4 dimensions includes fifteen generators
  Mack, Salam ‘69; Treiman, Jackiw, Gross ‘72

  $\mathbf{P}_\mu$ (4 translations) $\mathbf{M}_{\mu\nu}$ (6 Lorentz rotations)
  $\mathbf{D}$ (dilatation) $\mathbf{K}_\mu$ (4 special conformal transformations)

Collinear subgroup $SL(2,\mathbb{R})$

- Leading order evolution operator $\mathcal{H}^{(1)}$ commutes with (canonical) generators of collinear conformal transformations
- Evolution kernel $h^{(1)}(\alpha,\beta) = \bar{h}(\tau)$ effectively function of one variable
  $\tau = \frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}$ (conformal ratio) Braun, Derkachov, Korchemsky, Manashov ‘99

$$h^{(1)}(\alpha,\beta) = -4C_F \left[\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\right],$$

- Conformal symmetry is broken in any realistic four-dimensional QFT
  - $\beta(a_s) \neq 0$

QCD in conformal window

- Instead of considering consequences of broken conformal symmetry in QCD make use of exact conformal symmetry of modified theory
  - $\beta(a_s) = 2a_s(-\epsilon - \beta_0 a_s - \beta_1 a_s^2 - \dots)$ with $a_s = \dfrac{\alpha_s}{4\pi}$
  - large-n_f QCD in $4 - 2\epsilon$ dimensions at critical coupling a_* with $\beta(a_*) = 0$ Banks, Zaks '82
- Maintain exact conformal symmetry, but the generators of $SL(2,\mathbb{R})$ are modified by quantum corrections

Results

- $\hat{\gamma}^{\mathcal{G}}$ in Gegenbauer basis Müller '93, Belitsky, Müller '99

$$\hat{\gamma}^{\mathcal{G}}(a_s) \;=\; \mathbf{G}\left\{[\hat{\gamma}^{\mathcal{G}}(a_s),\hat{b}]\left(\frac{1}{2}\hat{\gamma}^{\mathcal{G}}(a_s)+\beta(a_s)\right)+[\hat{\gamma}^{\mathcal{G}}(a_s),\hat{w}(a_s)]\right\}$$

  - matrix commutators denoted as $[*,*]$ and $\mathbf{G}\{\hat{M}\}_{N,k} \;=\; -\frac{M_{N,k}}{a(N,k)}$

$$a(N,k) \;=\; (N-k)(N+k+3)\,,$$

$$\hat{b}_{N,k} \;=\; -2k\delta_{N,k} - 2(2k+3)\vartheta_{N,k}$$

  - Conformal anomaly $\hat{w}(a_s)$ up to two-loops Braun, Manashov '13, Braun, Manashov, S.M., Strohmaier '16, Braun, Manashov, S.M., Strohmaier '17

Total derivative basis (I)

- Total derivative basis

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = \mathcal{S}\, \partial^{\mu_1} \dots \partial^{\mu_p} \left((D^{\nu_1} \dots D^{\nu_q} \overline{\psi})\, \lambda^{\alpha} \gamma_{\mu}\, (D^{\sigma_1} \dots D^{\sigma_r} \psi) \right)$$

  - expansion in terms of powers of derivatives (left, right and total)
- Total derivative basis used in computations of operator correlation functions for DIS Gracey '09, Kniehl, Veretin '20
  - renormalization schemes $\overline{\text{MS}}$ and RI (for comparison to lattice QCD)
- Action of partial derivatives

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = \mathcal{O}^{\mathcal{D}}_{p-1,q+1,r} + \mathcal{O}^{\mathcal{D}}_{p-1,q,r+1}$$

  - left and right derivative operators renormalize with the same renormalization constants

$$\mathcal{O}^{\mathcal{D}}_{p,0,r} = \sum_{j=0}^{r} Z_{r,r-j}\, [\mathcal{O}^{\mathcal{D}}_{p+j,0,r-j}]$$

- Anomalous dimensions $\gamma^{\mathcal{D}}_{N,k}$ govern scale dependence

$$\gamma^{\mathcal{D}}_{N,k} = -\left(\frac{d}{d \ln \mu^2}\, Z_{N,j} \right) Z^{-1}_{j,k} \qquad \text{with } \gamma^{\mathcal{D}}_{N,j} = 0 \text{ if } j > N$$

Total derivative basis (II)

- Operator bases related using light-ray operators as generating functions

$$\mathcal{O}(z_1, z_2) = \sum_{m,k} \frac{z_1^m z_2^k}{m!\, k!} \mathcal{O}^{\mathcal{D}}_{0,m,k}$$

  - expansion of Gegenbauer polynomials yields

$$\mathcal{O}^{\mathcal{G}}_{N,k} = \frac{1}{2N!} \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(N+l+2)!}{(l+1)!} \mathcal{O}^{\mathcal{D}}_{k-l,0,l}$$

- Evolution equations relate anomalous dimension matrices $\gamma^{\mathcal{G}}_{N,j}$ and $\gamma^{\mathcal{D}}_{N,j}$

$$\sum_{j=0}^{N} (-1)^j \frac{(j+2)!}{j!} \gamma^{\mathcal{G}}_{N,j} = \frac{1}{N!} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{(N+j+2)!}{(j+1)!} \sum_{l=0}^{j} \gamma^{\mathcal{D}}_{j,l}$$

Task

- Exploit relation of $\gamma^{\mathcal{G}}_{N,j}$ and $\gamma^{\mathcal{D}}_{N,j}$ for known results
- Analyze constraints on $\gamma^{\mathcal{D}}_{N,j}$ in total derivative basis

Constraints on the anomalous dimensions (I)

- Recursion for bare operators $\mathcal{O}^{\mathcal{D}}_{p,q,r} = \sum_{i=0}^{p} \binom{p}{i} \mathcal{O}^{\mathcal{D}}_{0,p+q-i,r+i}$ leads to relation between sums of elements of the mixing matrix $\hat{\gamma}^{\mathcal{D}}_N$

$$\forall k: \qquad \sum_{j=k}^{N} \left\{ (-1)^k \binom{j}{k} \gamma^{\mathcal{D}}_{N,j} - (-1)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,k} \right\} = 0$$

  - $k = N-1$ relates next-to-diagonal elements to forward anomalous dimensions $\gamma_{N,N}$

$$\gamma^{\mathcal{D}}_{N,N-1} = \frac{N}{2} \left(\gamma_{N-1,N-1} - \gamma_{N,N} \right)$$

  - mixing matrix

$$\hat{\gamma}^{\mathcal{D}} = \begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \dots & \gamma^{\mathcal{D}}_{N,N-k} & \dots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \dots & \gamma^{\mathcal{D}}_{N-1,N-k} & \dots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \gamma^{\mathcal{D}}_{N-k,N-k} & \dots & \gamma^{\mathcal{D}}_{N-k,0} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

Constraints on the anomalous dimensions (II)

- $k = 0$ relates sum of elements in N-th row to the conjugate $\mathcal{C}\,\gamma^{\mathcal{D}}_{N,0}$

$$\sum_{j=0}^{N} \left\{ \gamma^{\mathcal{D}}_{N,j} - (-1)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,0} \right\} = 0$$

  - mixing matrix

$$\hat{\gamma}^{\mathcal{D}} = \begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \cdots & \gamma^{\mathcal{D}}_{N,N-k} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \cdots & \gamma^{\mathcal{D}}_{N-1,N-k} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \gamma^{\mathcal{D}}_{N-k,N-k} & \cdots & \gamma^{\mathcal{D}}_{N-k,0} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

- arbitrary k

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}$$

  - bootstrapping $\gamma^{\mathcal{D}}_{N,k}$
  - solution of sums with ansatz for $\gamma^{\mathcal{D}}_{N,k}$ by means of **Sigma** Schneider '07

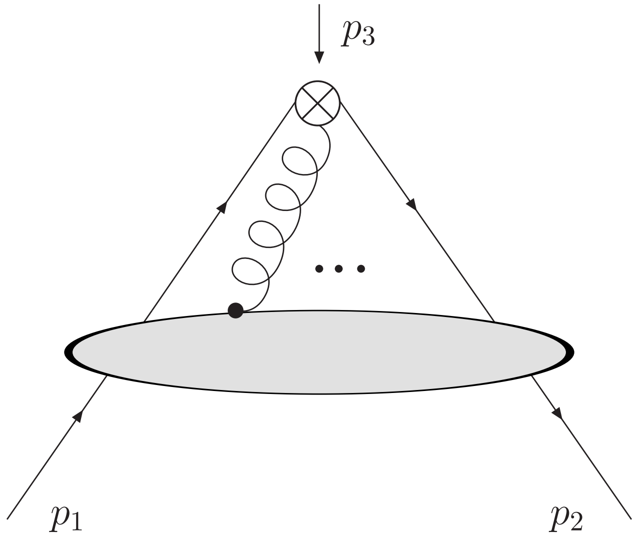
Calculation (I)

- OMEs in off-forward kinematics
  - momentum-flow through the operator vertex
  - choice $p_2 = 0$ maps OMEs to two-point functions

$$\Delta^{\mu_1} \ldots \Delta^{\mu_N} \langle \psi(p_1) | O^{NS}_{\mu_1 \ldots \mu_N}(-p_1) | \overline{\psi}(0) \rangle$$

- OMEs

$$\mathcal{O}_N \equiv \Delta^{\mu_1} \ldots \Delta^{\mu_N} \mathcal{O}^{NS}_{\mu_1 \ldots \mu_N}$$
$$\mathcal{O}_1 = \Delta^{\mu} \overline{\psi} \lambda^{\alpha} \gamma_{\mu} \psi$$



Work flow

- Anomalous dimensions $\gamma^{\mathcal{D}}_{N,k}$ from ultraviolet divergence of loop corrections to operator in (anti-)quark two-point function
- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with **Forcer** Ruijl, Ueda, Vermaseren '17

Calculation (II)

- Renormalized OMEs in off-forward kinematics

$$\mathcal{O}_{N+1} = Z_\psi \left(Z_{N,N} [O_{N+1}] + Z_{N,N-1} [\partial \mathcal{O}_N] + \cdots + Z_{N,0} [\partial^N \mathcal{O}_1] \right)$$

$$\begin{pmatrix} \mathcal{O}_{N+1} \\ \partial \mathcal{O}_N \\ \vdots \\ \partial^k \mathcal{O}_{N+1-k} \\ \vdots \\ \partial^N \mathcal{O}_1 \end{pmatrix} = Z_\psi \begin{pmatrix} Z_{N,N} & \dots & Z_{N,N-k} & \dots & Z_{N,0} \\ 0 & \dots & Z_{N-1,N-k} & \dots & Z_{N-1,0} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & Z_{N-k,N-k} & \dots & Z_{N-k,0} \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{N+1}] \\ [\partial \mathcal{O}_N] \\ \vdots \\ [\partial^k \mathcal{O}_{N+1-k}] \\ \vdots \\ [\partial^N \mathcal{O}_1] \end{pmatrix}$$

- Disentangle elements of anomalous dimensions matrix $\gamma^{\mathcal{D}}_{N,k}$ from
- Use additional constraints on $\gamma^{\mathcal{D}}_{N,k}$ in total derivative basis

Results (I)

Gegenbauer basis

- One-loop results for $\gamma^{\mathcal{G}}_{N,k}$ Makeenko '81

$$\gamma^{\mathcal{G},(0)}_{N,k} = 0$$

- Matrix for $N = 5$

$$\hat{\gamma}^{\mathcal{G},(0)}_{N=5} = C_F \begin{pmatrix} \frac{91}{15} & 0 & 0 & 0 & 0 \\ 0 & \frac{157}{30} & 0 & 0 & 0 \\ 0 & 0 & \frac{25}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Results (II)

Total derivative basis

- One-loop results for $\gamma_{N,k}^{\mathcal{D}}$

$$\gamma_{N,k}^{\mathcal{D},(0)} = C_F \left(\frac{2}{N+2} - \frac{2}{N-k} \right)$$

- Matrix for $N = 5$

$$\hat{\gamma}_{N=5}^{\mathcal{D},(0)} = C_F \begin{pmatrix} \frac{91}{15} & -\frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{157}{30} & -\frac{8}{5} & -\frac{3}{5} & -\frac{4}{15} \\ 0 & 0 & \frac{25}{6} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{8}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Results (III)

- General structure of matrix for $N = 5$

$$\begin{pmatrix} \gamma_{4,4} & \gamma_{4,3} & \gamma_{4,2} & \gamma_{4,1} & \gamma_{4,0} \\ 0 & \gamma_{3,3} & \gamma_{3,2} & \gamma_{3,1} & \gamma_{3,0} \\ 0 & 0 & \gamma_{2,2} & \gamma_{2,1} & \gamma_{2,0} \\ 0 & 0 & 0 & \gamma_{1,1} & \gamma_{1,0} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Three-loop results for $\gamma^{\mathcal{D}}_{N,k}$

$$\begin{aligned} \gamma_{2,2} &= 5.55556\, a_s + (70.8848 - 5.12346\, n_f)\, a_s^2 \\ &\quad + (1244.91 - 199.637\, n_f - 1.762\, n_f^2)\, a_s^3 + O(a_s^4) \\ \gamma^{\mathcal{D}}_{2,1} &= -2\, a_s + (-22.5556 + 1.96296\, n_f)\, a_s^2 \\ &\quad + (-385.466 + 66.1992\, n_f + 0.532922\, n_f^2)\, a_s^3 + O(a_s^4) \\ \gamma^{\mathcal{D}}_{2,0} &= -0.666667\, a_s + (-9.50617 + 0.481481\, n_f)\, a_s^2 \\ &\quad + (-170.654 + 24.8232\, n_f + 0.3107\, n_f^2)\, a_s^3 + O(a_s^4) \end{aligned}$$

- comparision with Kniehl, Veretin ‘20

$$\gamma^{\mathcal{D},(2)}_{2,0}\Big|_{\text{Kniehl, Veretin ‘20}} = -170.641(12) + 24.822(2)\, n_f + 0.3107(1)\, n_f^2$$

Results (IV)

- Four- and five-loop results for $\gamma^{\mathcal{G}}_{N,k}$ and $\gamma^{\mathcal{D}}_{N,k}$ in large n_f limit
  - forward anomalous dimension known $\gamma_{N,N}$ Gracey '94

$$
\begin{aligned}
\gamma^{\mathcal{D},(3)}_{N,k} = \frac{8}{27} n_f^3 C_F \Bigg\{ & \frac{1}{3}\Big(S_1(N)-S_1(k)\Big)^3 \Big(\frac{1}{N+2}-\frac{1}{N-k}\Big) \\
& + \Big(S_1(N)-S_1(k)\Big)^2 \Big(\frac{5}{3}\frac{1}{N-k}+\frac{2}{N+1}-\frac{11}{3}\frac{1}{N+2}+\frac{1}{(N+2)^2}\Big) \\
& + \Big(S_1(N)-S_1(k)\Big)\Big(S_2(N)-S_2(k)\Big)\Big(\frac{1}{N+2}-\frac{1}{N-k}\Big) \\
& + 2\Big(S_1(N)-S_1(k)\Big)\Big(\frac{1}{3}\frac{1}{N-k}-\frac{13}{3}\frac{1}{N+1}+\frac{2}{(N+1)^2}+\frac{4}{N+2}-\frac{11}{3}\frac{1}{(N+2)^2} \\
& + \frac{1}{(N+2)^3}\Big) + \Big(S_2(N)-S_2(k)\Big)\Big(\frac{5}{3}\frac{1}{N-k}+\frac{2}{N+1}-\frac{11}{3}\frac{1}{N+2}+\frac{1}{(N+2)^2}\Big) \\
& + \frac{2}{3}\Big(S_3(N)-S_3(k)\Big)\Big(\frac{1}{N+2}-\frac{1}{N-k}\Big) + \frac{2}{3}\frac{1}{N-k}+\frac{2}{N+1}-\frac{26}{3}\frac{1}{(N+1)^2} \\
& + \frac{4}{(N+1)^3}-\frac{8}{3}\frac{1}{N+2}+\frac{8}{(N+2)^2}-\frac{22}{3}\frac{1}{(N+2)^3}+\frac{2}{(N+2)^4}\Bigg\} + n_f^3 C_F \zeta_3 \ldots
\end{aligned}
$$

Summary

- Experimental precision of $\lesssim 1\%$ motivates computations at higher order in perturbative QCD
  - hard scattering in forward and off-forward kinematics
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at N^{3}LO and N^{4}LO
  - evolutions equations and inclusive cross sections
  - massive use of computer algebra
- Novel structural insights into QCD from integrabilty and conformal symmetry
  - Key parts of QCD inherited from $N = 4$ Super Yang-Mills theory
  - Conformal symmetry in QCD evolution equations for light-ray operators
- Precision studies of hadron structure
  - great prospects for DIS at future colliders