Gravitational Dynamics from Scattering Amplitudes

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Seminar QMUL SAGEX (June 2022)

Based on work in Collaboration with Poul Damgaard, Andrea Cristofoli, Ludovic Plante, Pierre Vanhove

N. Emil J. Bjerrum-Bohr

SAGEX
Scattering Amplitudes:
from Geometry to Experiment

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General Relativity

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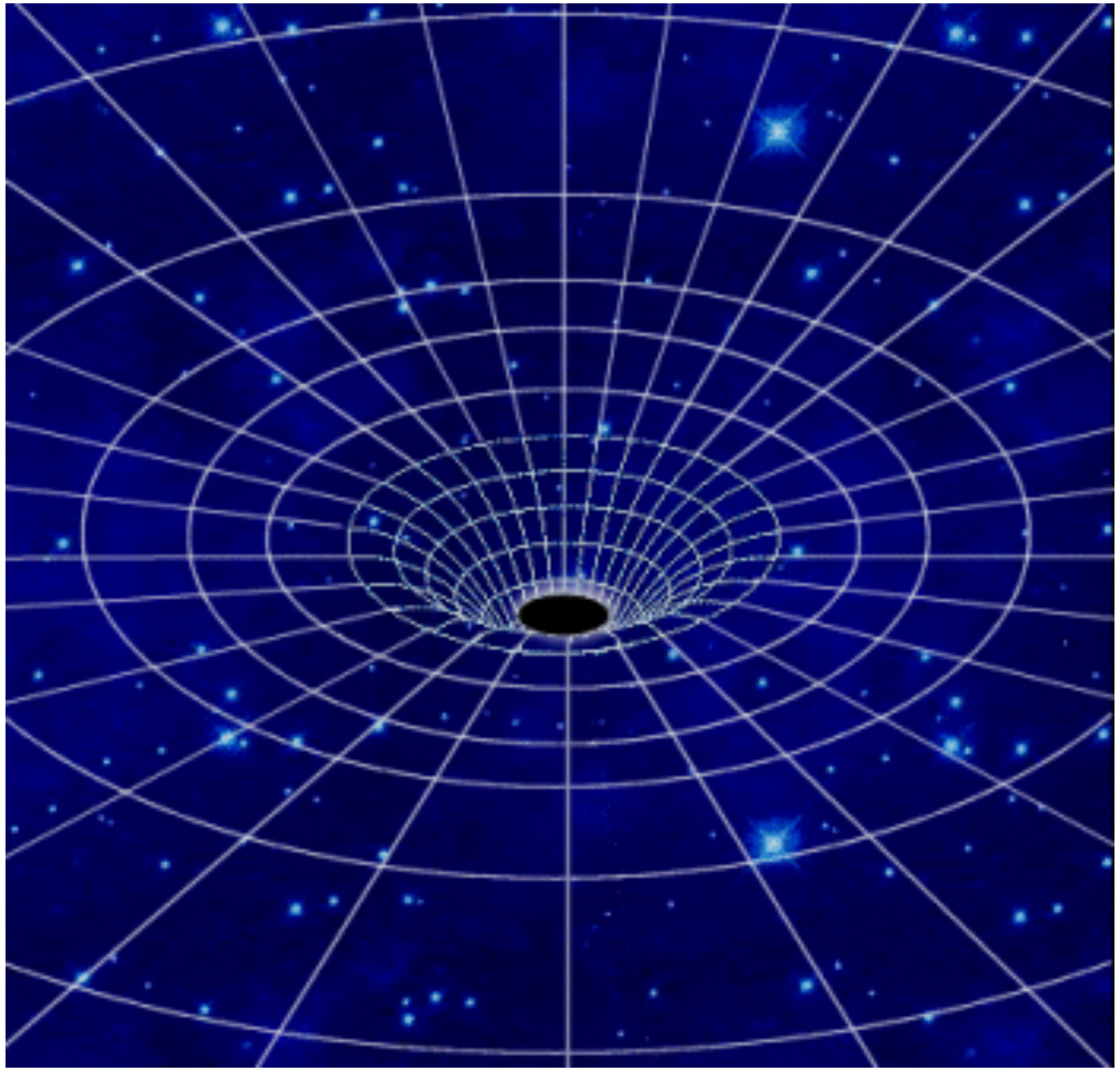
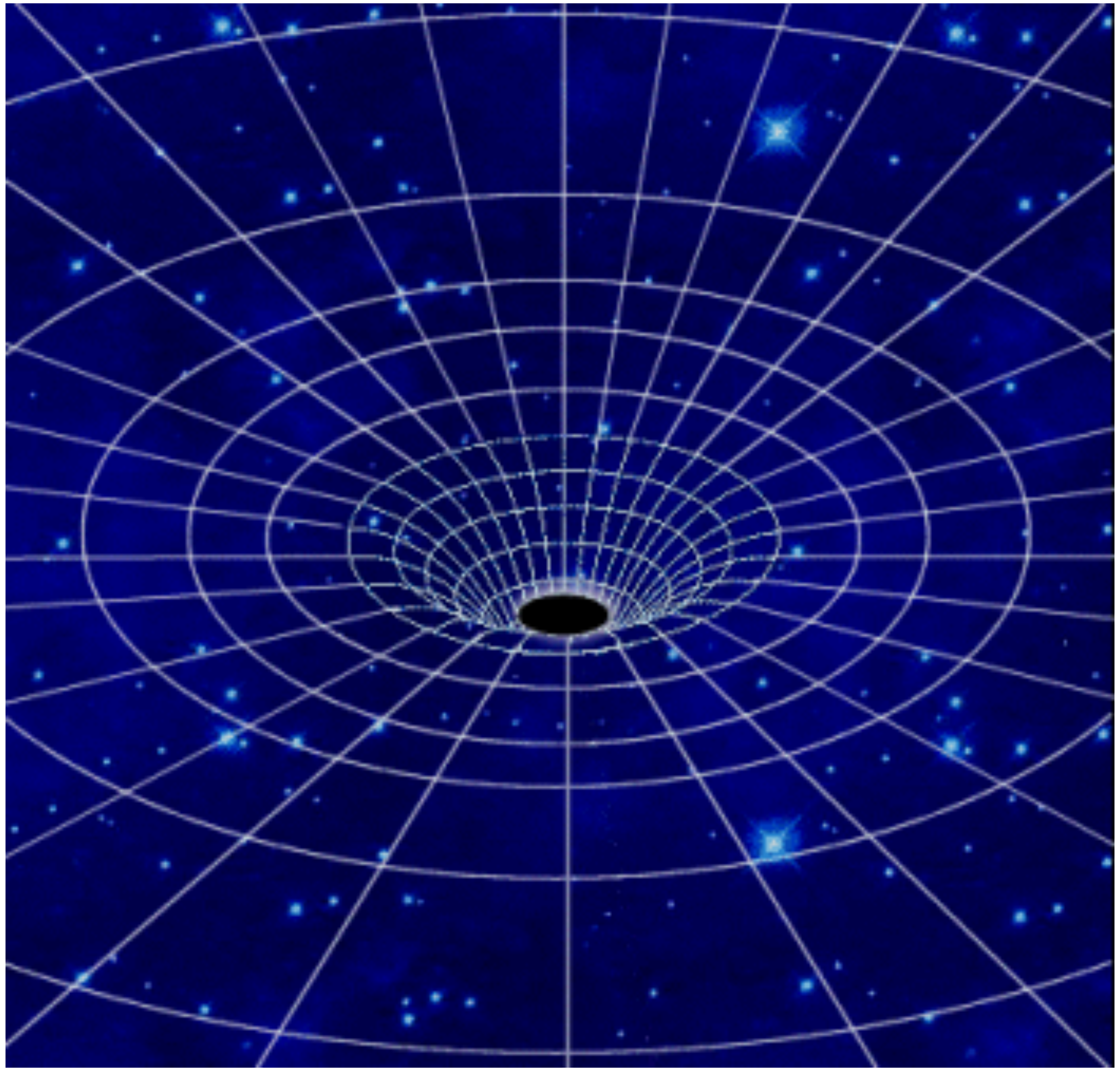


General Relativity

- Einstein’s theory presents us with a beautiful theory for gravity. Many exciting questions to study:



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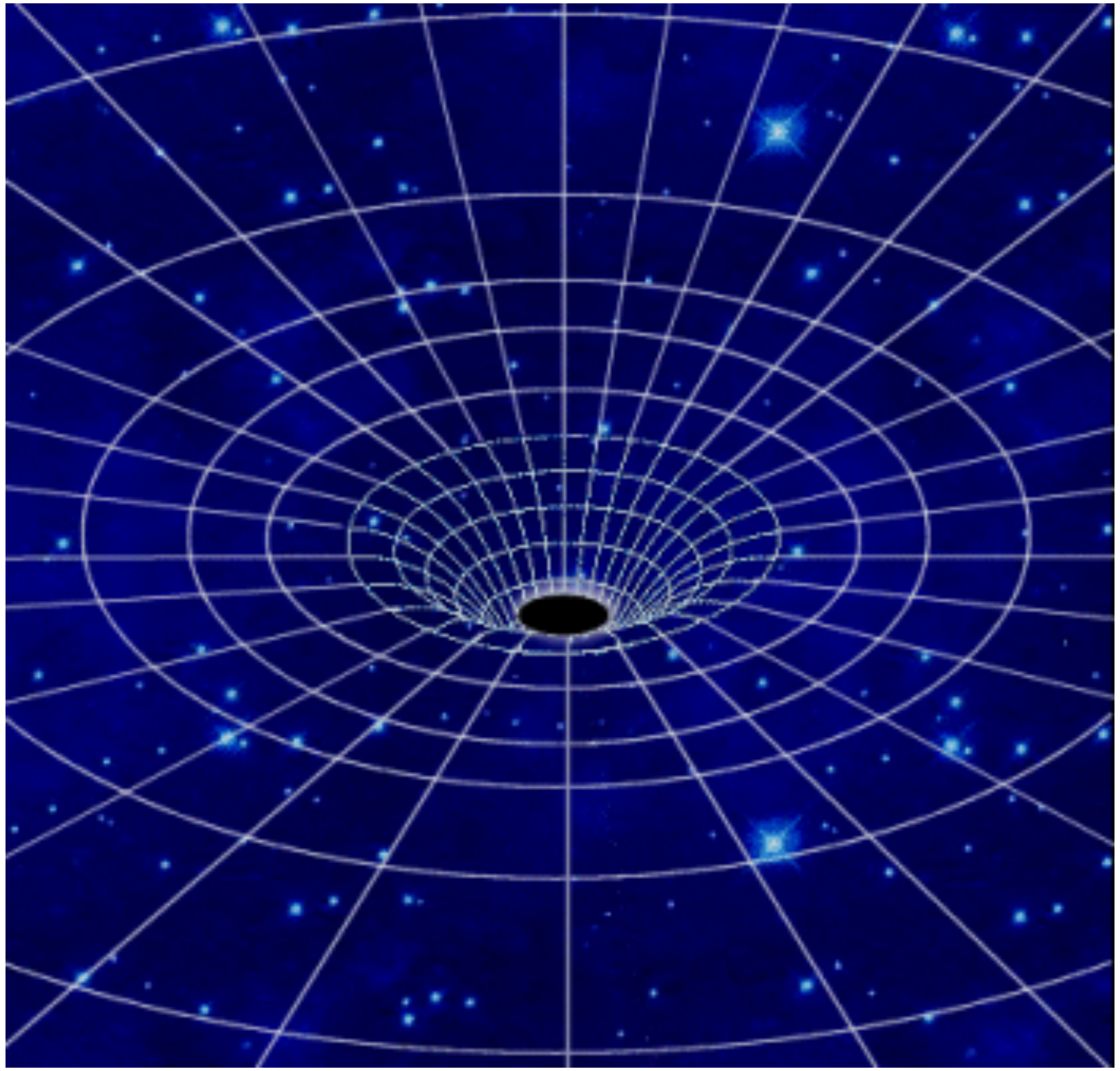

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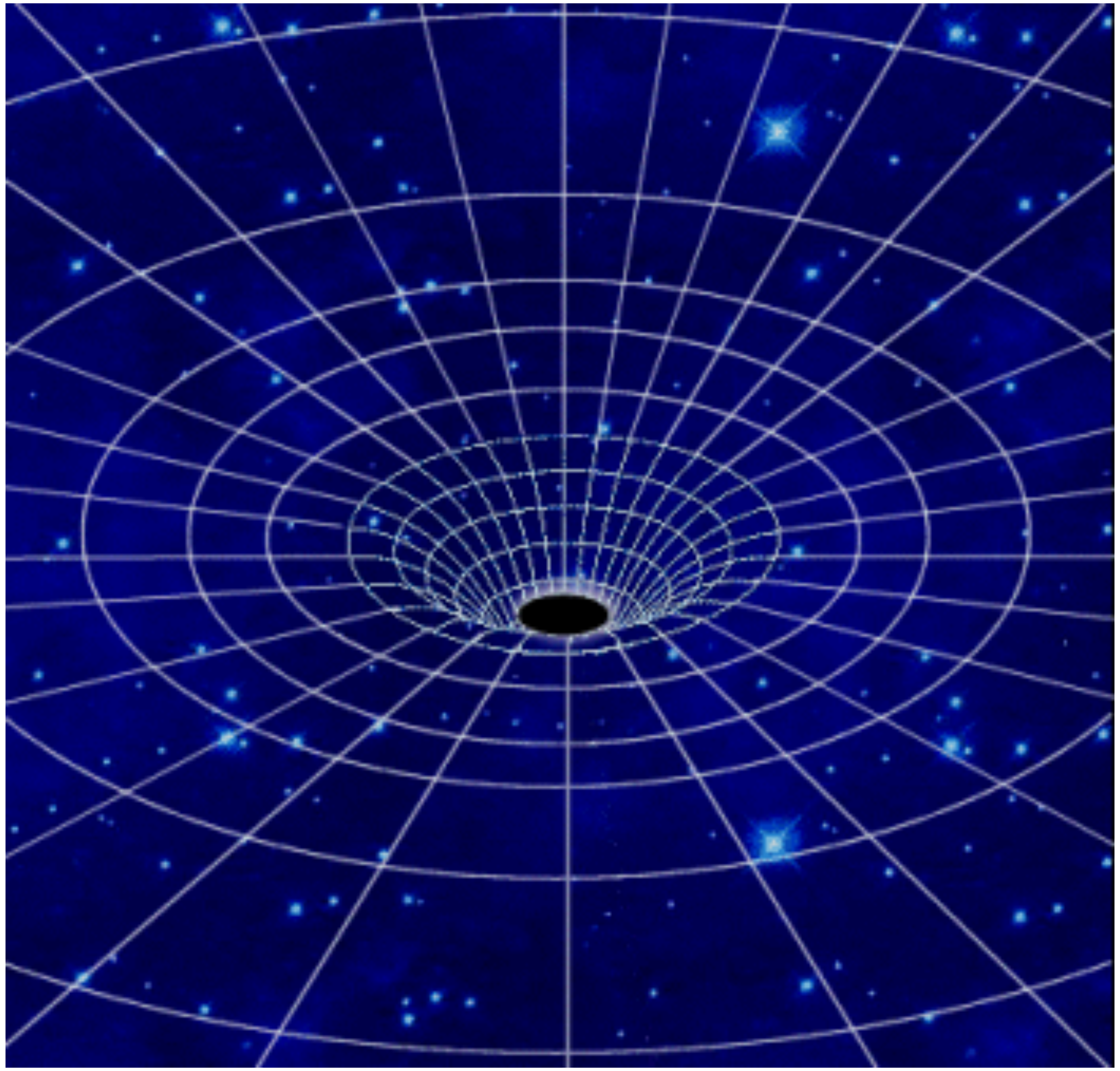

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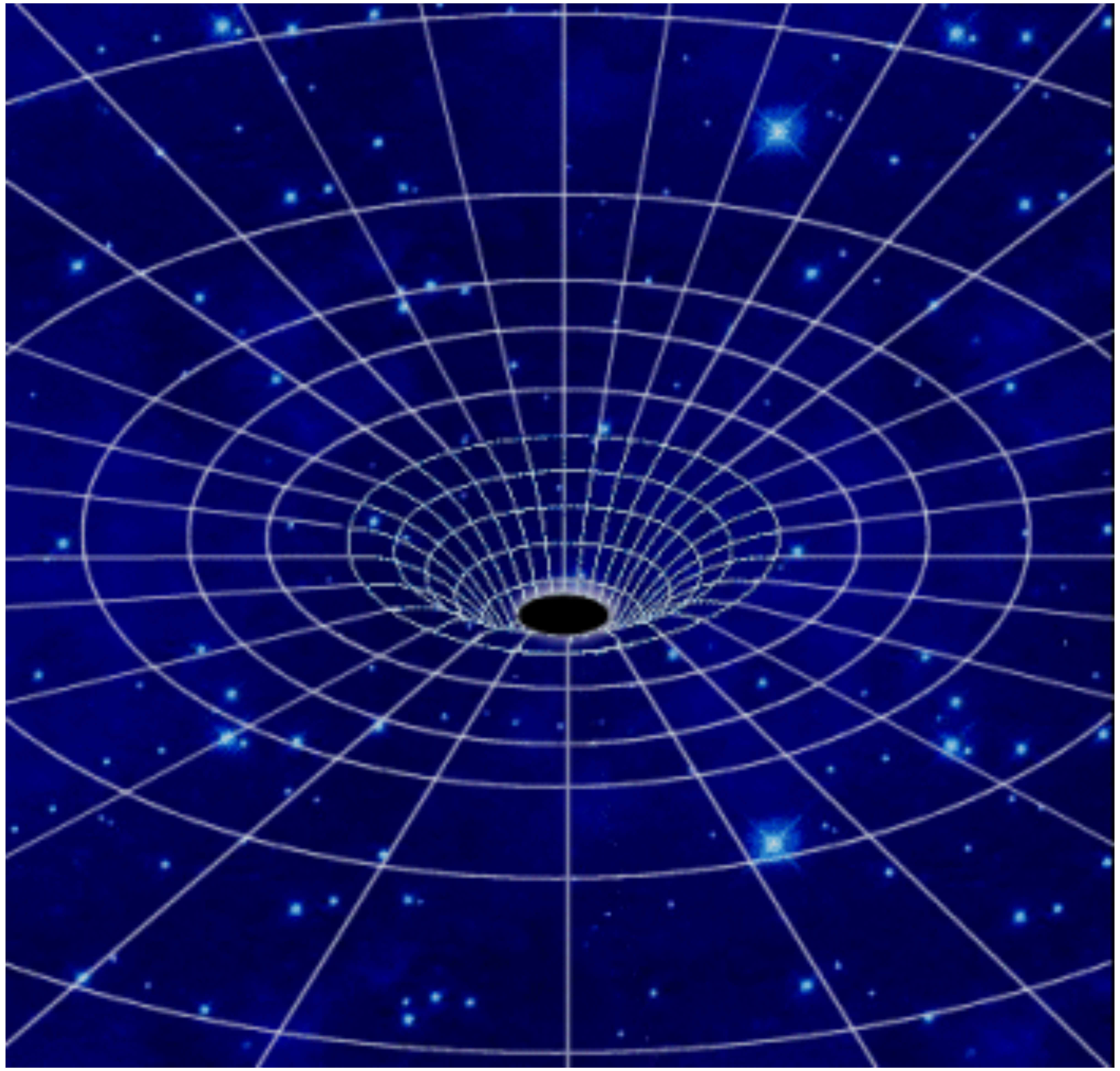

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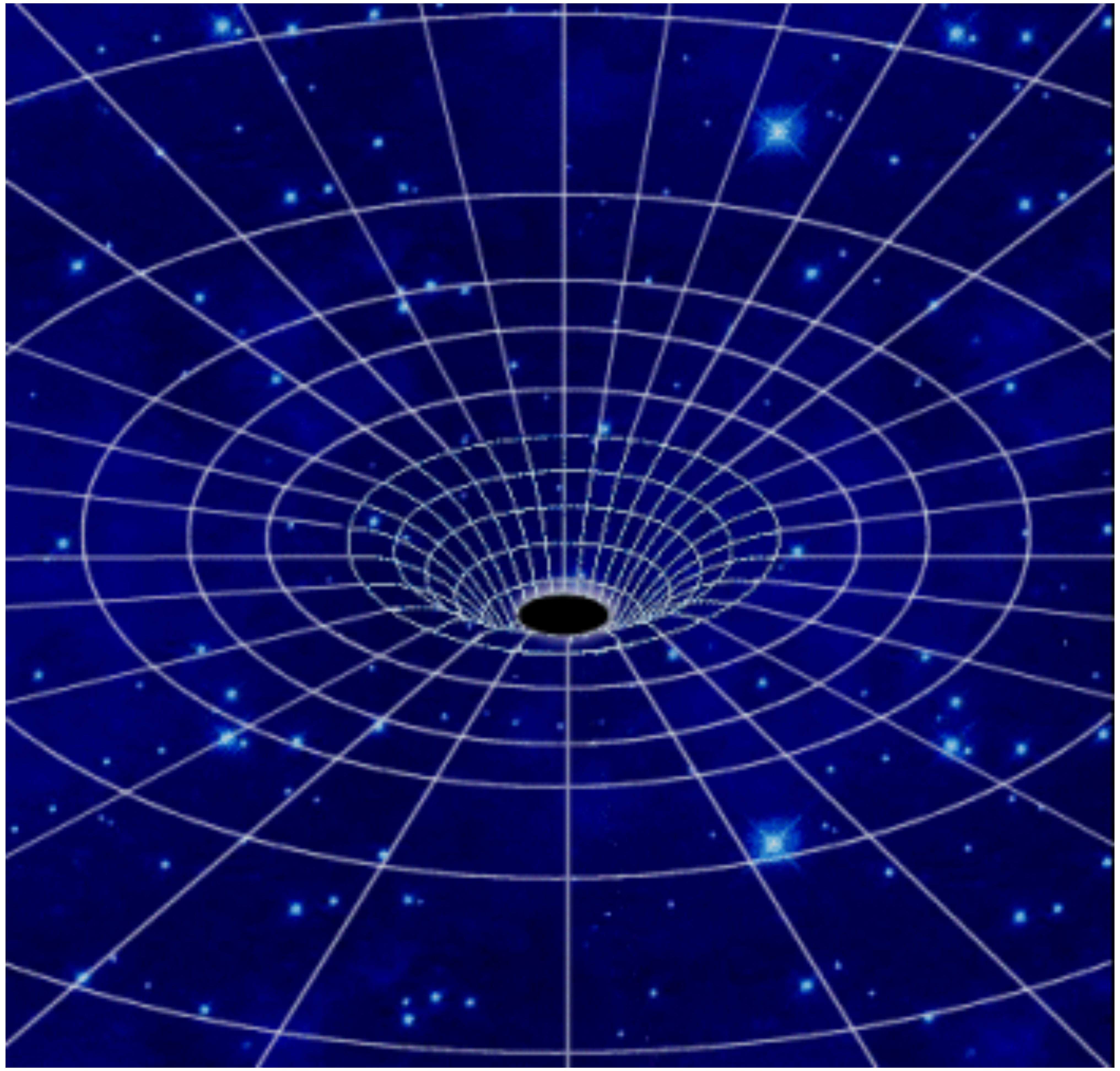

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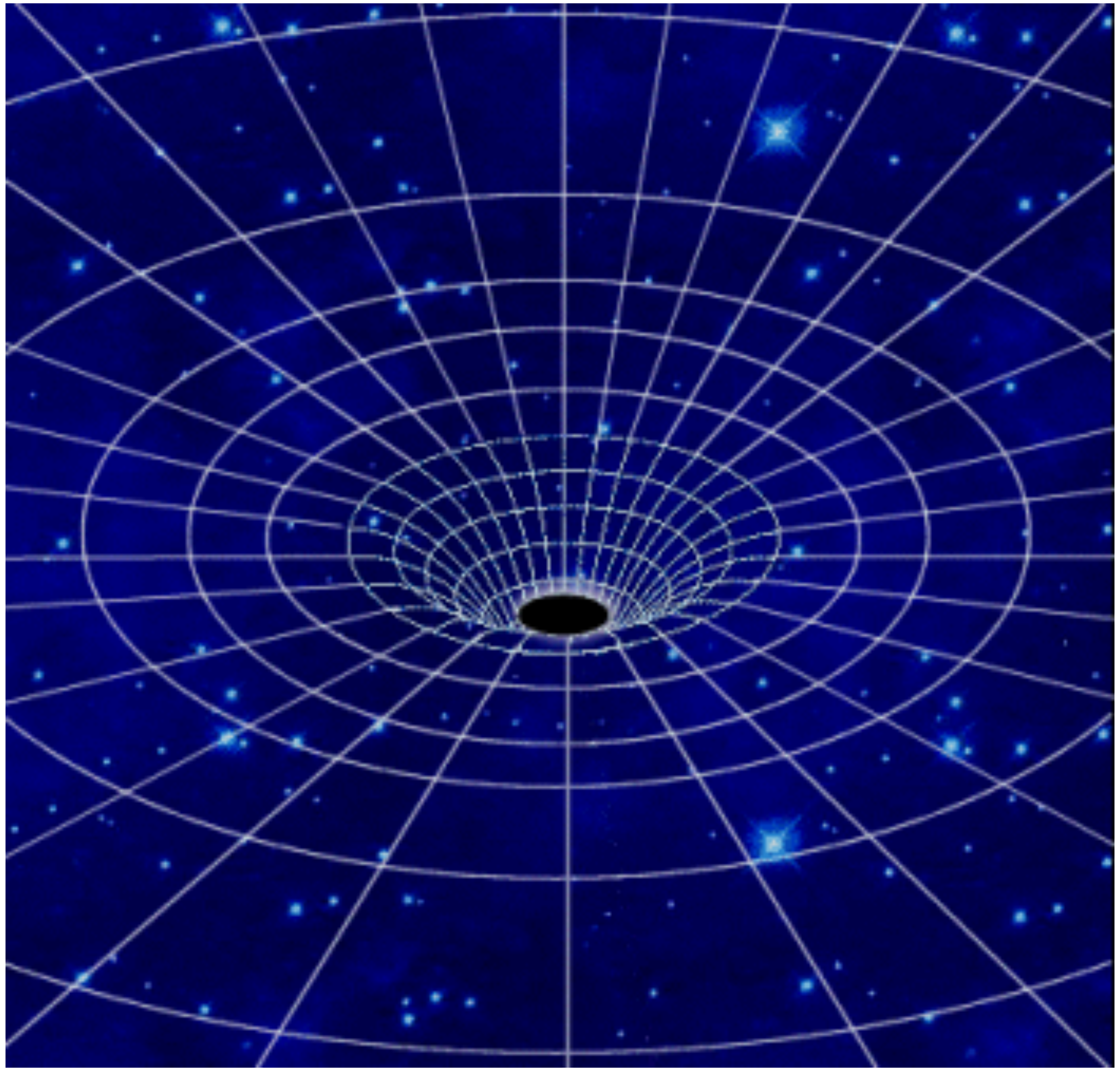

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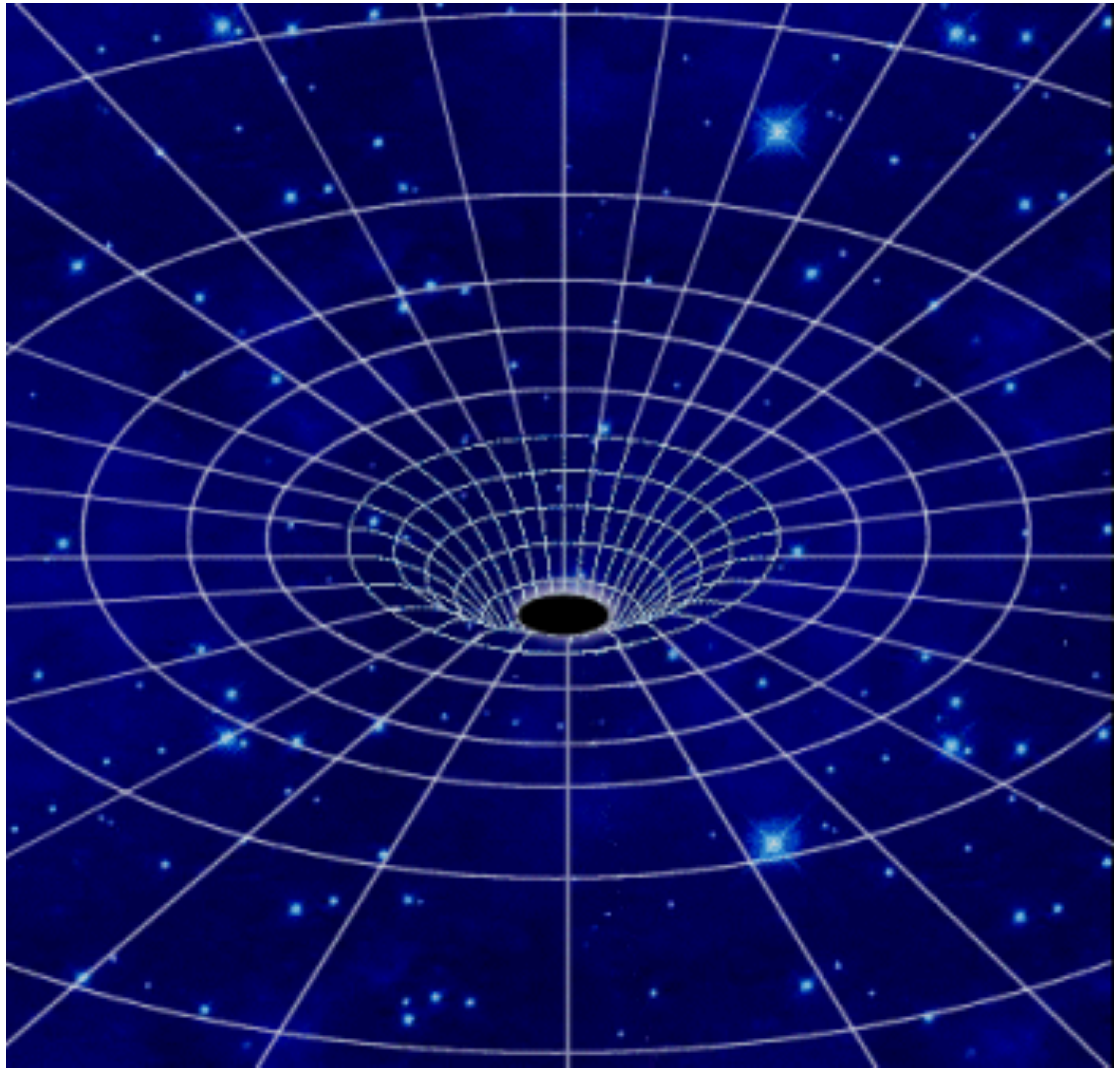

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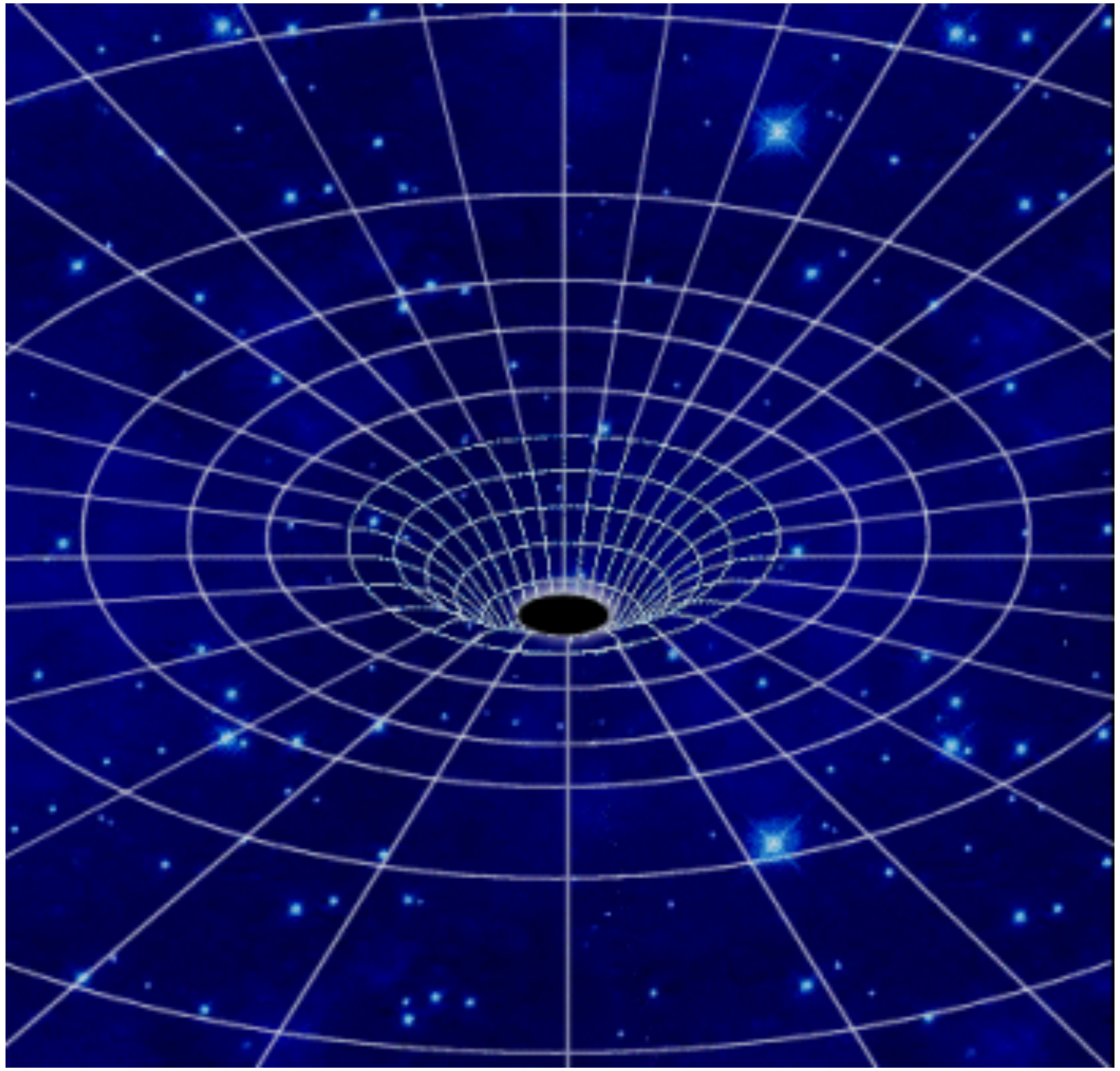

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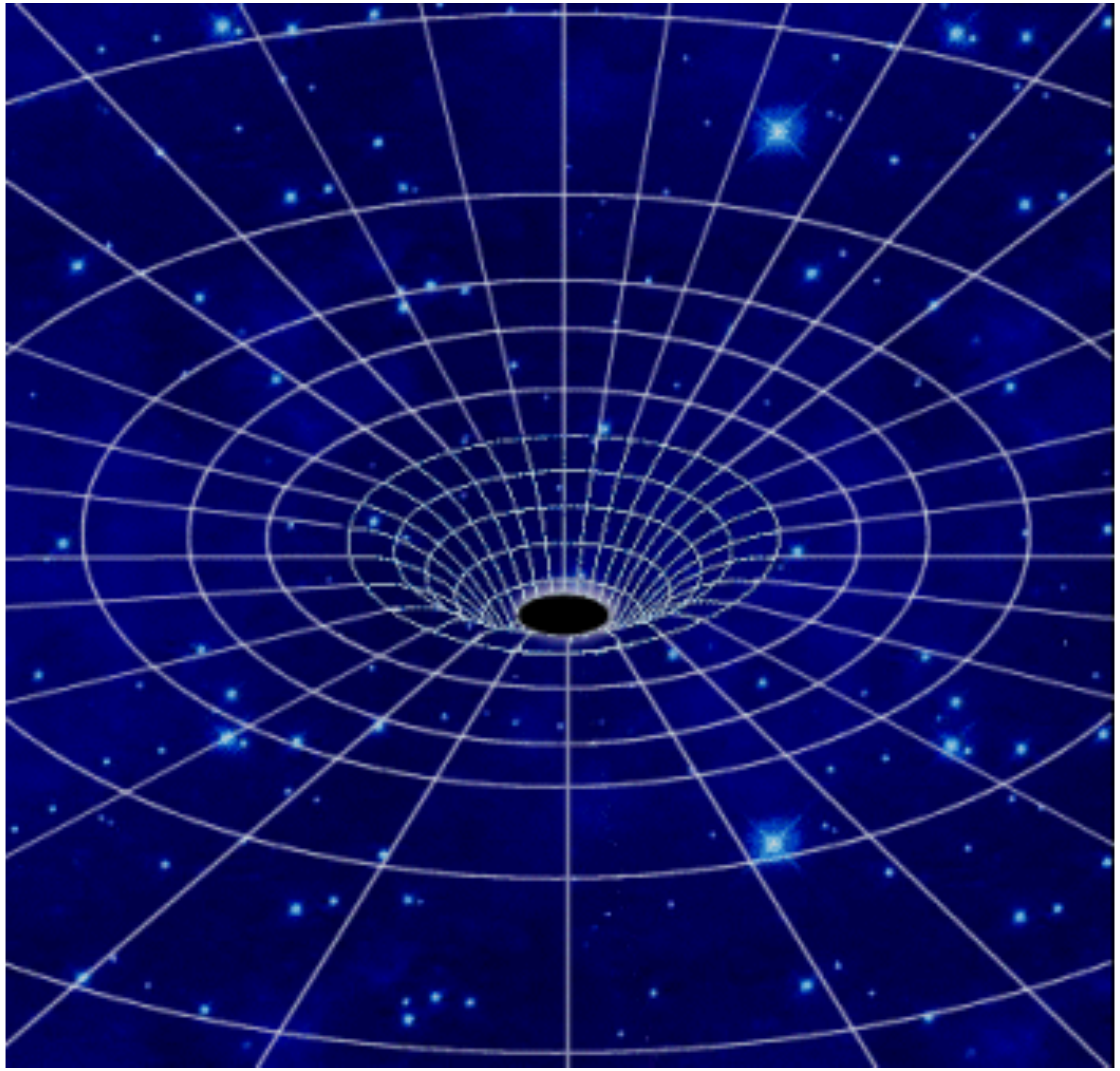

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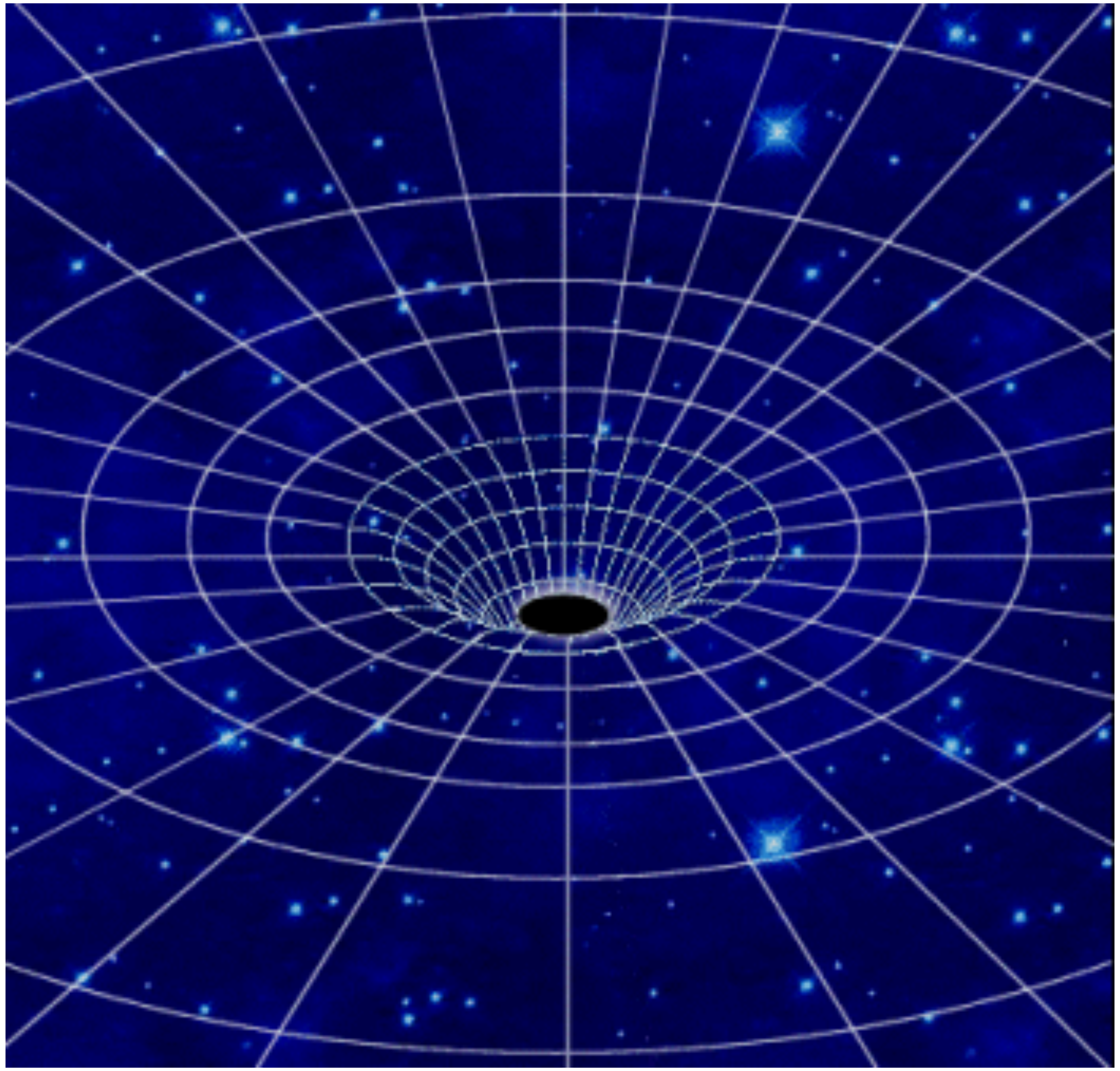

New data - new window

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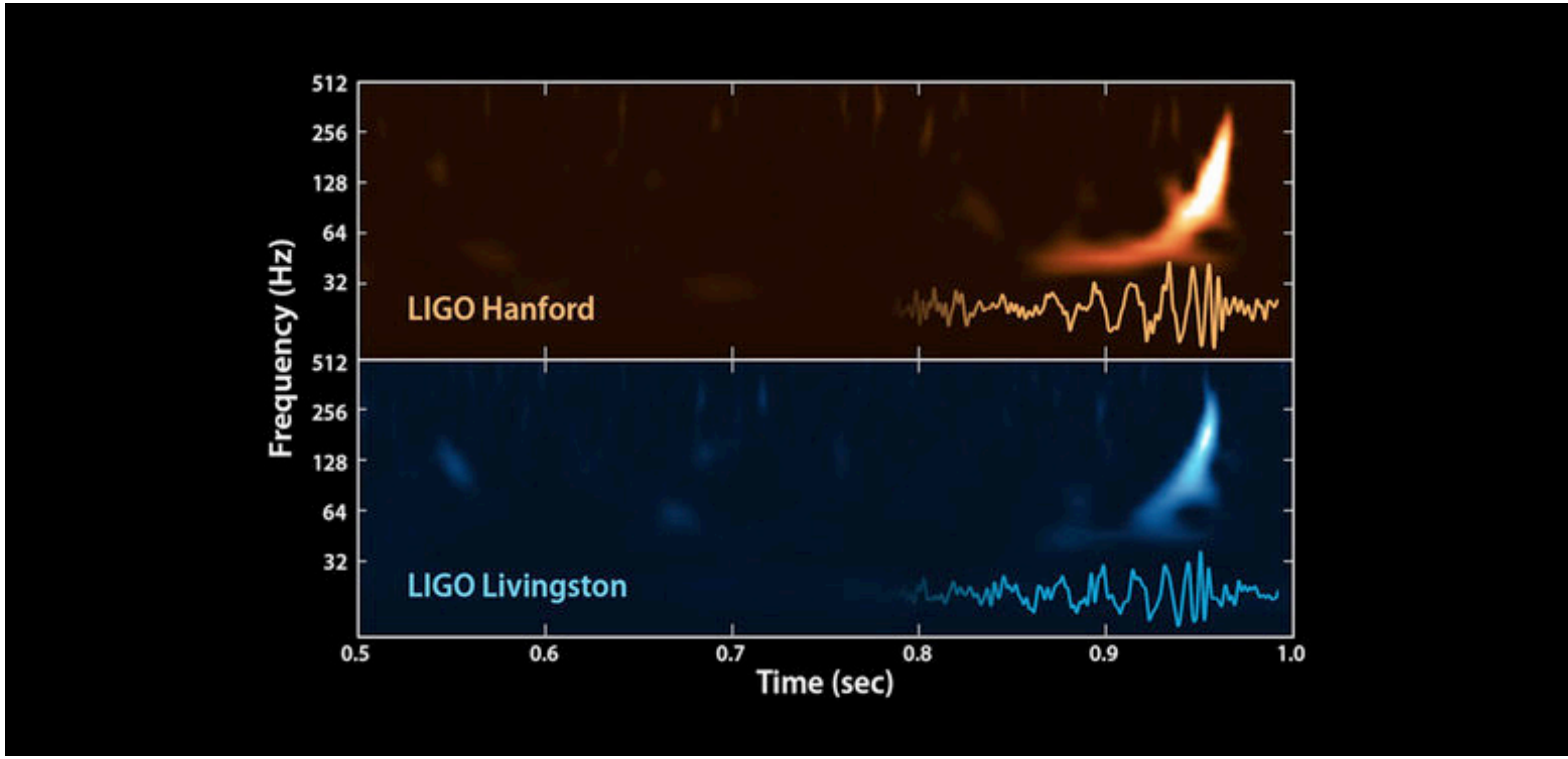

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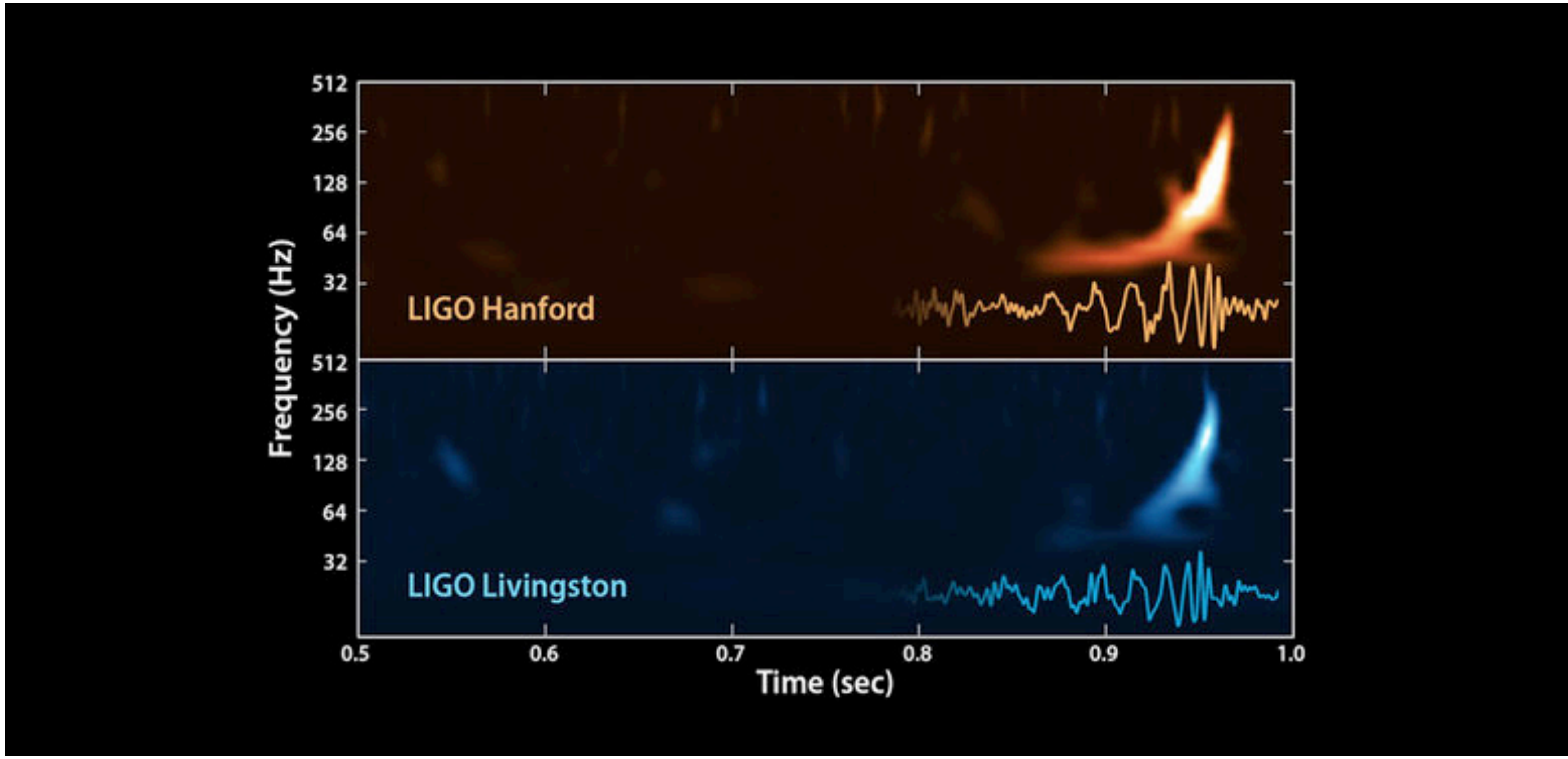

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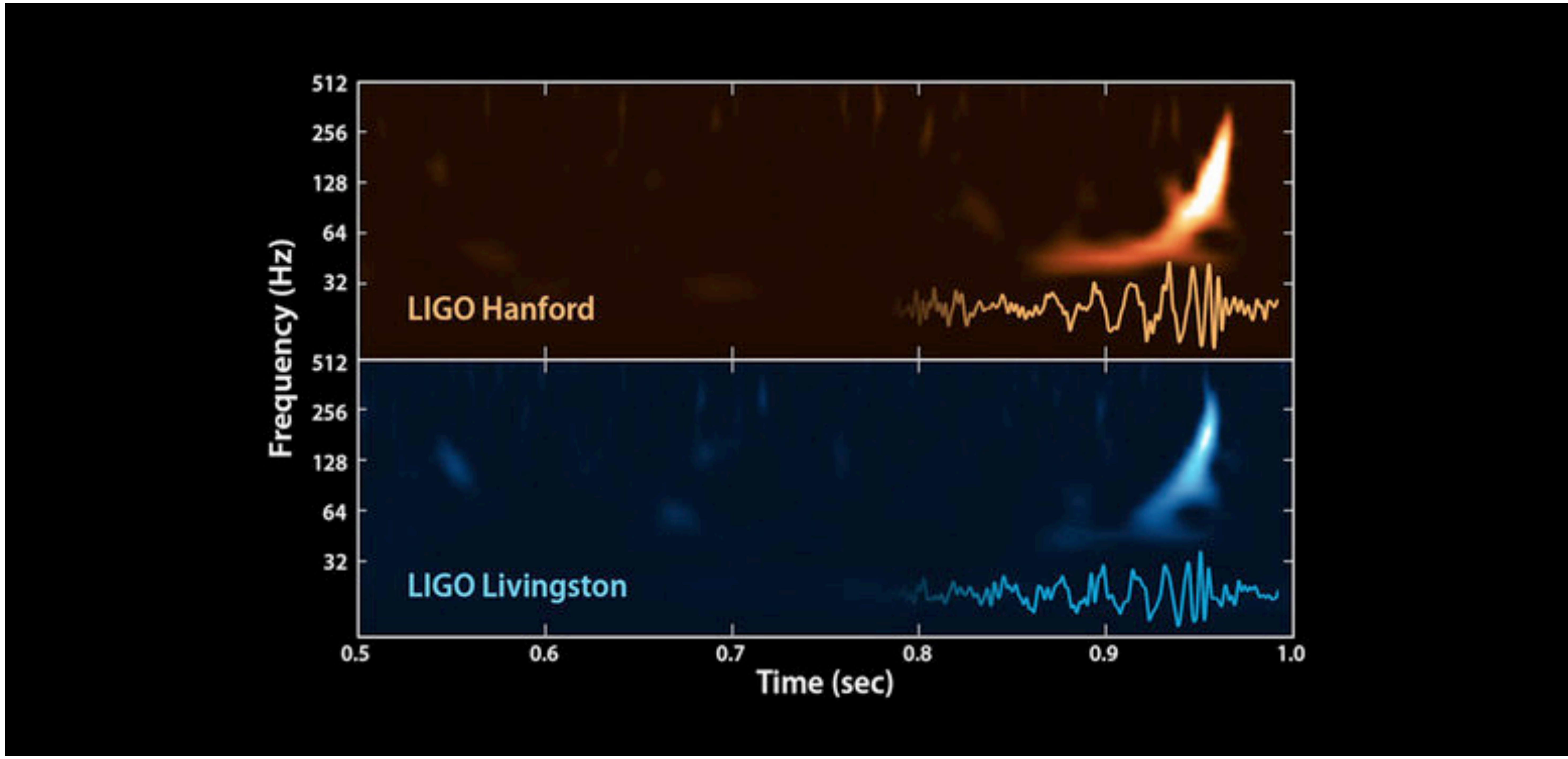

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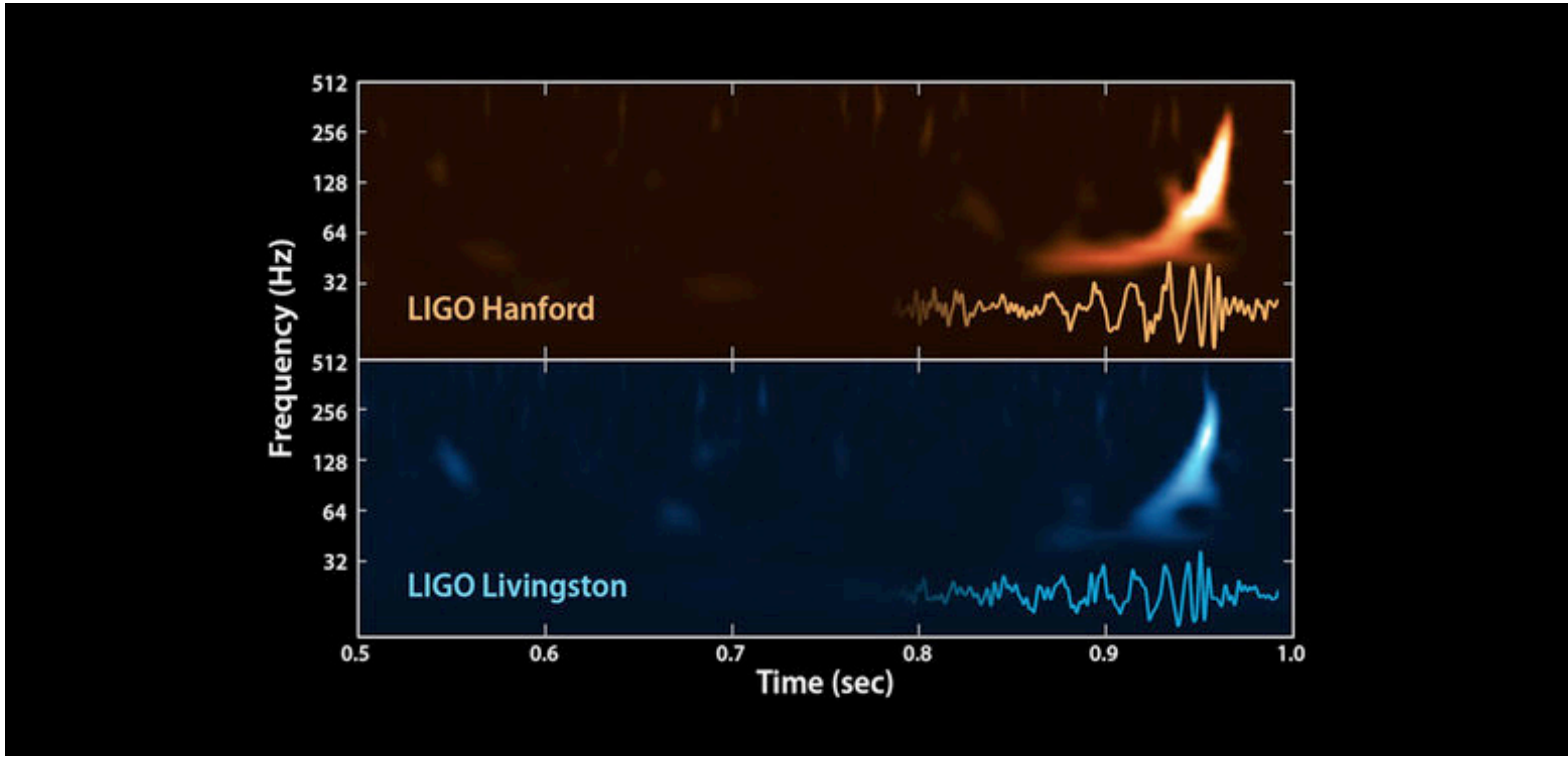

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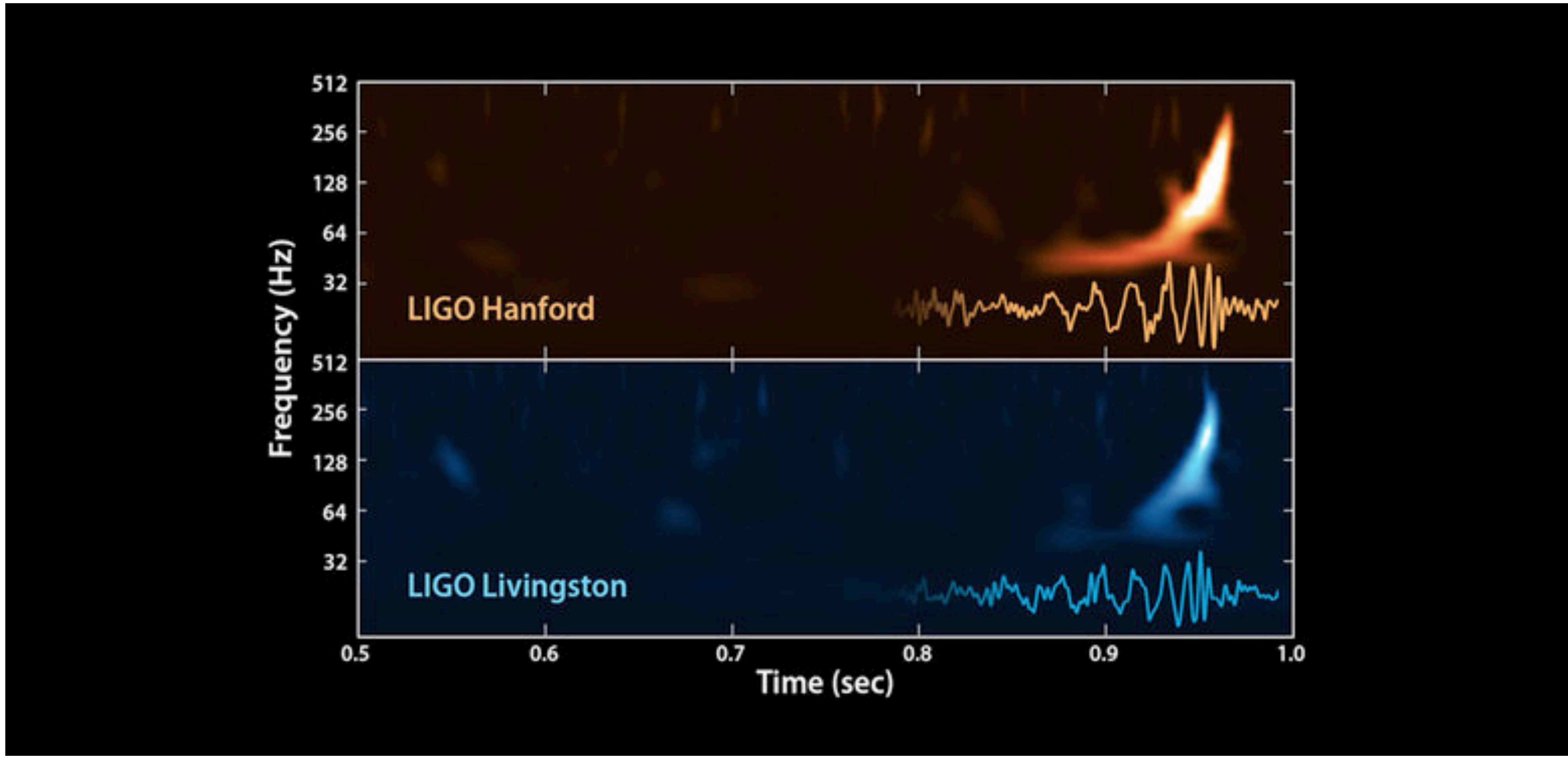

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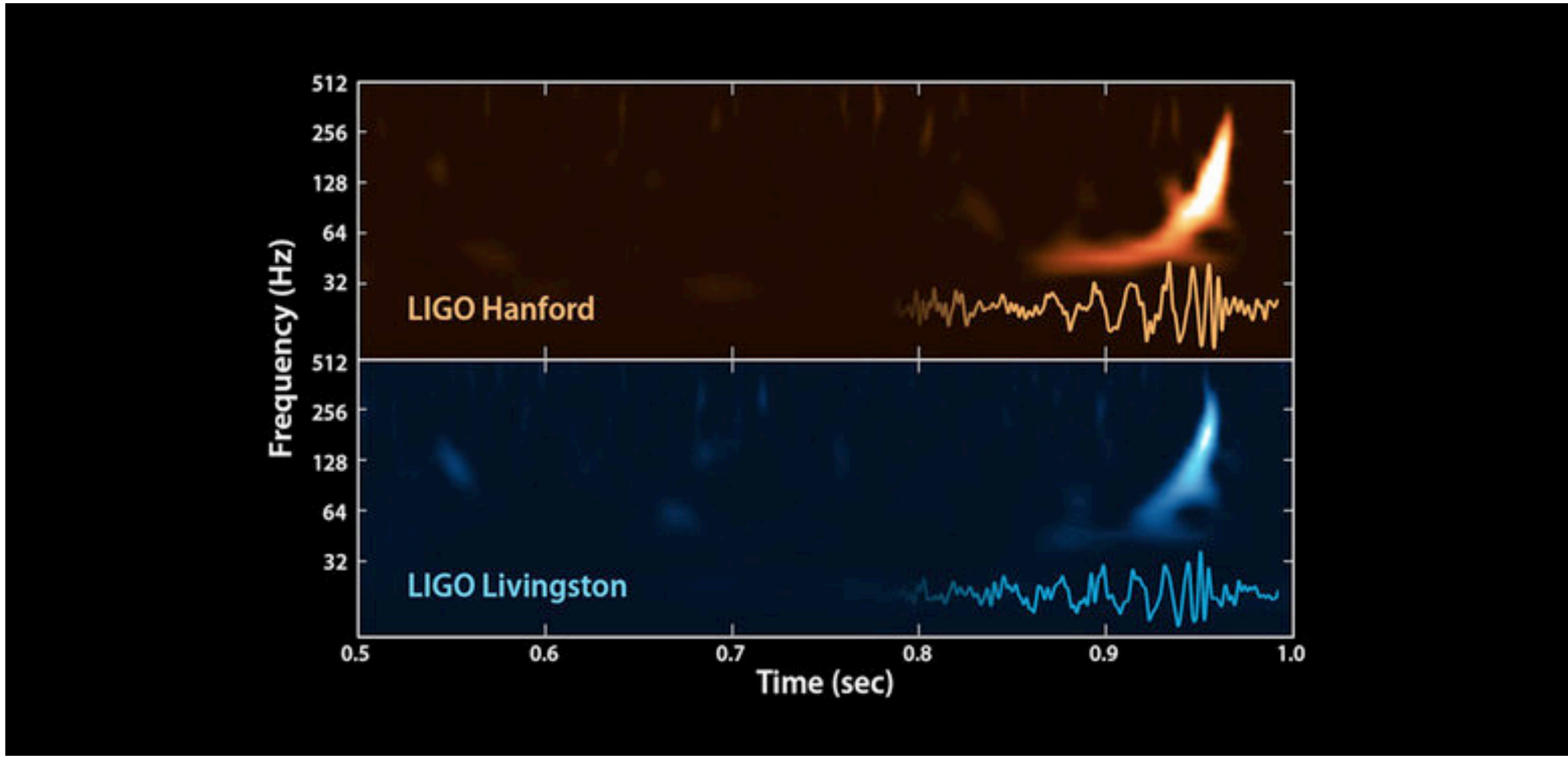

Amplitudes methods allow refined computation and increased precision!

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Test of general relativity in certain regimes of binary mergers where GR observables are extracted from QFT methods.

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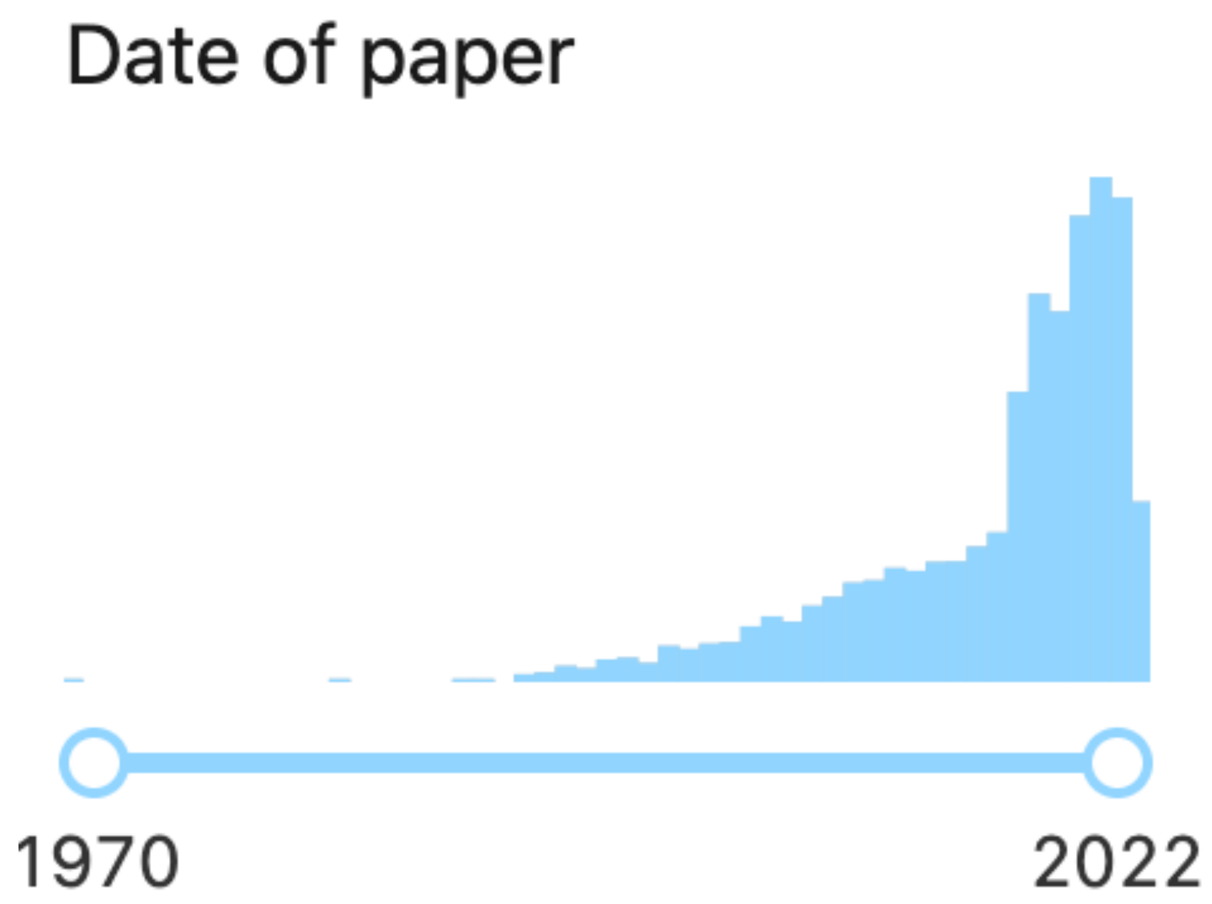

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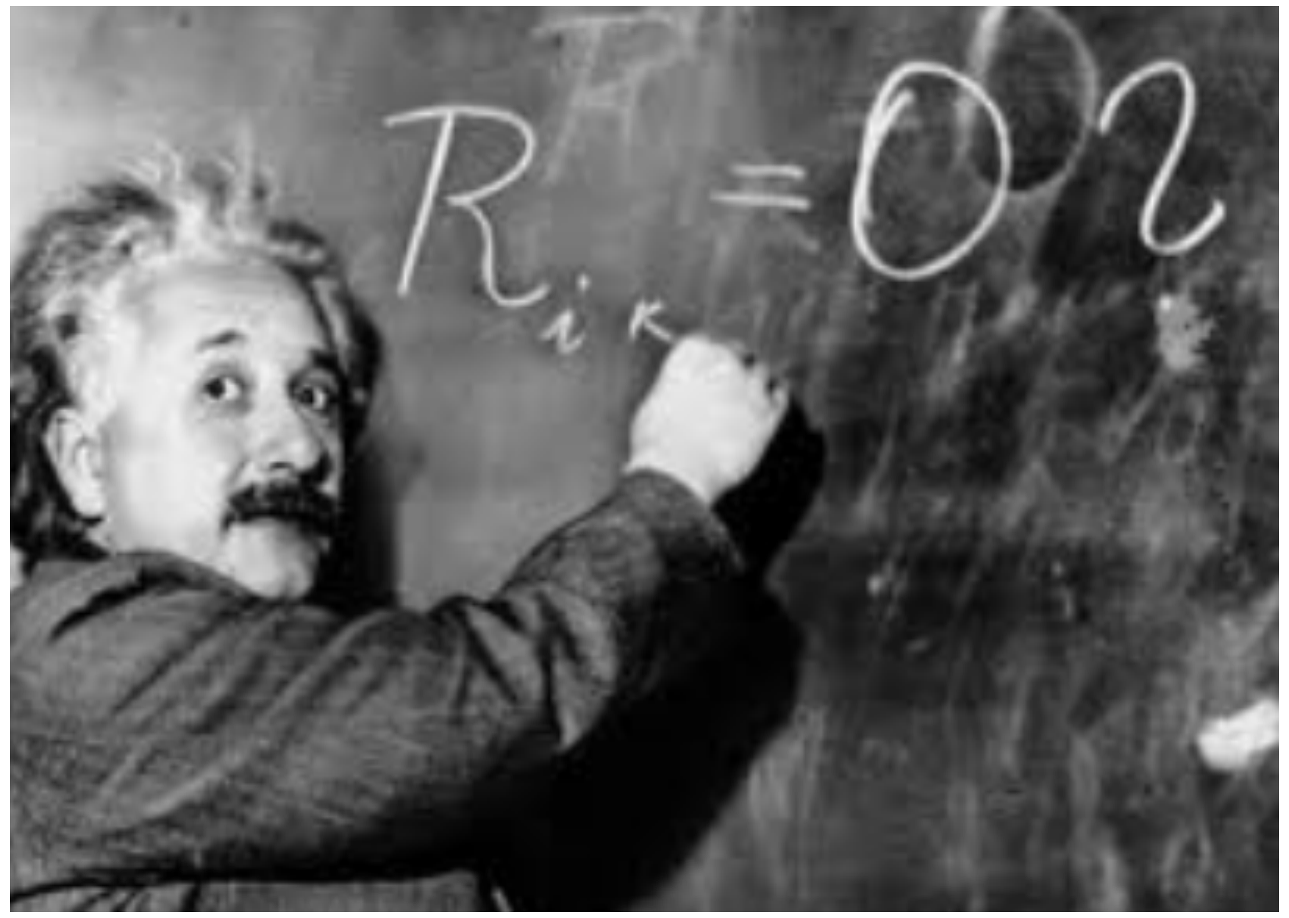

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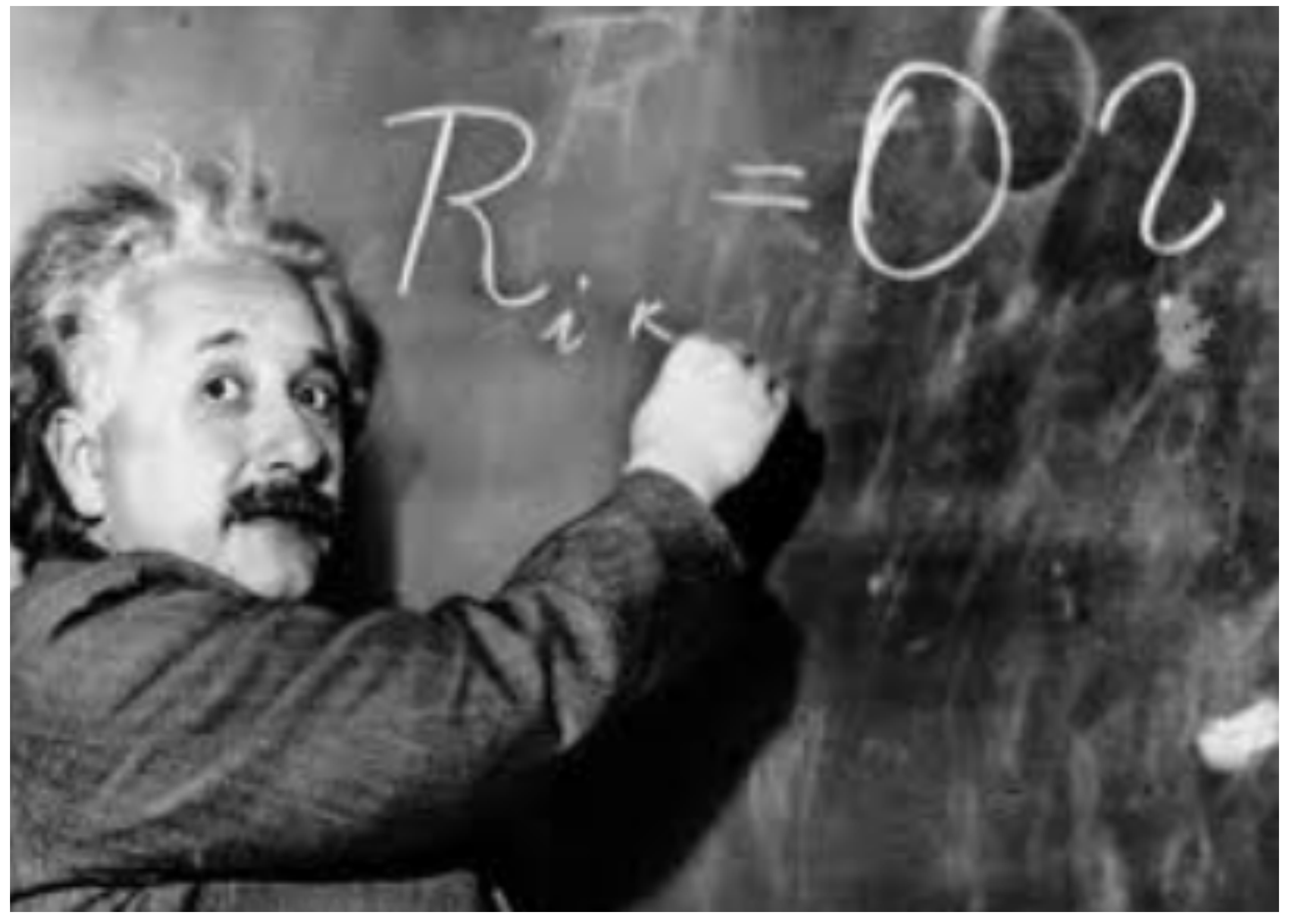

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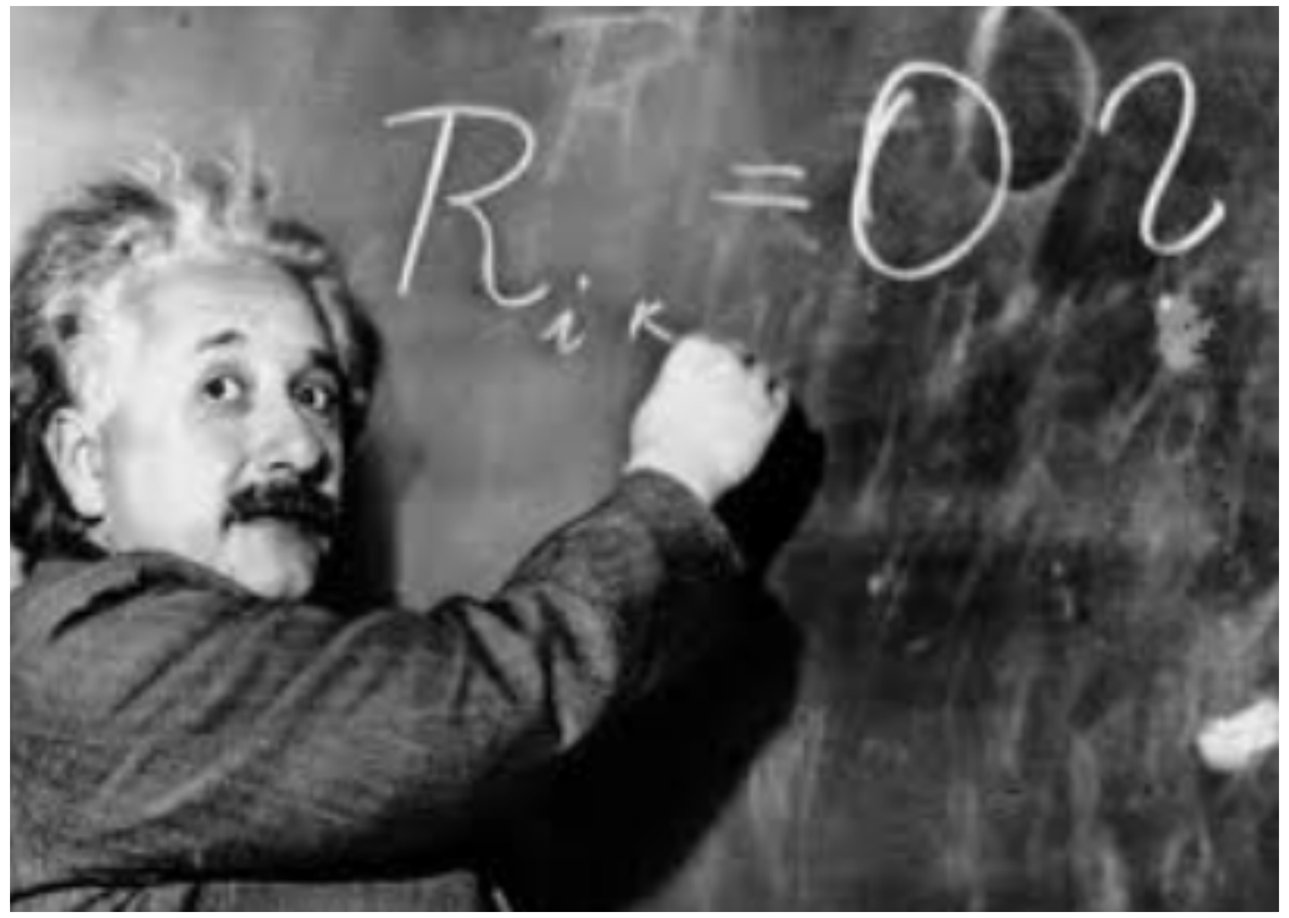

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Off-shell QFT methods: not very computationally efficient!

Amplitudes and Feynman diagrams

- Feynman's method not flawless
- Diagrammatic expansion : huge permutational problem!
  - Scalar field theory : constant vertex (~1 term)
  - Gluons : momentum dependent vertex (~3 terms)
  - Gravitons : momentum dependent vertex (~100 terms)
- Naïve basic 4pt diagram count (graviton exchange) :

100 x 100 ~ 10^4 terms + index contractions (~ 36 pr diagram)
Number of diagrams: (~ 4 !) ~ 10^5 terms ~ 10^6 index contractions
n-point: (~ n !) ~ more atoms in your brain!

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  - GR-EFT is attractive for investigating quantum aspects

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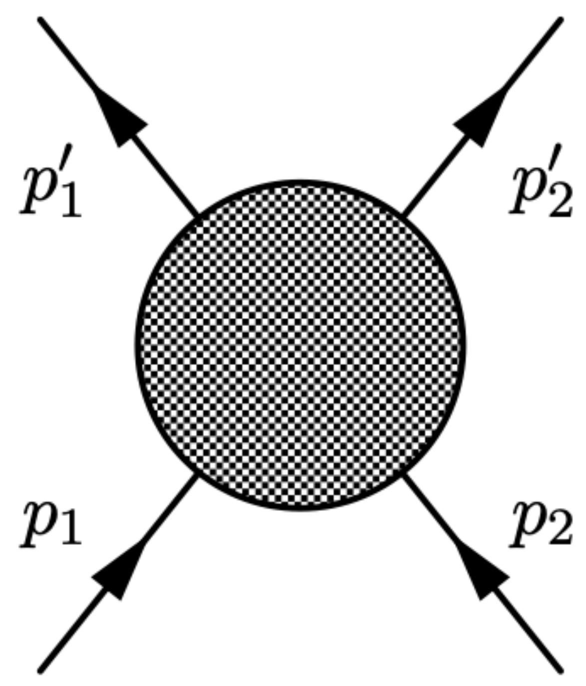
- Consider the 2 -> 2 process from path integral

$$\varphi_1(p_1, m_1) + \varphi_2(p_2, m_2) \to \varphi_1(p_1', m_1) + \varphi_2(p_2', m_2)$$

$$= \sum_{L=0}^{\infty} \mathcal{M}_L(p_1, p_2, p_1', p_2') =$$



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We will also assume (classical) long-distance scattering (this has the consequence that we can focus on non-analytic contributions -> ideal for unitarity)

(NEJBB, Donoghue, Holstein; Cristofoli, NEJBB, Damgaard, Vanhove)

Classical potential from a Lippmann-Schwinger equation

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$$\mathcal{V}(r,p) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \mathcal{V}(p,q) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \tilde{\mathcal{M}}(p,q)$$

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$$\tilde{\mathcal{M}}(p,p') = \mathcal{V}(p,p') + \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{V}(p,k)\mathcal{M}(k,p')}{E_p - E_k + i\varepsilon}$$

Tree level

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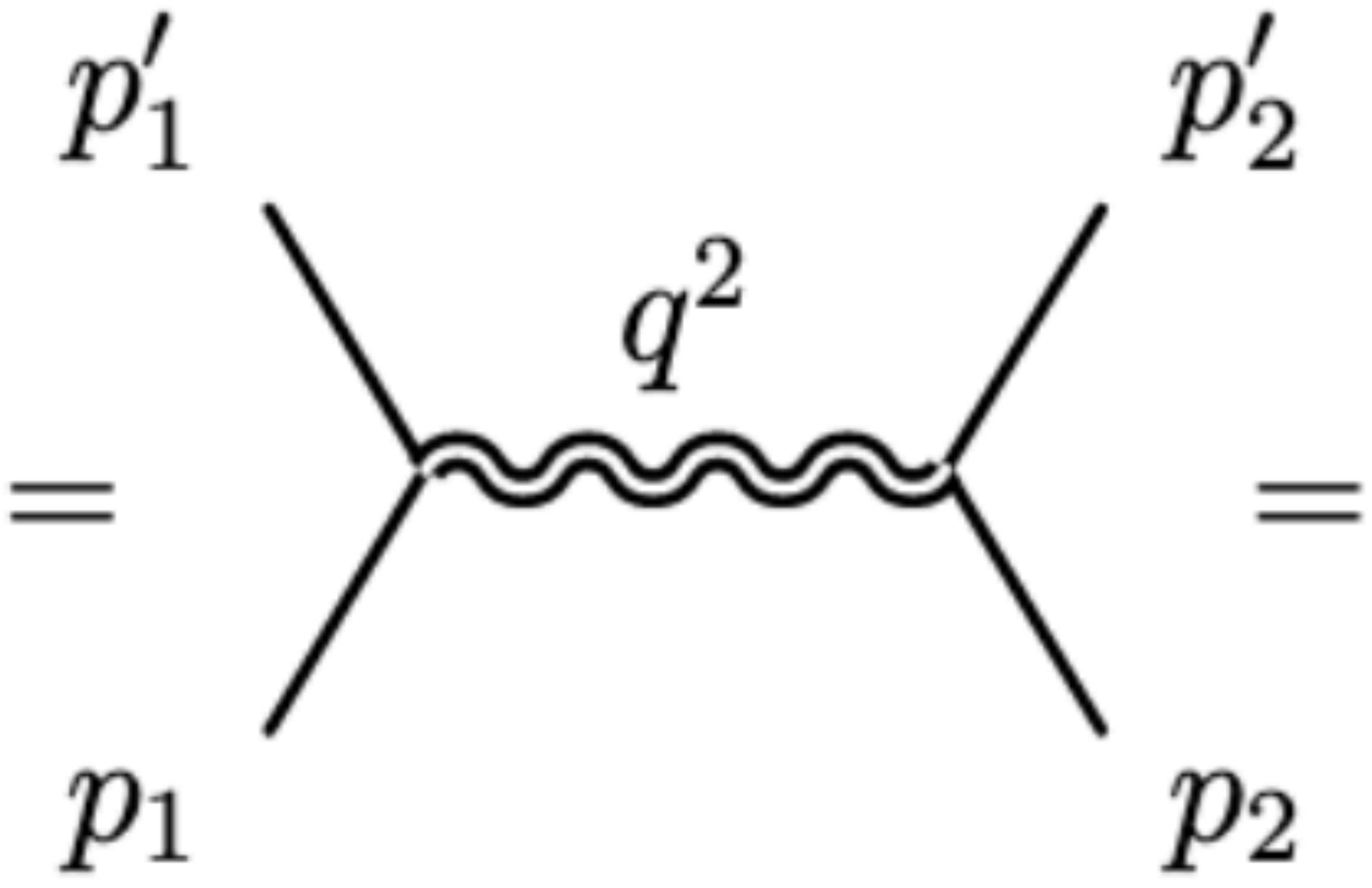
$$\mathcal{M}_0(\gamma, \underline{q}^2, \hbar) =$$



$$= \hbar \frac{2\pi m_1^2 m_2^2 G_N (2\gamma^2 - 1)}{|\underline{q}|^2} + O(\hbar^0)$$

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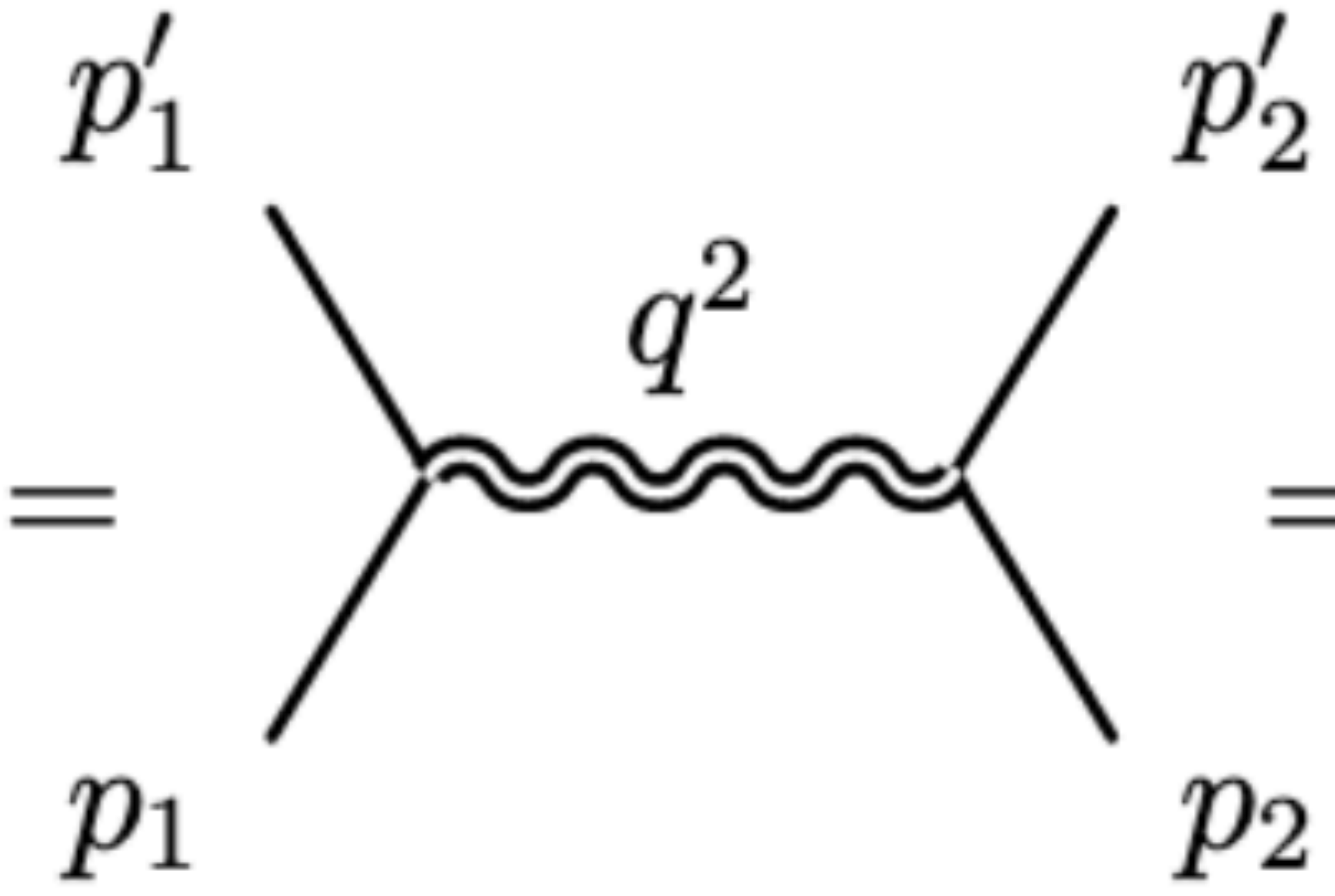

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Newton's law through Fourier transform

$$V(r) = -\frac{Gm_1 m_2}{r}$$

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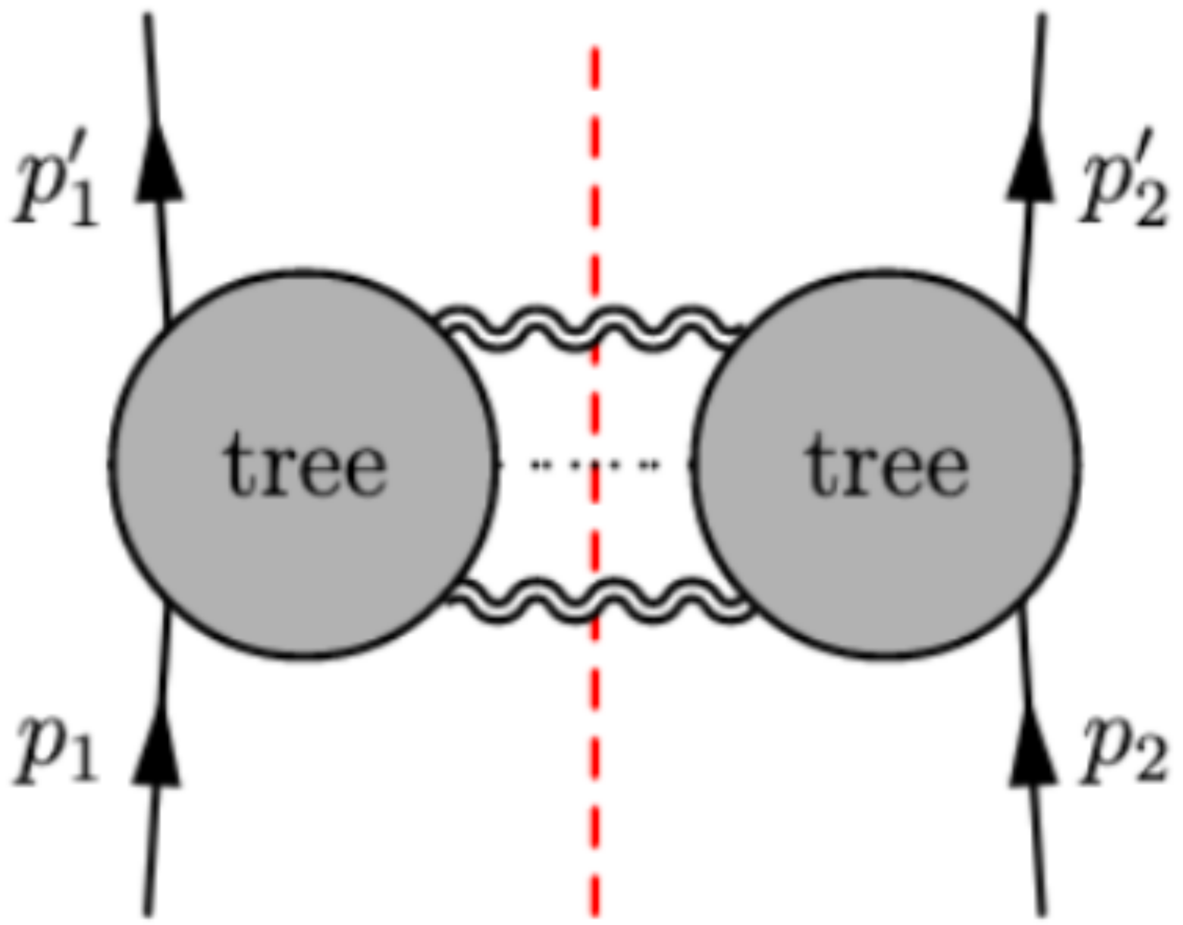

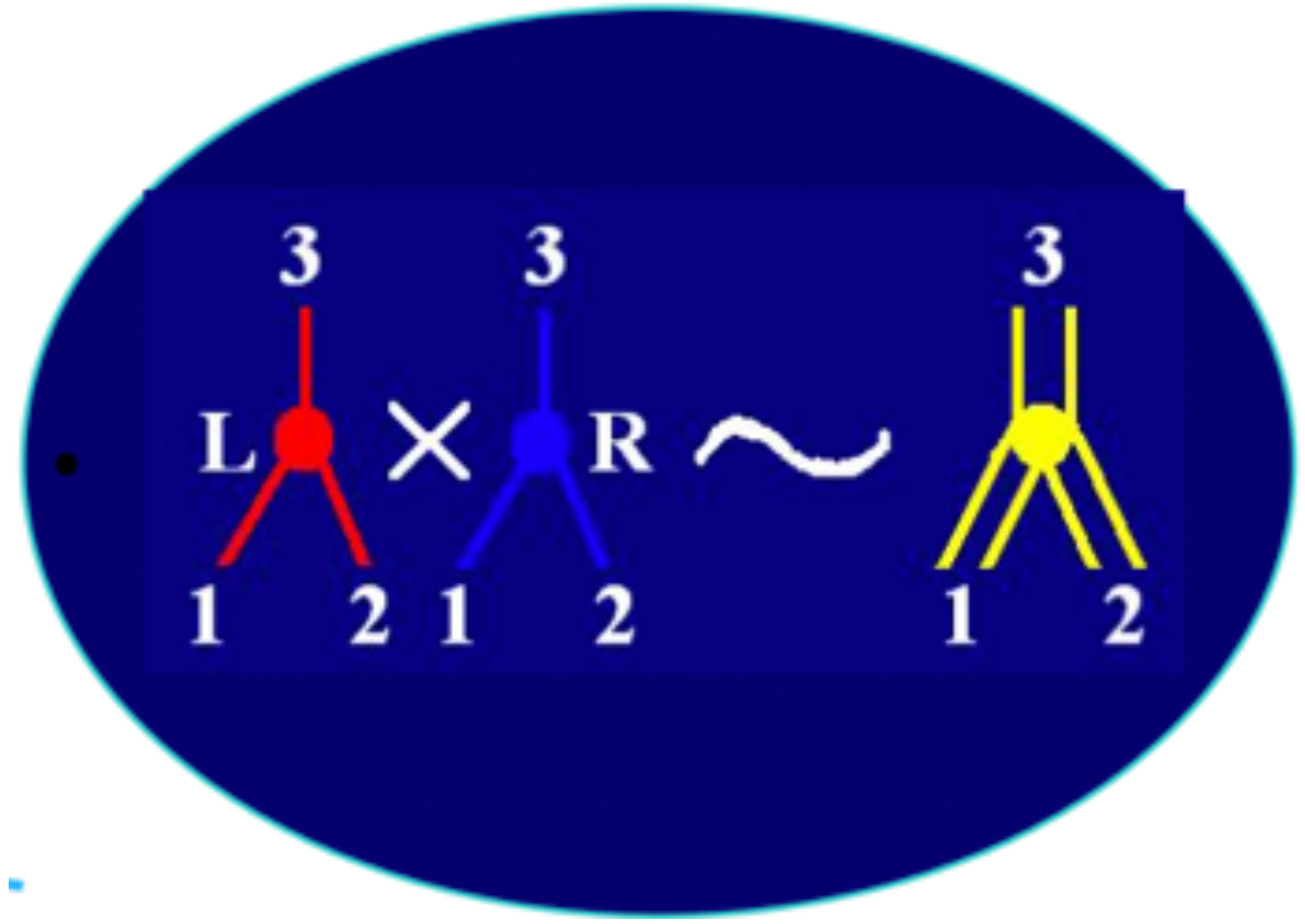

(Neill, Rothstein; NEJBB, Donoghue, Vanhove)

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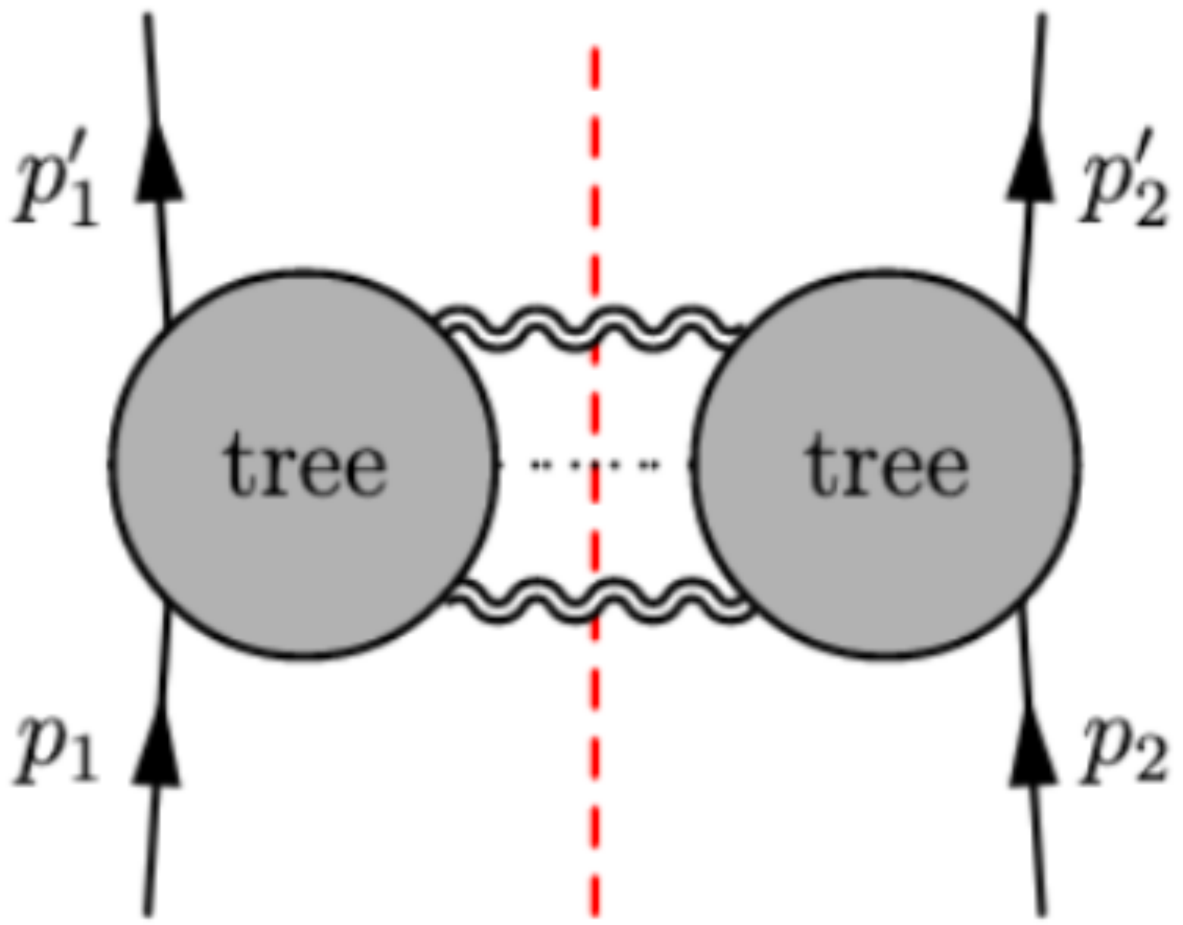
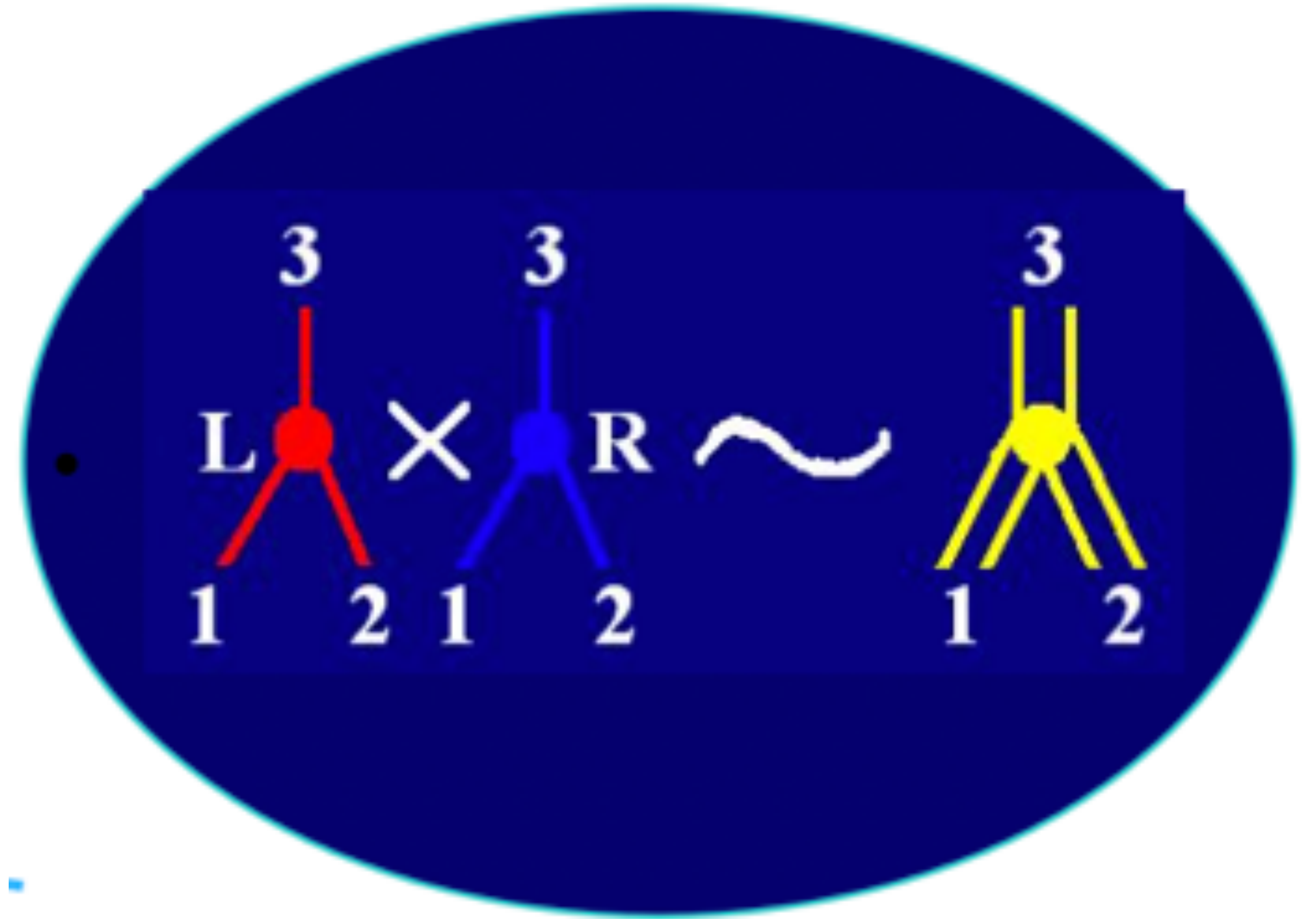




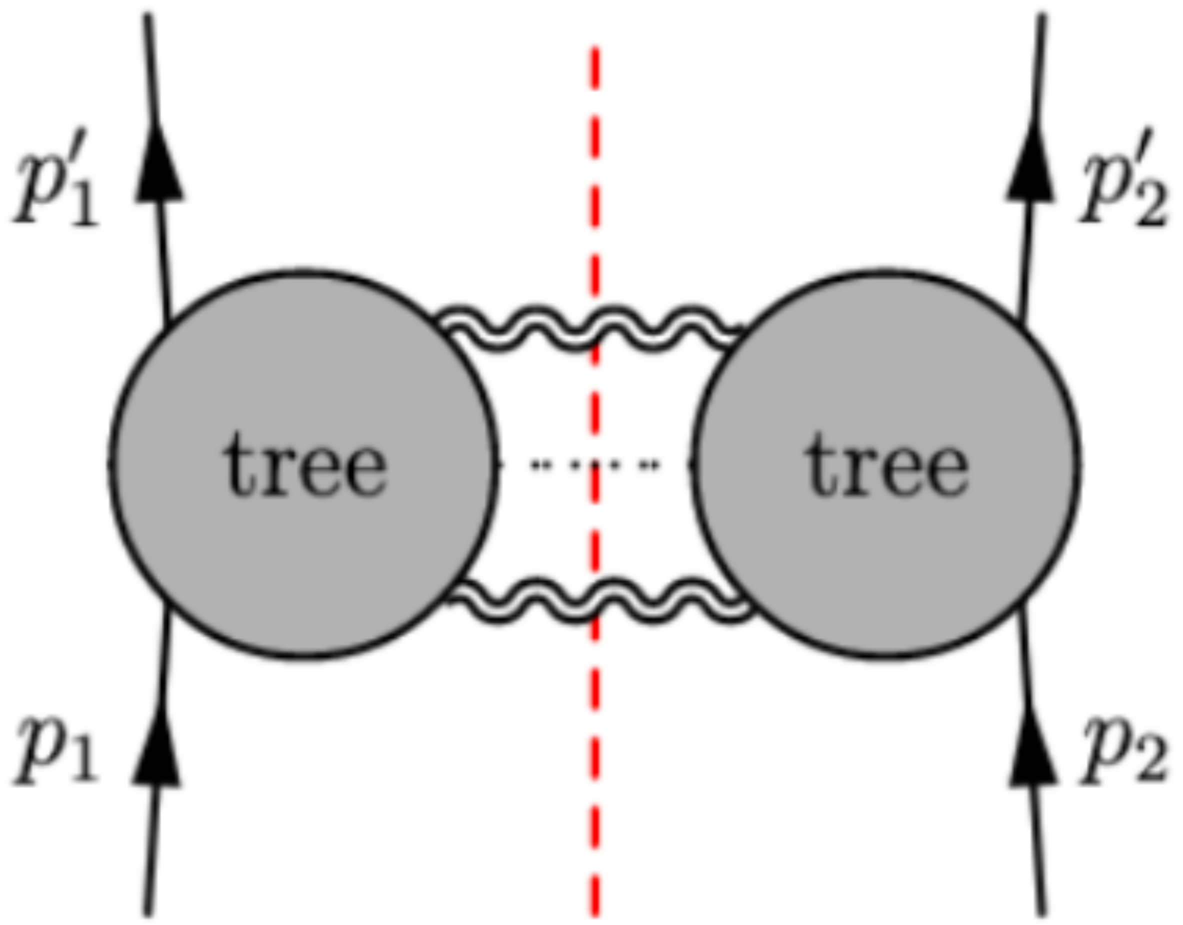
KLT+on-shell input trees (e.g. Badger et al., Forde, Kosower) recycled from Yang-Mills -> gravity

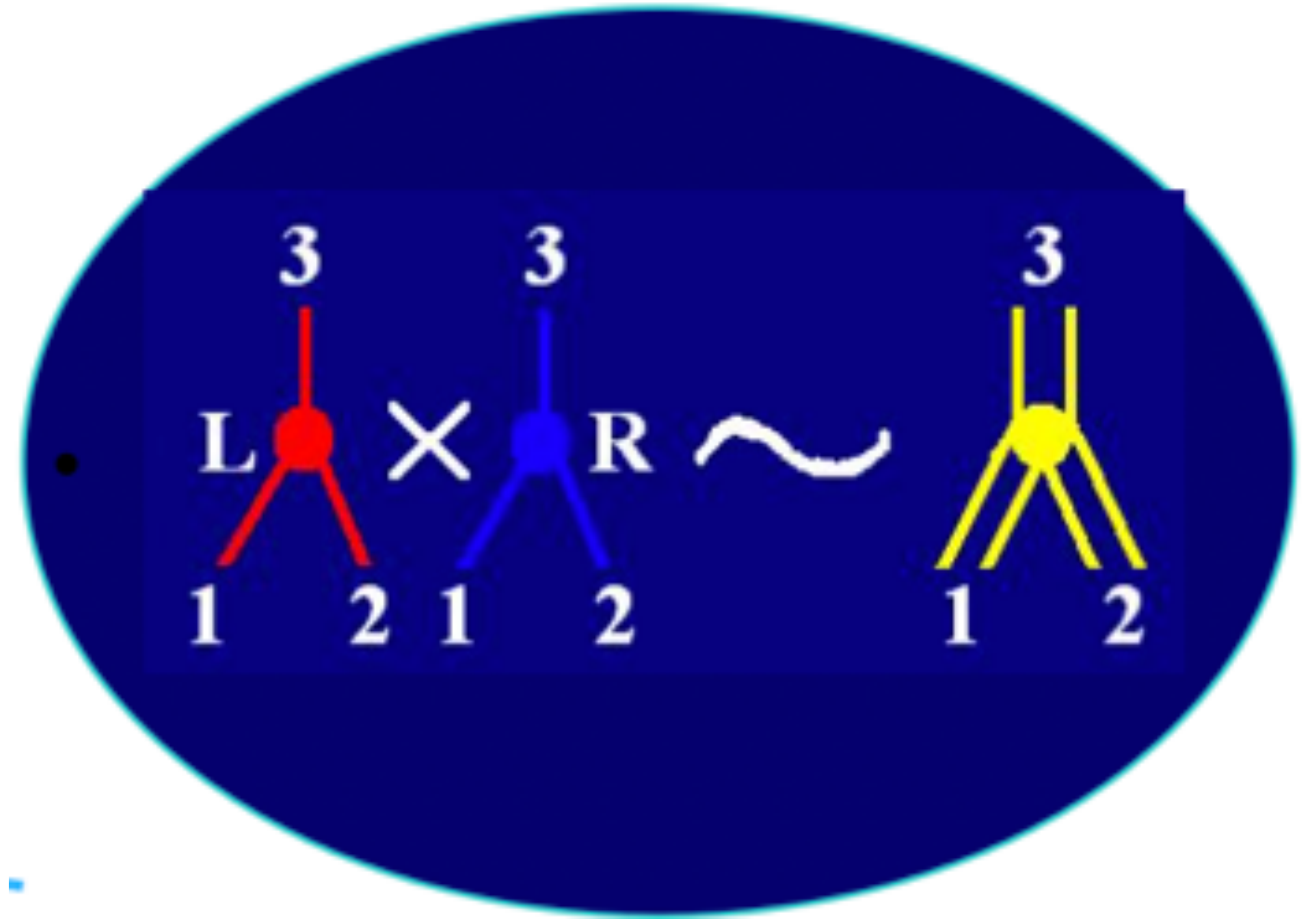
(Neill, Rothstein; NEJBB, Donoghue, Vanhove)

Computations: Loop level

Long range behaviour can be efficiently be captured from the evaluation of unitarity cuts for using on-shell tree amplitudes

$$C_{i,\ldots,j} = \mathrm{Im}_{K_{i,\ldots,j}>0}\, M^{\text{1-loop}}$$





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In D-dimensions from CHY (NEJBB, Cristofoli, Damgaard, Gomez; NEJBB, Plante, Vanhove)

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Classical gravitational scattering from quantum field theory

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- Surprise: Non-linear (classical) corrections from loop diagrams!
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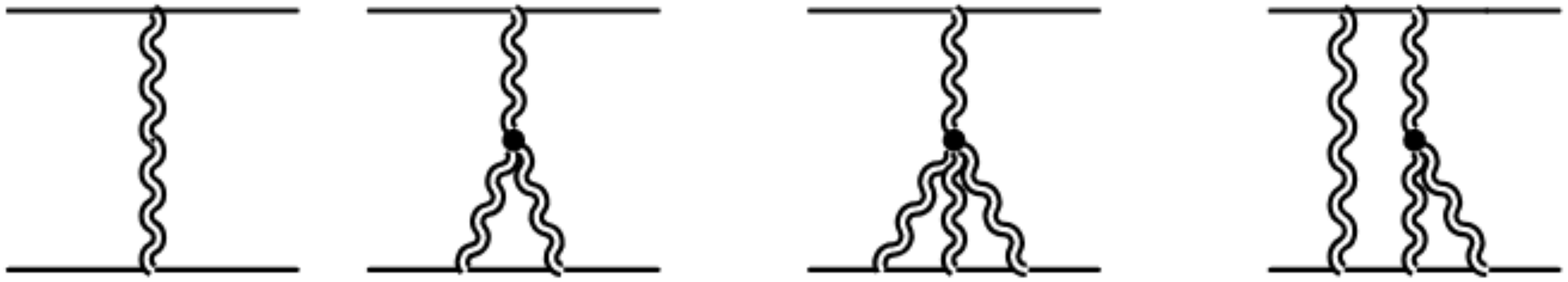

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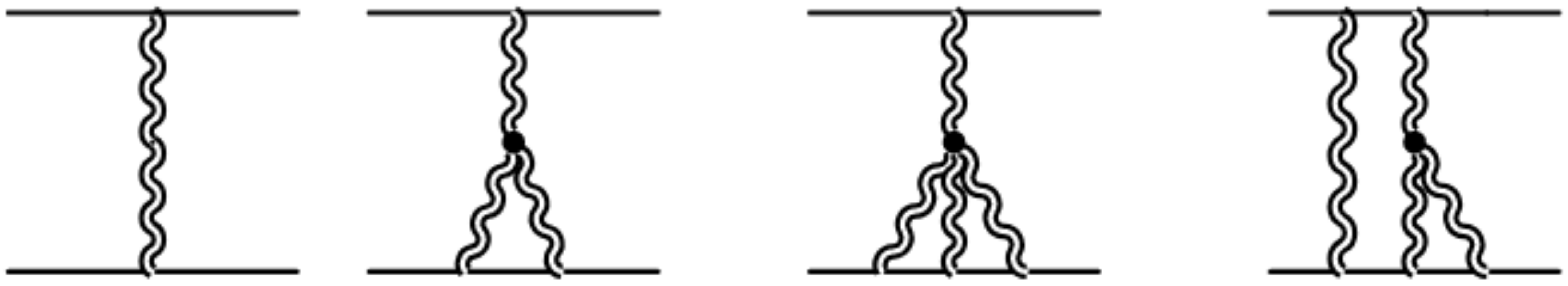

- Define transfer momentum, CM energy

$$q^2 \equiv (p_1 - p_1')^2 \qquad \gamma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}$$

$$\mathcal{E}_{CM}^2 \equiv (p_1 + p_2)^2 \equiv (p_1' + p_2')^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma$$

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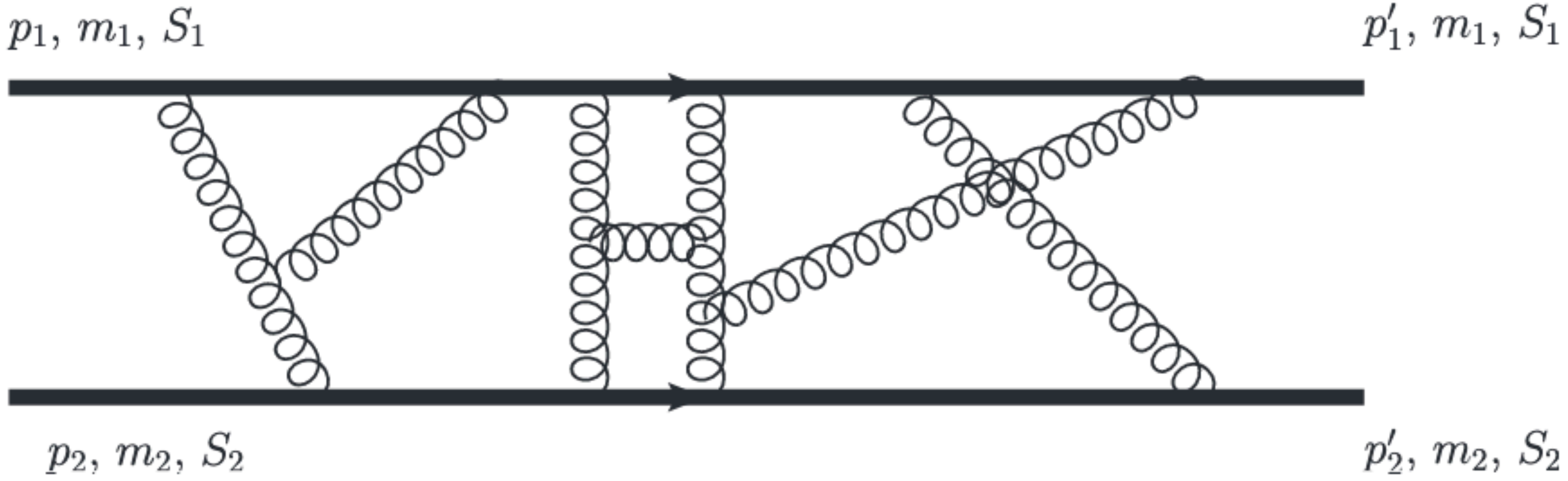

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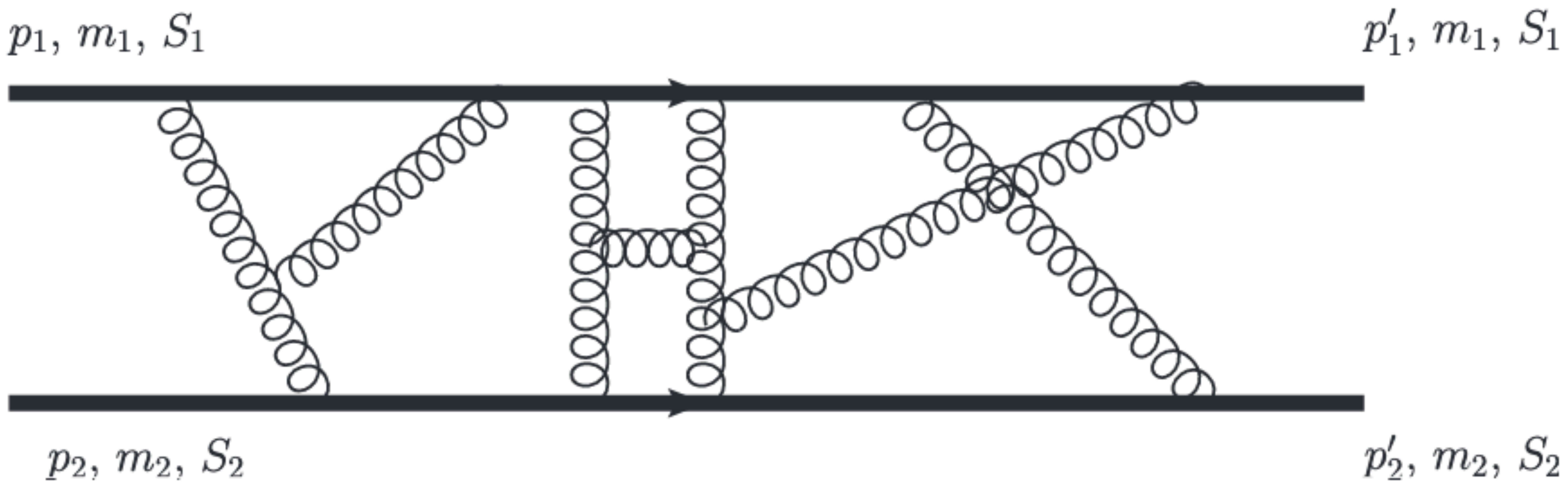

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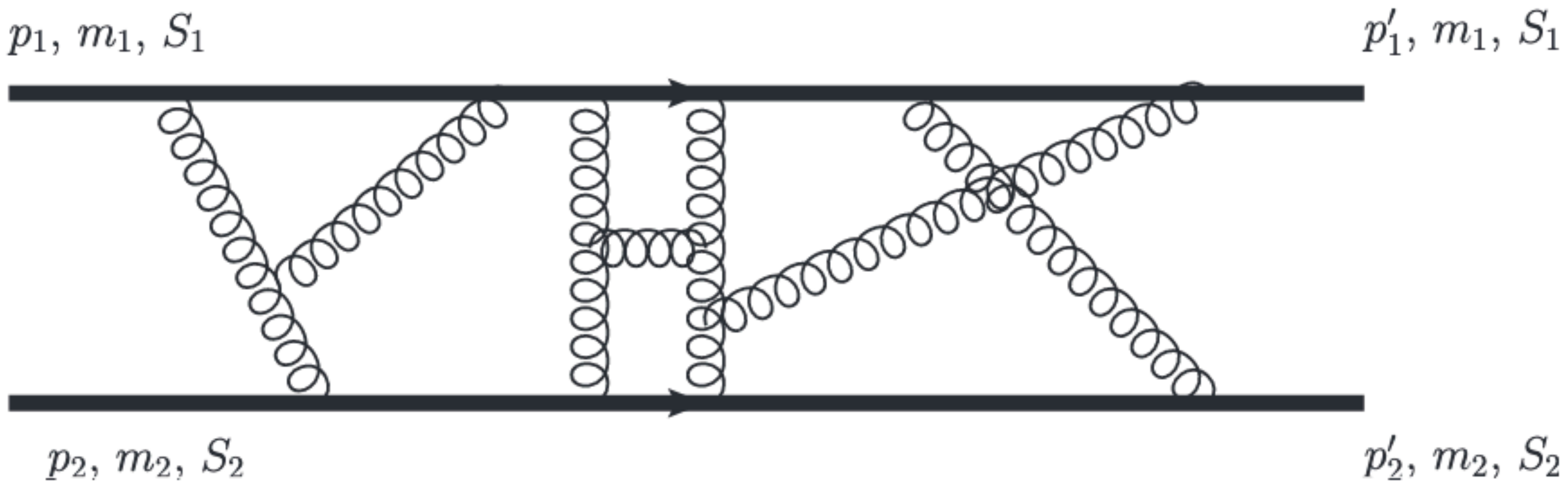

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$$\mathcal{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0) \qquad 1/\hbar$$

Origin: classical pieces in loops

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Expansion of massive propagators

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Expansion of massive propagators $(\ell + p_1)^2 - m_1^2 = \ell^2 + 2\ell \cdot p_1 \simeq 2m_1\ell_0$

$$\frac{1}{2m_1}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2 + i\epsilon}\frac{1}{(\ell + q)^2 + i\epsilon}\frac{1}{\ell_0 + i\epsilon}$$

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Close contour

(NEJBB, Damgaard, Festuccia, Plante, Vanhove)

$$\int_{|\vec{\ell}| \ll m} \frac{d^3\vec{\ell}}{(2\pi)^3}\frac{i}{4m}\frac{1}{\vec{\ell}^2}\frac{1}{(\vec{\ell}+q)^2} = -\frac{i}{32m|\vec{q}|}$$

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The result for the amplitude (in coordinate space) after summing all diagrams in (leading in small momentum transfer contribution):

$$-\frac{Gm_1m_2}{r}\left[1+3\frac{G(m_1+m_2)}{r}+\frac{41}{10\pi}\frac{G\hbar}{r^2}\right]$$

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Classical Quantum

Post-Newtonian term in complete accordance with general relativity (Iwasaki; Holstein and Ross; Neill and Rothstein; NEJBB, Damgaard, Festuccia, Plante, Vanhove)

Einstein-Infeld-Hoffman Potential

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Solve for potential in non-relativistic limit (Born subtraction)

$$\begin{aligned} i\langle f|T|i\rangle &= -2\pi i\delta(E-E') \\ &\times \left[\langle f|\tilde{V}_{bs}(\mathbf{q})|i\rangle + \sum_n \frac{\langle f|\tilde{V}_{bs}(\mathbf{q})|n\rangle\langle n|\tilde{V}_{bs}(\mathbf{q})|i\rangle}{E-E_n+i\epsilon} + \ldots\right] \end{aligned}$$

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$$
\tilde{V}_{bs}(r) = V(r) + \frac{7Gm_1m_2(m_1+m_2)}{2c^2r^2}
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Post-Newtonian interaction potentials

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$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_4^2}{2m_2} - \frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_4^4}{8m_2^3}$$

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$$3 - \frac{7}{2} = -\frac{1}{2}$$

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- **NB:** Many other problems can be considered in this framework

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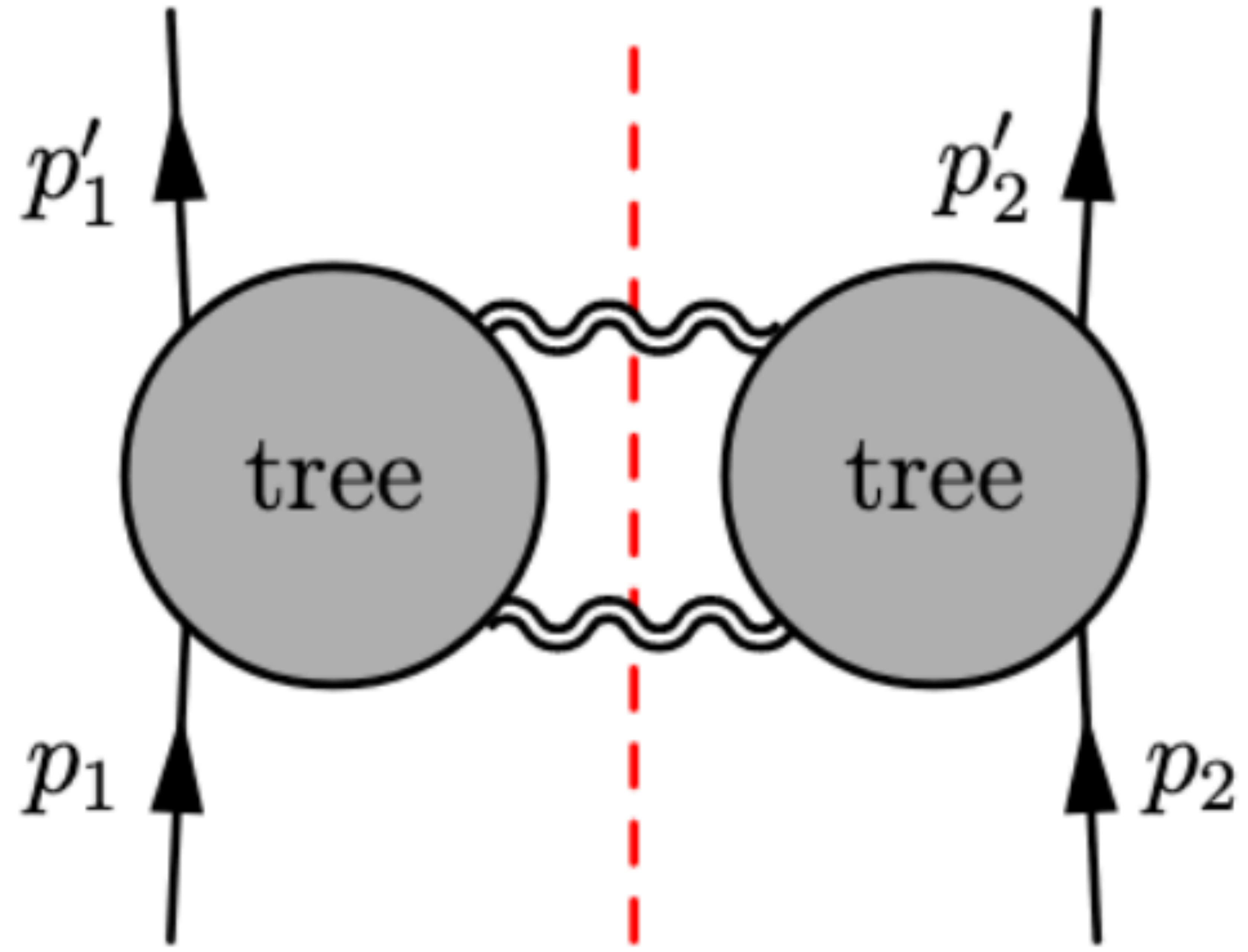

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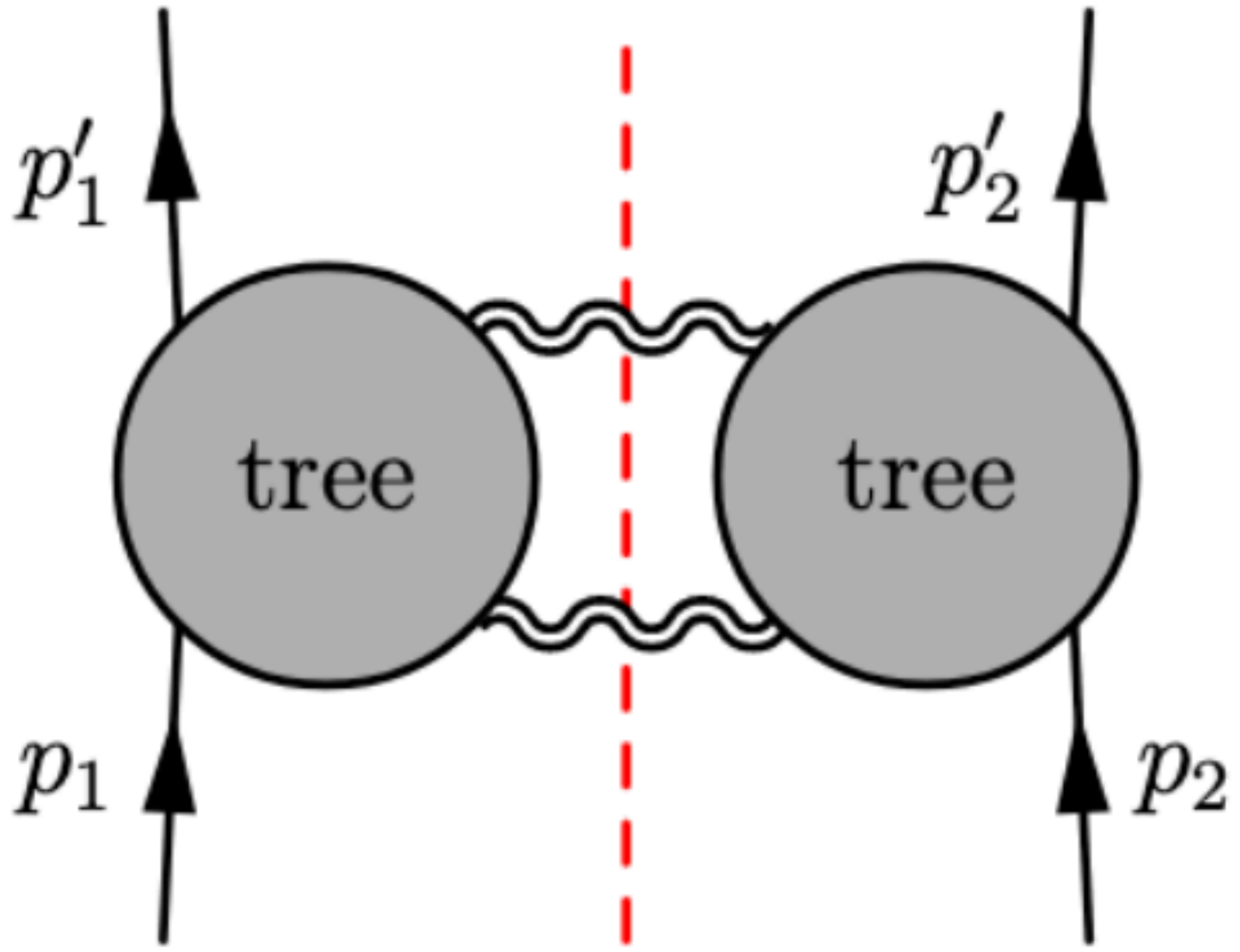

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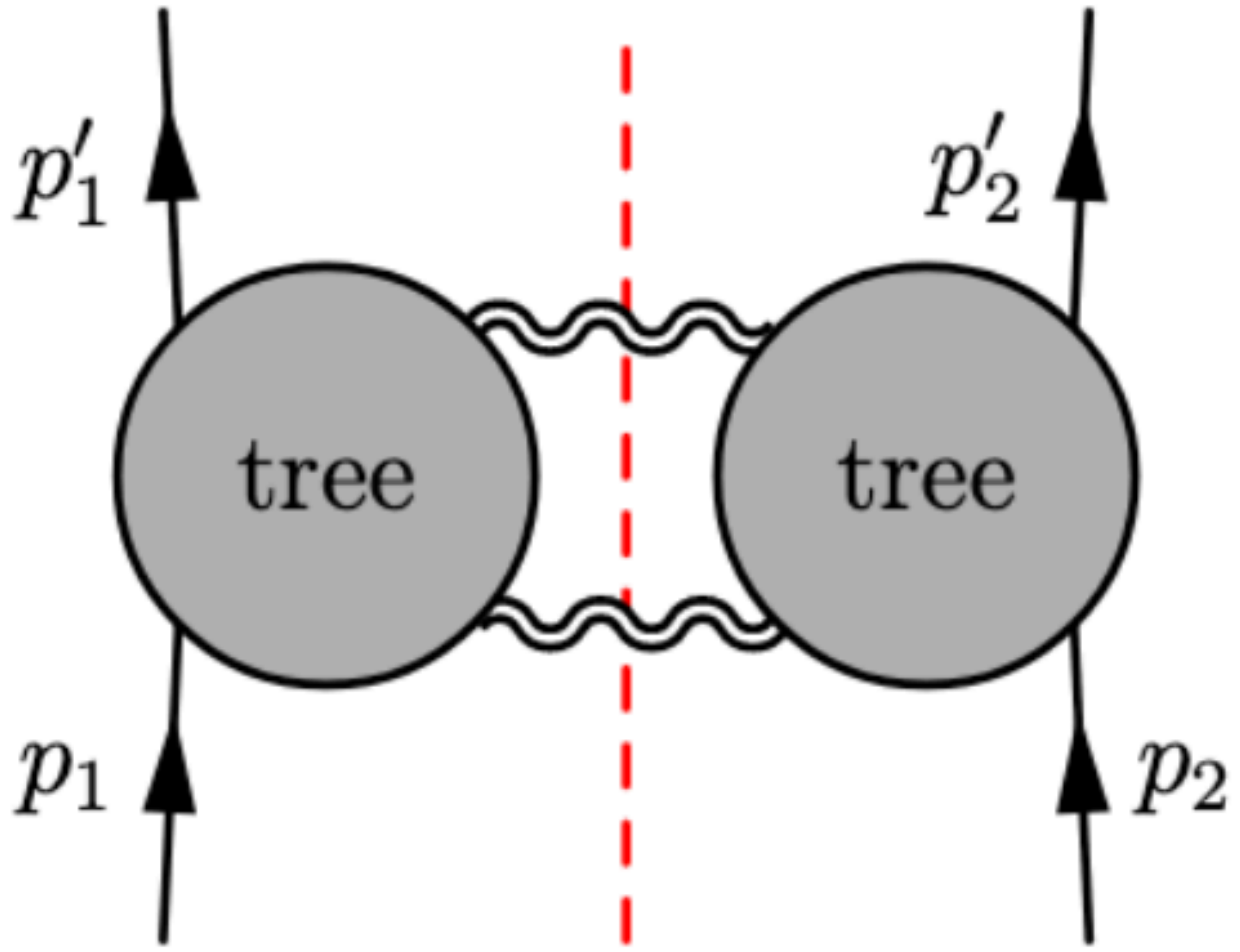

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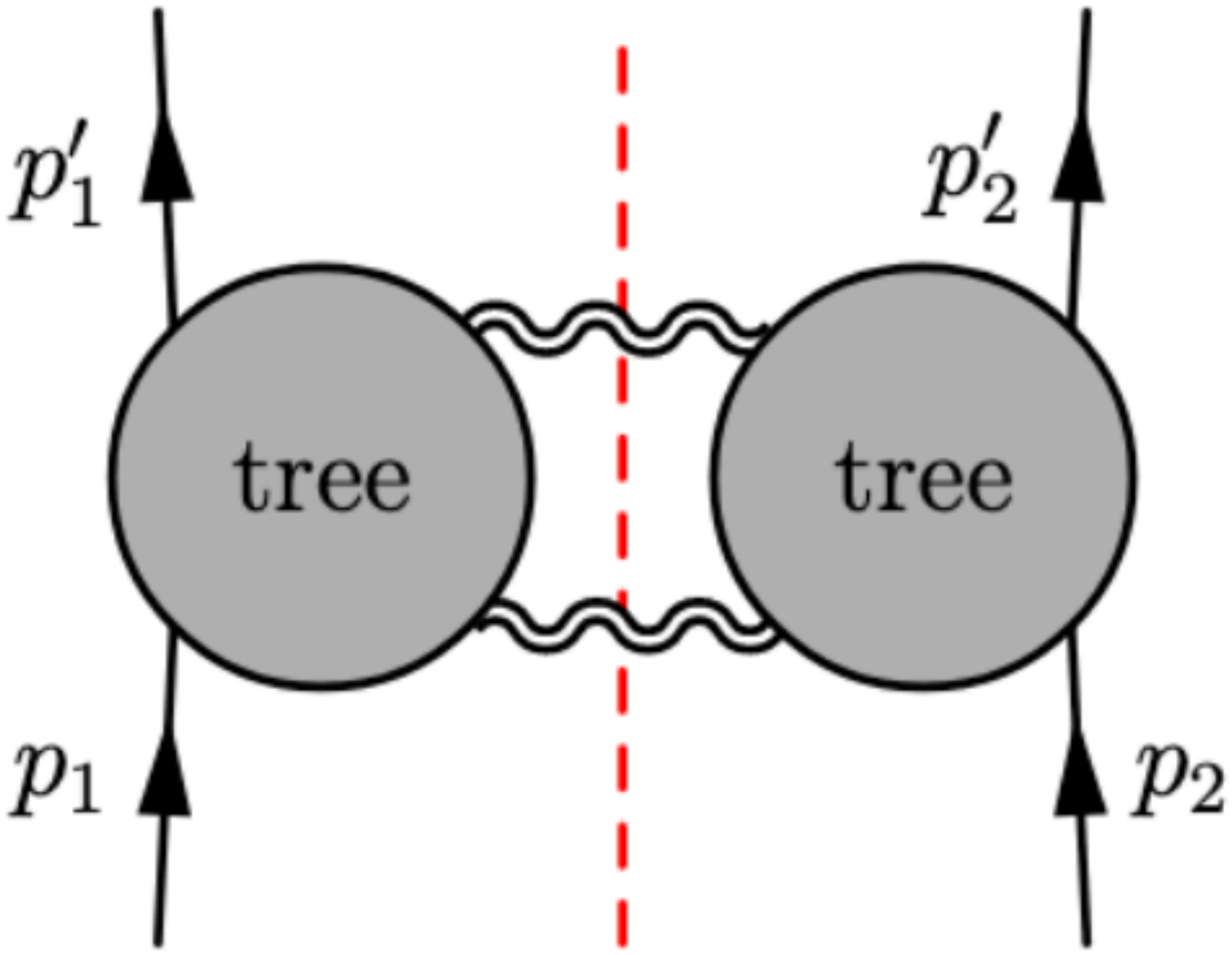

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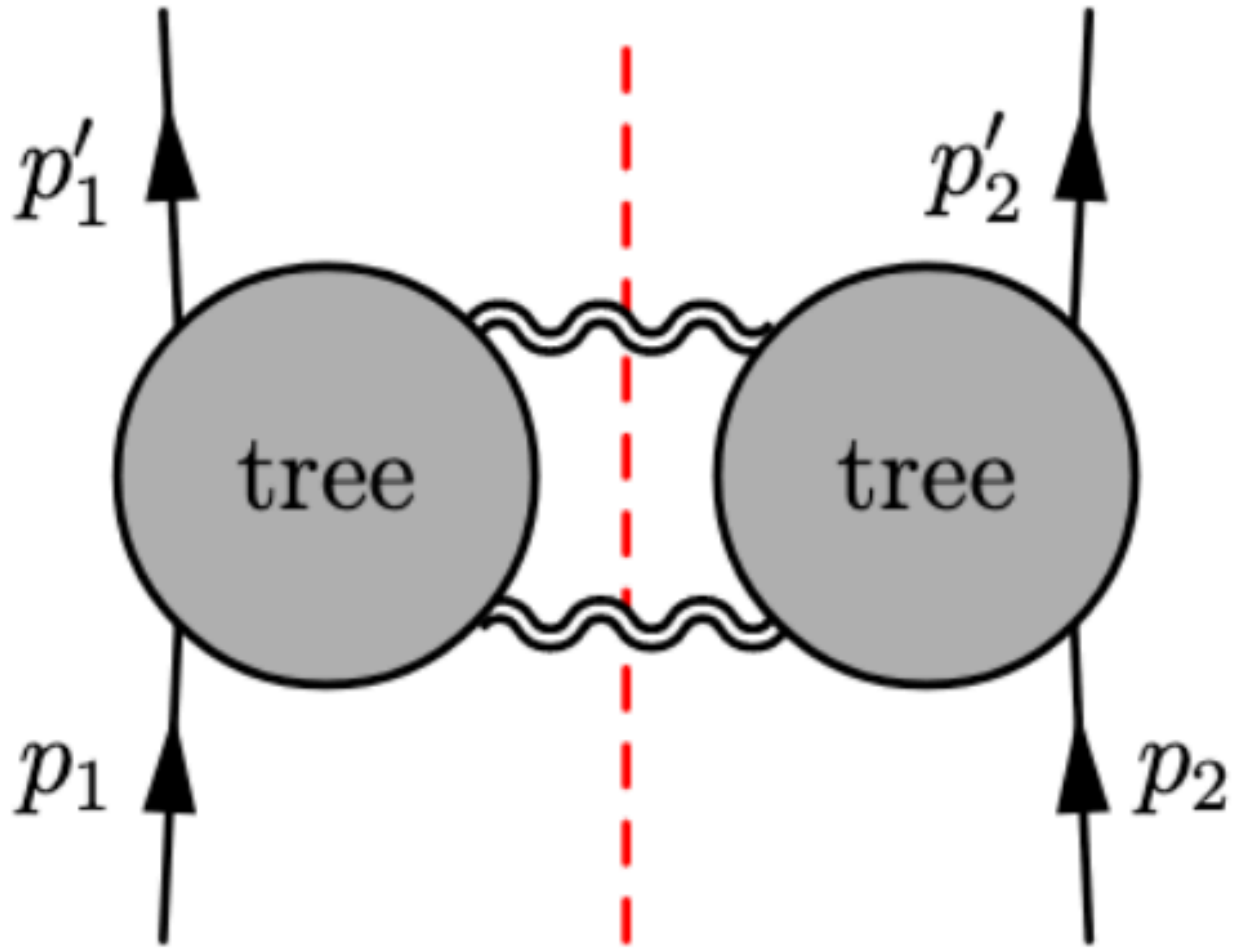

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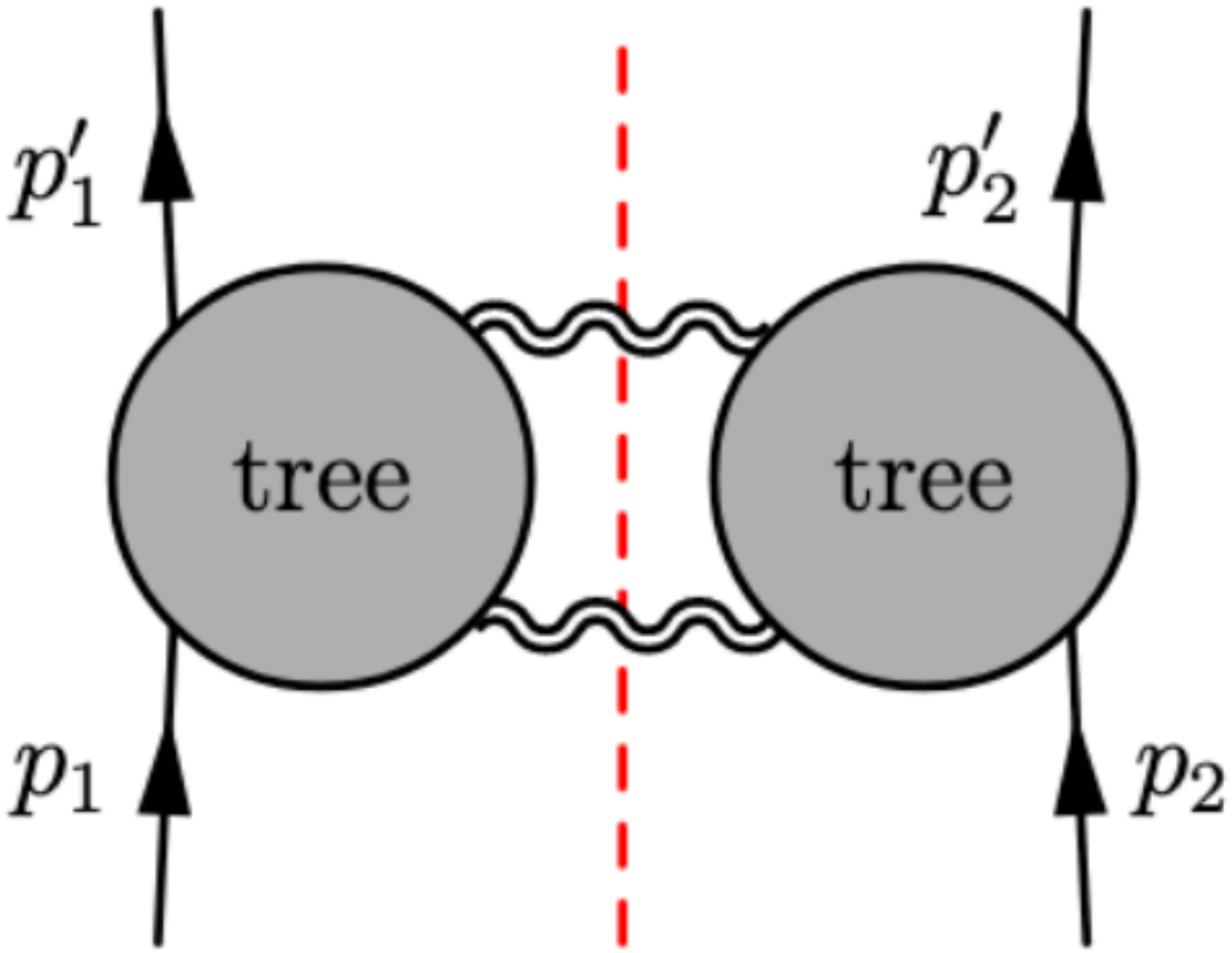

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$$\begin{aligned}
\mathcal{M}_1^{(-2)}(\gamma,\underline{q}^2) &= \mathcal{M}_1^{\square(-2)}(\gamma,\underline{q}^2),\\
\mathcal{M}_1^{(-1)}(\gamma,\underline{q}^2) &= \mathcal{M}_1^{\square(-1)}(\gamma,\underline{q}^2) + \mathcal{M}_1^{\triangleright(-1)}(\gamma,\underline{q}^2) + \mathcal{M}_1^{\triangleleft(-1)}(\gamma,\underline{q}^2),\\
\mathcal{M}_1^{(0)}(\gamma,\underline{q}^2) &= \mathcal{M}_1^{\square(0)}(\gamma,\underline{q}^2) + \mathcal{M}_1^{\triangleright(0)}(\gamma,\underline{q}^2) + \mathcal{M}_1^{\triangleleft(0)}(\gamma,\underline{q}^2) + \mathcal{M}_1^{\circ(0)}(\gamma,\underline{q}^2)
\end{aligned}$$

Result for the one-loop amplitude

The amplitude has a Laurent expansion

$$\mathcal{M}_1(\gamma, \underline{q}^2, \hbar) = \frac{1}{|\underline{q}|^{4-D}} \left(\frac{\mathcal{M}_1^{(-2)}(\gamma, \underline{q}^2)}{\hbar^2} + \frac{\mathcal{M}_1^{(-1)}(\gamma, \underline{q}^2)}{\hbar} + \mathcal{M}_1^{(0)}(\gamma, \underline{q}^2) + \mathcal{O}(\hbar) \right)$$

$$\mathcal{M}_1^{(-2)}(\gamma, \underline{q}^2) = \mathcal{M}_1^{\square(-2)}(\gamma, \underline{q}^2),$$

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Order by order in Planck's constant

PM potential one-loop amplitude

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$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_{\square}\mathcal{I}_{\square} + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\right)$$

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$$\mathcal{I}_\Box = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_a^2 + i\varepsilon)((\ell-p_3)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}$$

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Putting it all together

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Ignore quantum keep only classical pieces

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$$c_\Box = c_{\bowtie} = 16 m_1^4 m_2^4 \frac{(1-(D-2)\sigma^2)^2}{(D-2)^2},$$

$$c_{\triangleright} = \frac{4m_1^4 m_2^2\,(D - 7 + (D(4D-17)+19)\sigma^2)}{(D-2)^2}$$

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Relation to a PM potential

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One-loop amplitude after summing all contributions

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(NEJBB, Cristofoli,
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Imaginary super-classical/singular ..

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One-loop

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Born subtraction important to make contact with classical physics

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$$\mathcal{M}^{\text{Iterated}} = \frac{i\pi G_N^2}{E_p^3\xi}\frac{4c_1^2}{|\vec{p}|}\frac{\left(\log|\vec{q}|^2 - \frac{2}{3-d}\right)}{|\vec{q}|^2} + \frac{2\pi^2 G_N^2}{E_p^3\xi^2|\vec{q}|}\left(\frac{c_1^2(\xi-1)}{2E_p^2\xi} - 4c_1 p_1\cdot p_3\right)$$

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$$V_{2\text{PM}}(p,q) = \mathcal{M}^{1-\text{loop}} + \mathcal{M}^{\text{Iterated}} = \frac{\pi^2 G_N^2}{E_p^2\xi|\vec{q}|}\left[\frac{1}{2}\left(\frac{c_\triangleright}{m_a} + \frac{c_\triangleleft}{m_b}\right) + \frac{2}{E_p\xi}\left(\frac{c_1^2(\xi-1)}{2E_p^2\xi} - 4c_1 p_1\cdot p_3\right)\right]$$

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Again same result as from matching, singular term gone!

Scalar interaction potentials
(one-loop)

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Important ‘empirical’ observation classical part of radial action that for the gravitational Hamiltonian is given by triangle diagrams only rest is cancelled in subtractions

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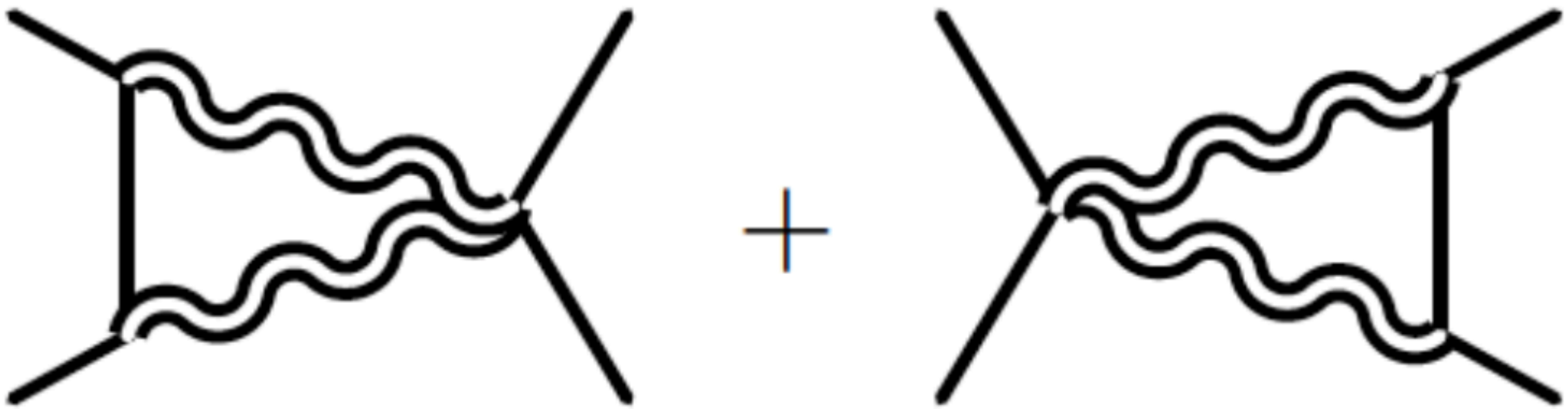
One-loop level

$$\mathcal{M}_2 =$$



$$= -i(8\pi G)^2\left(\frac{c(m_1,m_2)I_{\triangleright}(p_1,q)}{(q^2-4m_1^2)^2}+\frac{c(m_2,m_1)I_{\triangleright}(p_4,-q)}{(q^2-4m_2^2)^2}\right)$$

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$$\widetilde{\mathcal{M}}_1^{\text{Cl.}}(\gamma, b, \hbar) = \frac{3\pi G_N^2(m_1+m_2)m_1m_2(5\gamma^2-1)}{4b\sqrt{\gamma^2-1}\hbar}(\pi b^2 e^{\gamma_E})^{4-D} + \mathcal{O}(4-D)$$

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$$\widetilde{\mathcal{M}}_1^{\text{Qt.}}(\gamma, b) = \frac{G_N^2(\pi b^2 e^{\gamma_E})^{4-D}}{b^2}\Bigg(i\frac{4-D}{2}\frac{(2\gamma^2-1)^2\mathcal{E}_{\text{C.M.}}^2}{(\gamma^2-1)^2}$$
$$-\frac{m_1m_2}{\pi(\gamma^2-1)^{\frac{3}{2}}}\Big(\frac{1-49\gamma^2+18\gamma^4}{15}-\frac{2\gamma(2\gamma^2-1)(6\gamma^2-7)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}\Big)\Bigg)+\mathcal{O}((4-D)^2)$$

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We will now consider what happens at two-loops

Classical gravitational scattering:
Loop level

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- 1) compute multi-loop cuts and 2) use consistency of the representation in master integrals to generate the full non-analytics pieces of the amplitude (classical and super-classical contributions)

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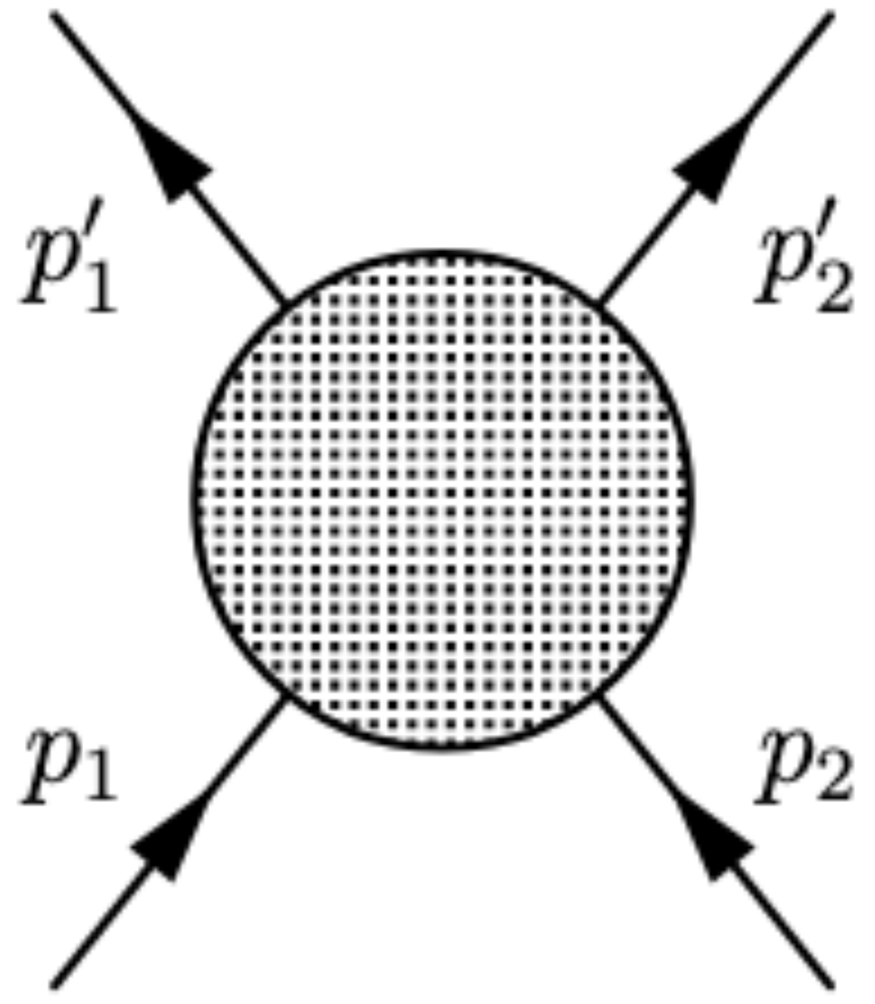

$$= \mathcal{M}(\gamma, q^2) = \sum_{L=0}^{\infty} \mathcal{M}_L(\gamma, q^2)\,.$$

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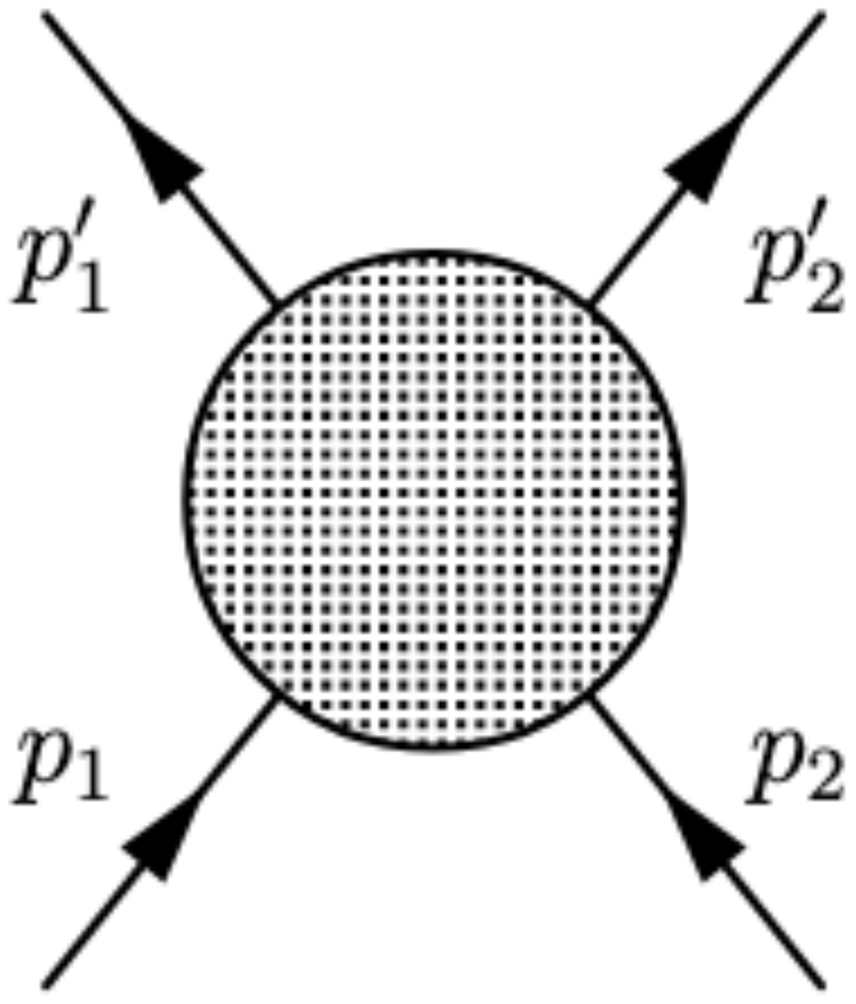

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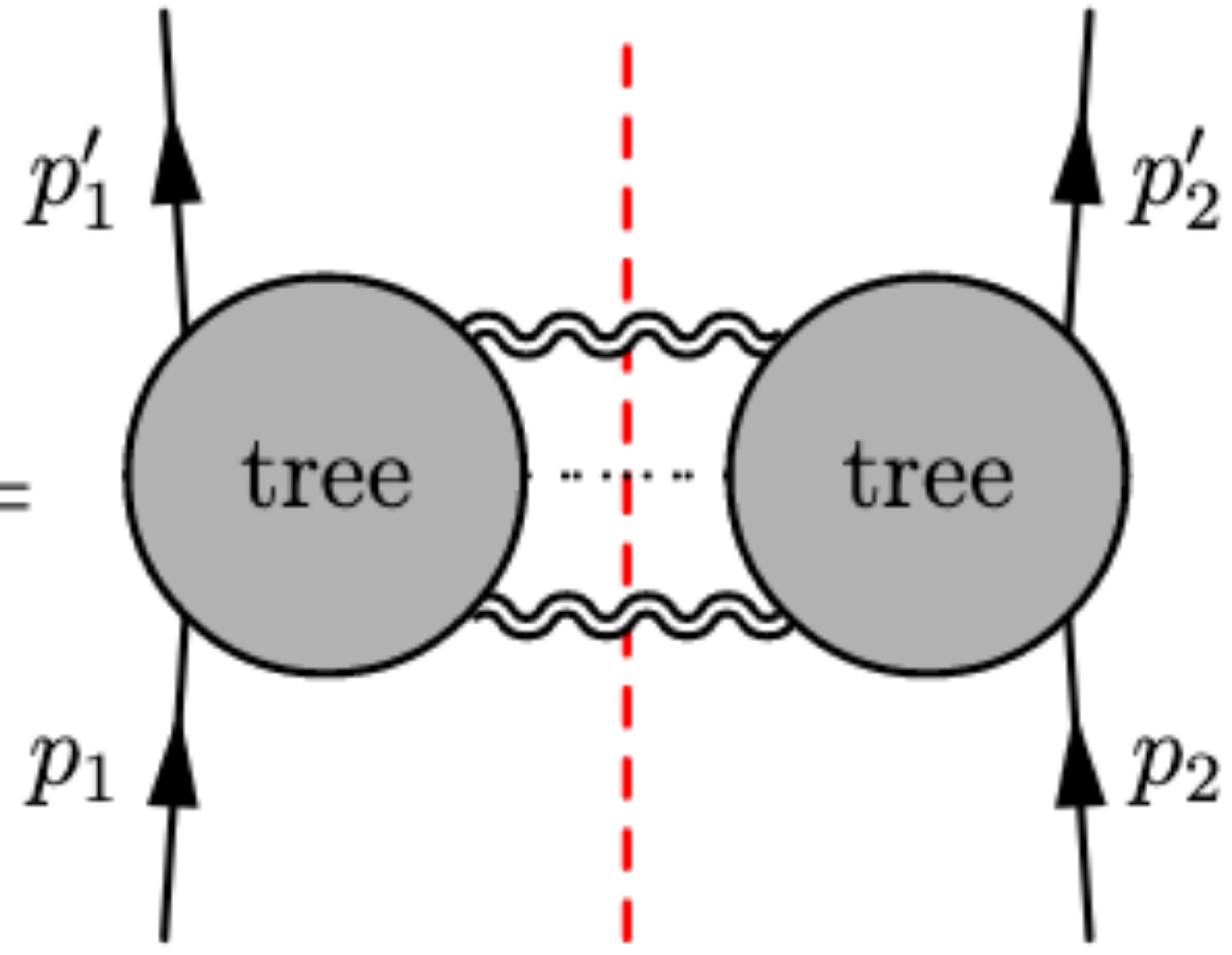

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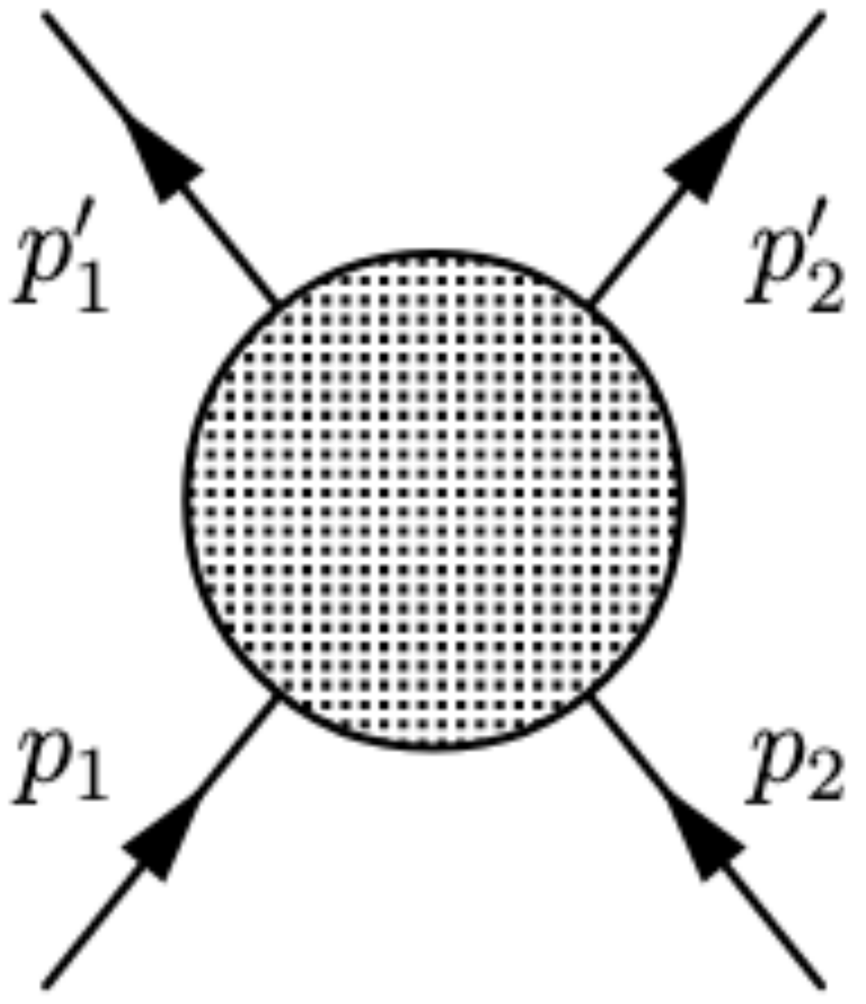
Extraction of integrand similar to QCD
Spinor-helicity and D-dimension
covariant tree
amplitudes can be used in cuts



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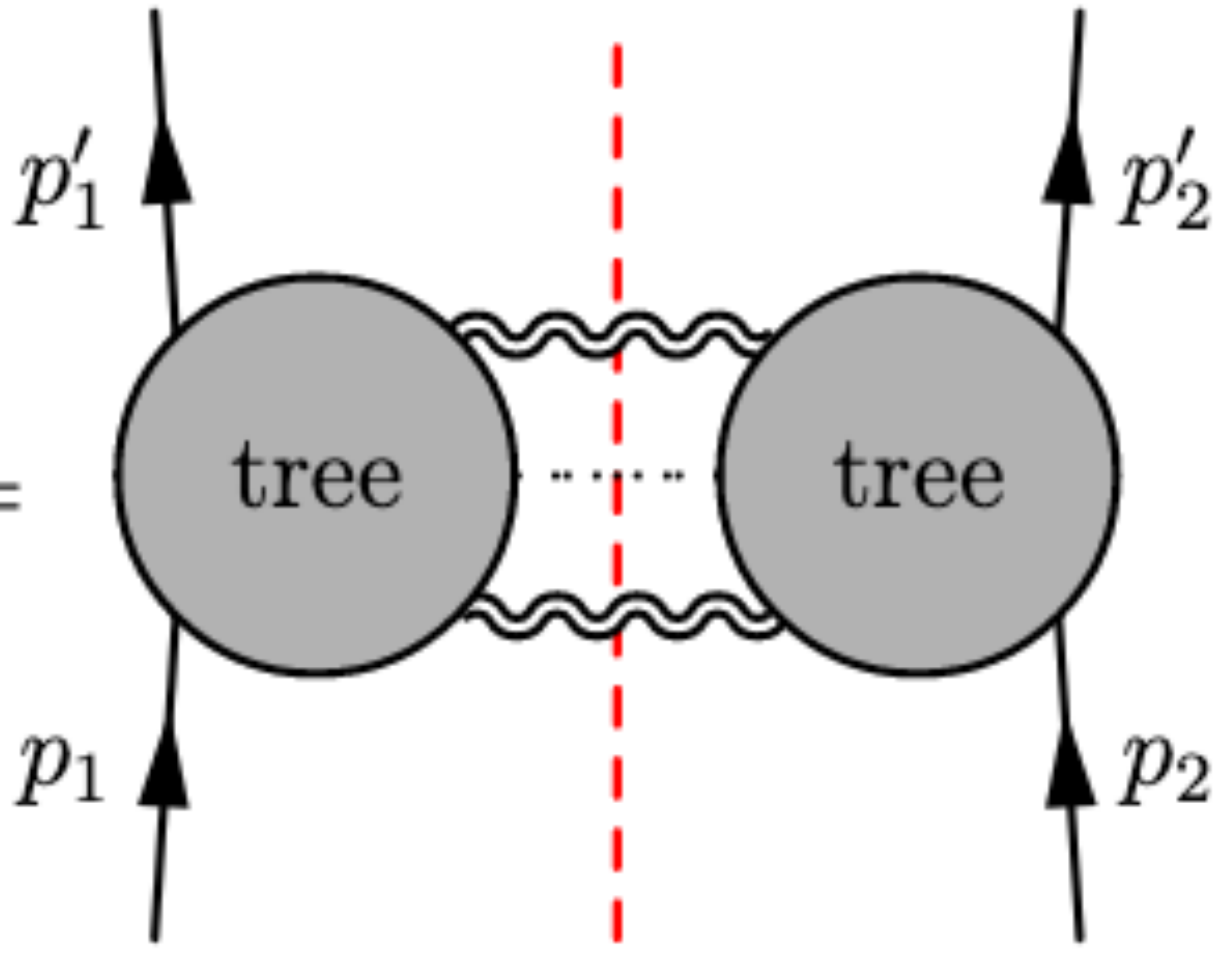

Example: Einstein gravity at two-loop order

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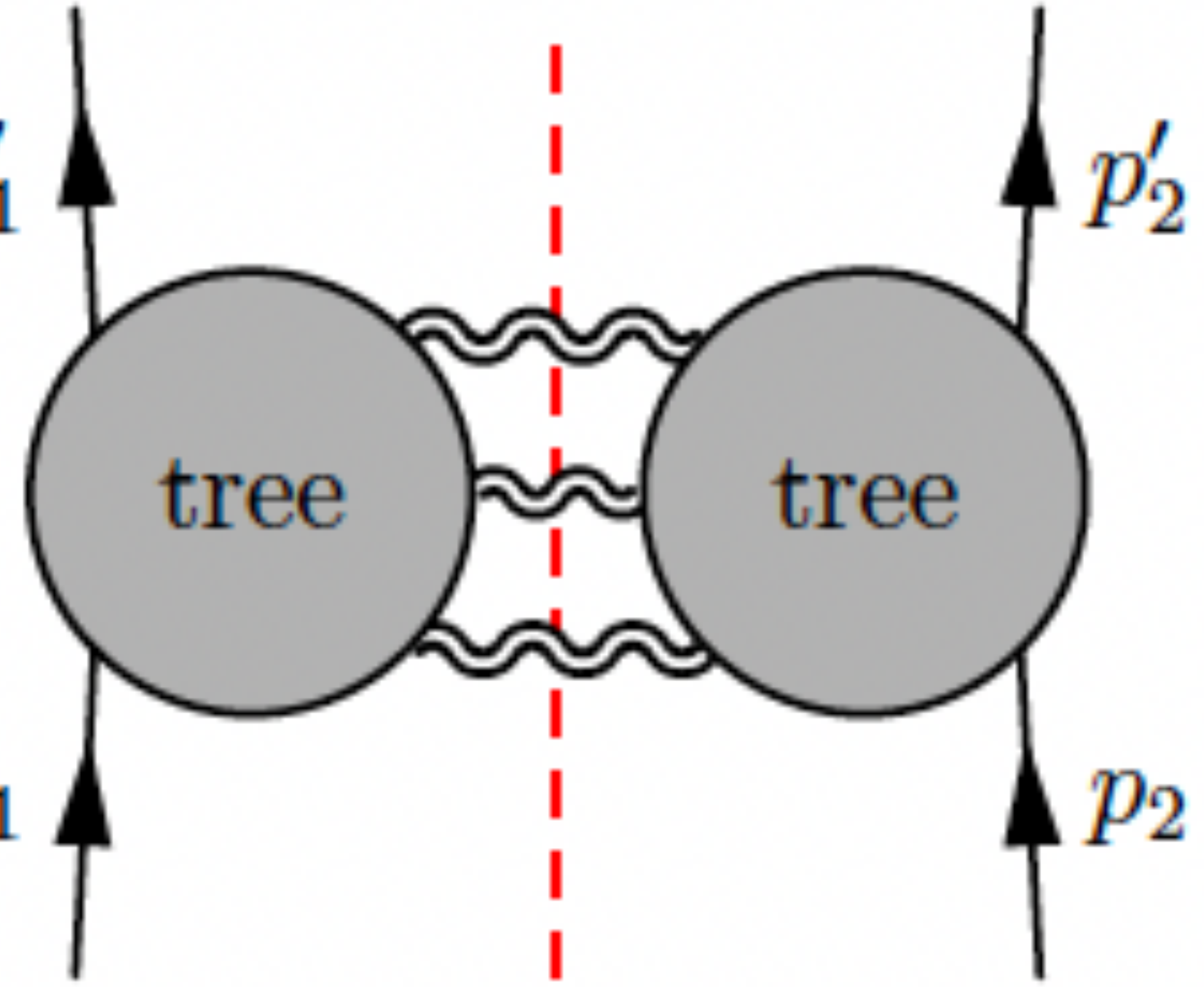
$$\mathcal{M}^2_{3-\mathrm{cut}}(\sigma, q^2) =$$



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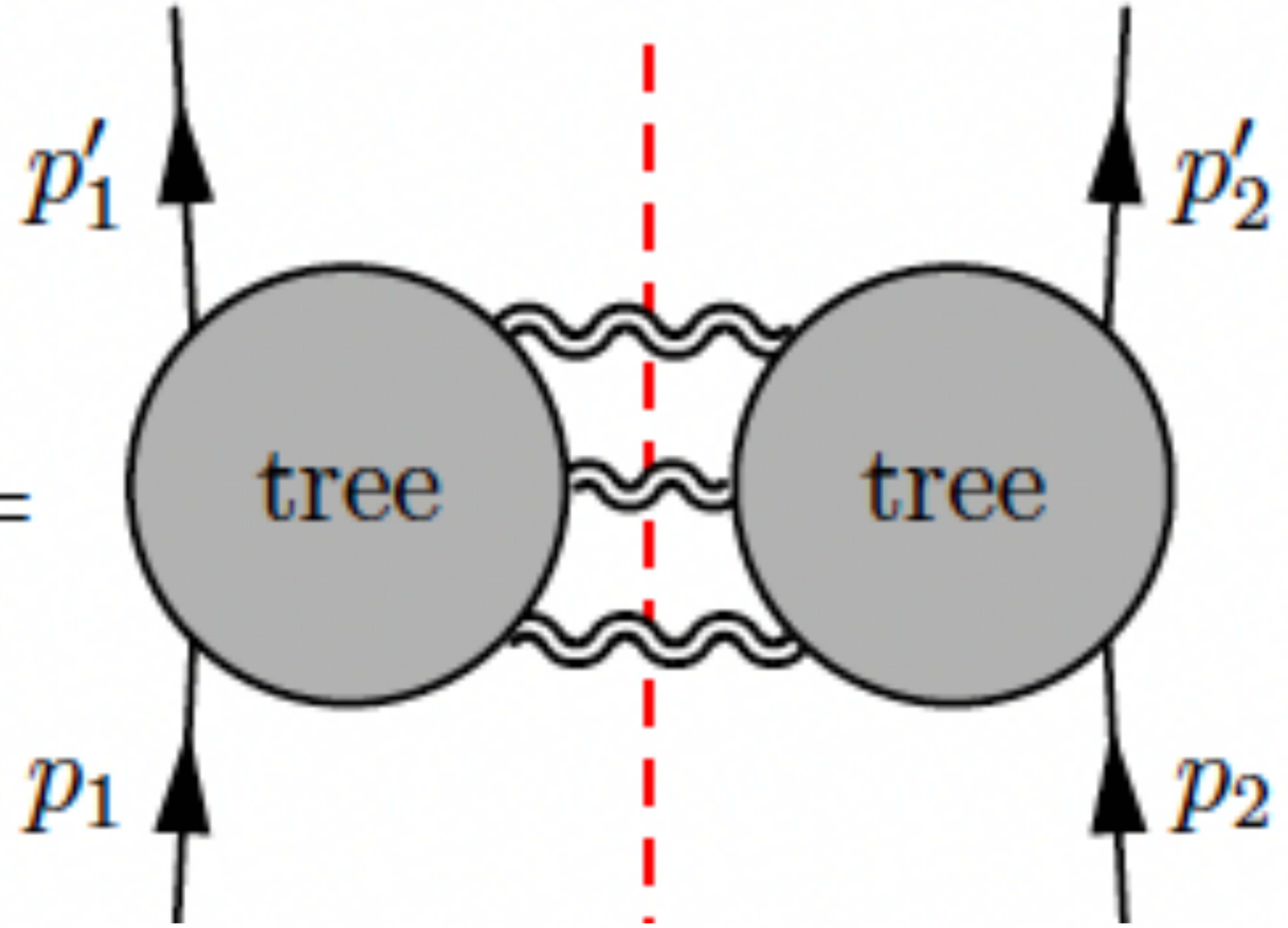
$\mathcal{M}^2_{3-\mathrm{cut}}(\sigma, q^2) =$ 

$$\mathcal{M}_2^{3-\mathrm{cut}}(\sigma, q^2) = \int \frac{d^D l_1 d^D l_2 d^D l_3}{(2\pi)^{3D}} (2\pi)^D \delta^{(D)}(l_1 + l_2 + l_3 + q) \frac{i^3}{l_1^2 l_2^2 l_3^2}$$
$$\times \frac{1}{3!} \sum_{\substack{\mathrm{Perm}(l_1,l_2,l_3) \\ \lambda_1=\pm,\lambda_2=\pm,\lambda_3=\pm}} \mathcal{M}_0(p_1, p_1', l_1^{\lambda_1}, l_2^{\lambda_2}, l_3^{\lambda_3})(\mathcal{M}_0(p_2, p_2', -l_1^{\lambda_1}, -l_2^{\lambda_2}, -l_3^{\lambda_3}))^*$$

Einstein gravity at two-loop order

Einstein gravity at two-loop order

Can e.g. use helicity formalism

Einstein gravity at two-loop order

Can e.g. use helicity formalism
to derive D=4 integrand — from traces..

Einstein gravity at two-loop order

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Einstein gravity at two-loop order

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$$i\mathcal{M}_0(p_1,p_1',l_1^+,l_2^+,l_3^+) = -\frac{(8\pi G_N)^{\frac{3}{2}} m_1^4}{\langle l_1\, l_2\rangle^2 \langle l_1\, l_3\rangle^2 \langle l_2\, l_3\rangle^2} \sum_{1\le i\ne j\ne k\le 3} \frac{(l_i\cdot l_j)(l_j\cdot l_k) tr_+[l_k,p_1,p_1',l_i]}{(p_1\cdot l_k)(p_1'\cdot l_i)}$$

Einstein gravity at two-loop order

Can e.g. use helicity formalism
to derive D=4 integrand — from traces..

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$$i\mathcal{M}_0(p_1,p_1',l_1^-,l_2^+,l_3^+) = \frac{(2\pi G_N)^{\frac{3}{2}}}{2}\left(\sum_{2\le j\neq k\le 3} \frac{\langle l_1|p_1|l_j]\,\langle l_1|p_1'|l_j]^2\,\langle l_1|p_1|l_k]^3}{\langle l_1\, l_j\rangle\,\langle l_1\, l_k\rangle\,(l_1\cdot l_j)(l_1\cdot l_k)(p_1\cdot l_1)(p_1'\cdot l_j)}\right.$$

$$-\frac{\langle l_1|p_1|l_2]^3\,\langle l_1|p_1'|l_3]^3}{\langle l_1\, l_2\rangle\,\langle l_1\, l_3\rangle\,(l_1\cdot l_2)(l_1\cdot l_3)(p_1\cdot l_2)(p_1'\cdot l_3)} - \frac{2\,[l_2\, l_3]\,\langle l_1|p_1|l_2]\,\langle l_1|p_1|l_3]\,\langle l_1|p_1|p_1'|l_1\rangle^2}{\langle l_1\, l_2\rangle\,\langle l_1\, l_3\rangle\,\langle l_2\, l_3\rangle\,(l_1\cdot l_2)(l_1\cdot l_3)(p_1\cdot l_1)}$$

$$\left.+\frac{2\,[l_2\, l_3]^3\,\langle l_1|p_1|p_1'|l_1\rangle^2}{\langle l_2\, l_3\rangle\,(l_1\cdot l_2)(l_1\cdot l_3)t}\right) + (p_1 \leftrightarrow -p_1')$$

Einstein gravity at two-loop order

Can e.g. use helicity formalism
to derive D=4 integrand — from traces..

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$$-\frac{\langle l_1|p_1|l_2]^3\,\langle l_1|p_1'|l_3]^3}{\langle l_1\, l_2\rangle\,\langle l_1\, l_3\rangle\,(l_1\cdot l_2)(l_1\cdot l_3)(p_1\cdot l_2)(p_1'\cdot l_3)} - \frac{2\,[l_2\, l_3]\,\langle l_1|p_1|l_2]\,\langle l_1|p_1|l_3]\,\langle l_1|p_1|p_1'|l_1\rangle^2}{\langle l_1\, l_2\rangle\,\langle l_1\, l_3\rangle\,\langle l_2\, l_3\rangle\,(l_1\cdot l_2)(l_1\cdot l_3)(p_1\cdot l_1)}$$
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Alternative is covariant tree - D-dimensional formalism

Einstein gravity at two-loop order

Einstein gravity at two-loop order

New integrals

Einstein gravity at two-loop order

New integrals

$$\mathcal{M}_2^{3-\text{cut}}(\sigma, q^2) = \mathcal{M}_2^{\square\square} + \mathcal{M}_2^{\triangleleft\square} + \mathcal{M}_2^{\square\triangleright} + \mathcal{M}_2^{\triangleleft\triangleleft} + \mathcal{M}_2^{\triangleright\triangleright} + \mathcal{M}_2^{H} + \mathcal{M}_2^{\square\circ}$$

Einstein gravity at two-loop order

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We use unitarity cut to fix coefficients in front of

Einstein gravity at two-loop order

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We use unitarity cut to fix coefficients in front of master-integrals. The full result can be written

$$\mathcal{M}_2(\gamma, q^2) = \mathcal{M}_2^{3-\text{cut}}(\gamma, q^2) + \mathcal{M}_2^{\text{SE}}(\gamma, q^2)$$

Einstein gravity at two-loop order

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Einstein gravity at two-loop order

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Einstein gravity at two-loop order

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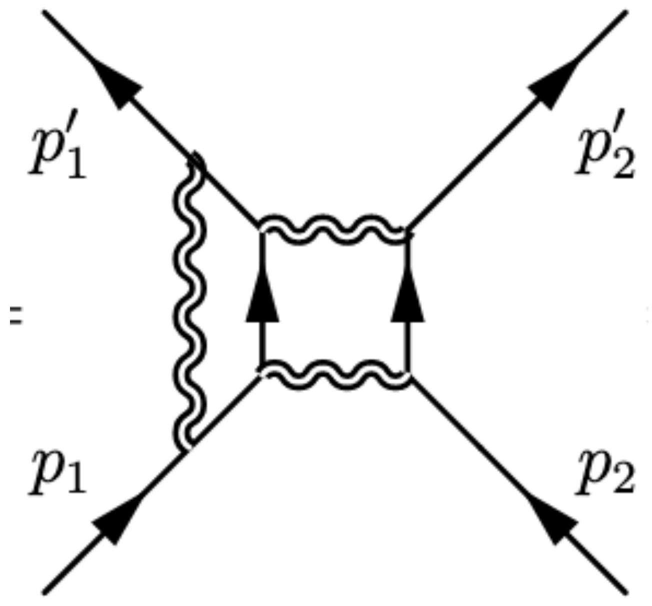

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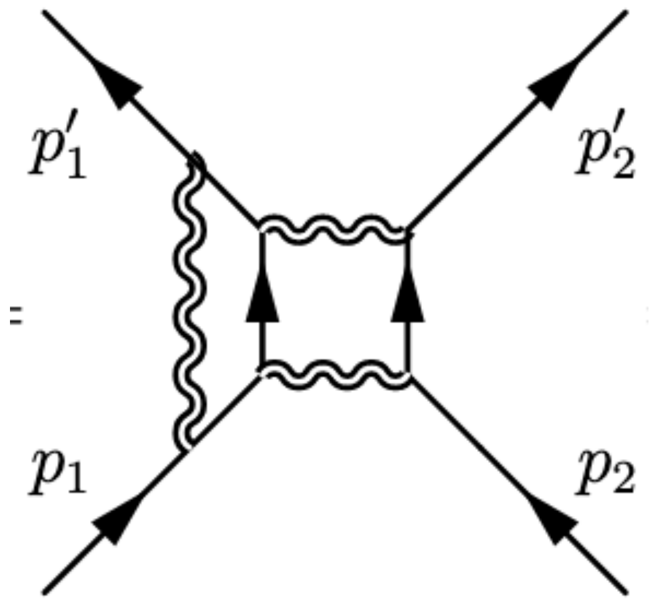

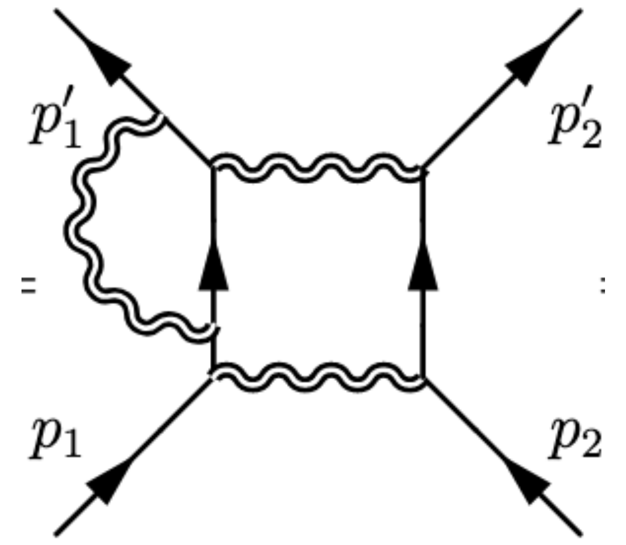

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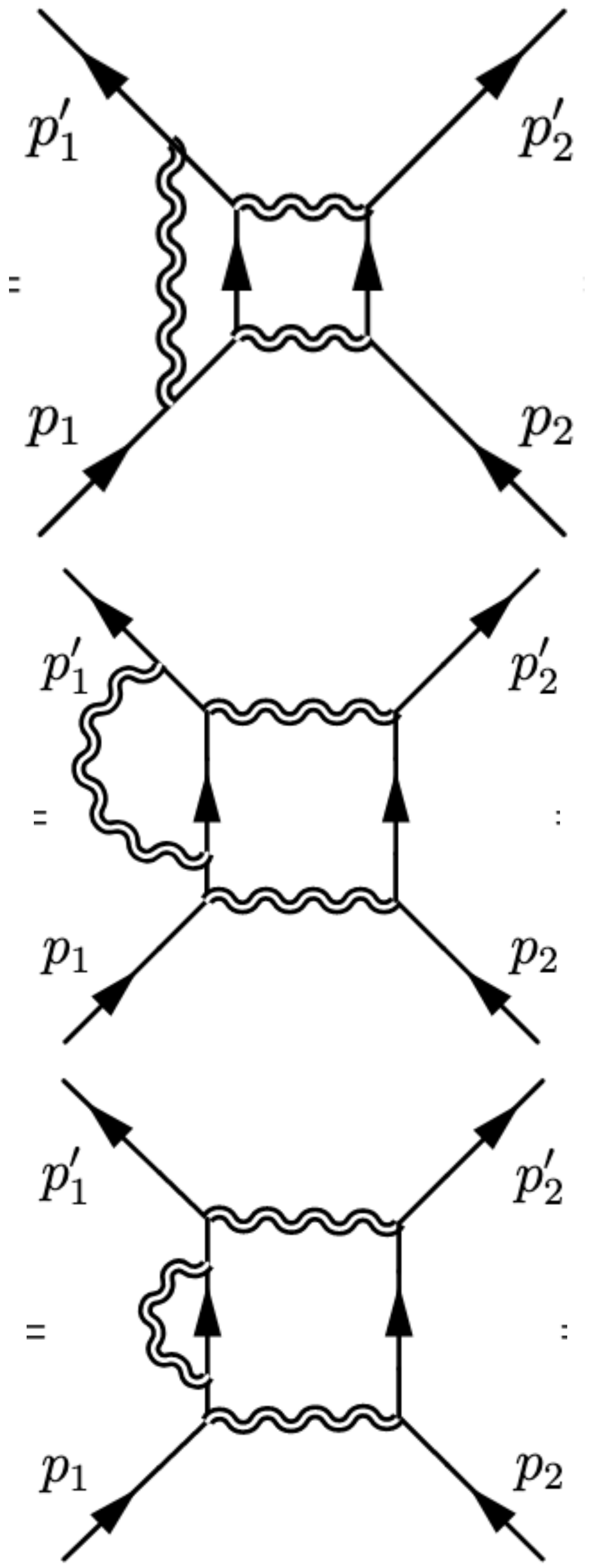

Einstein gravity at two-loop order

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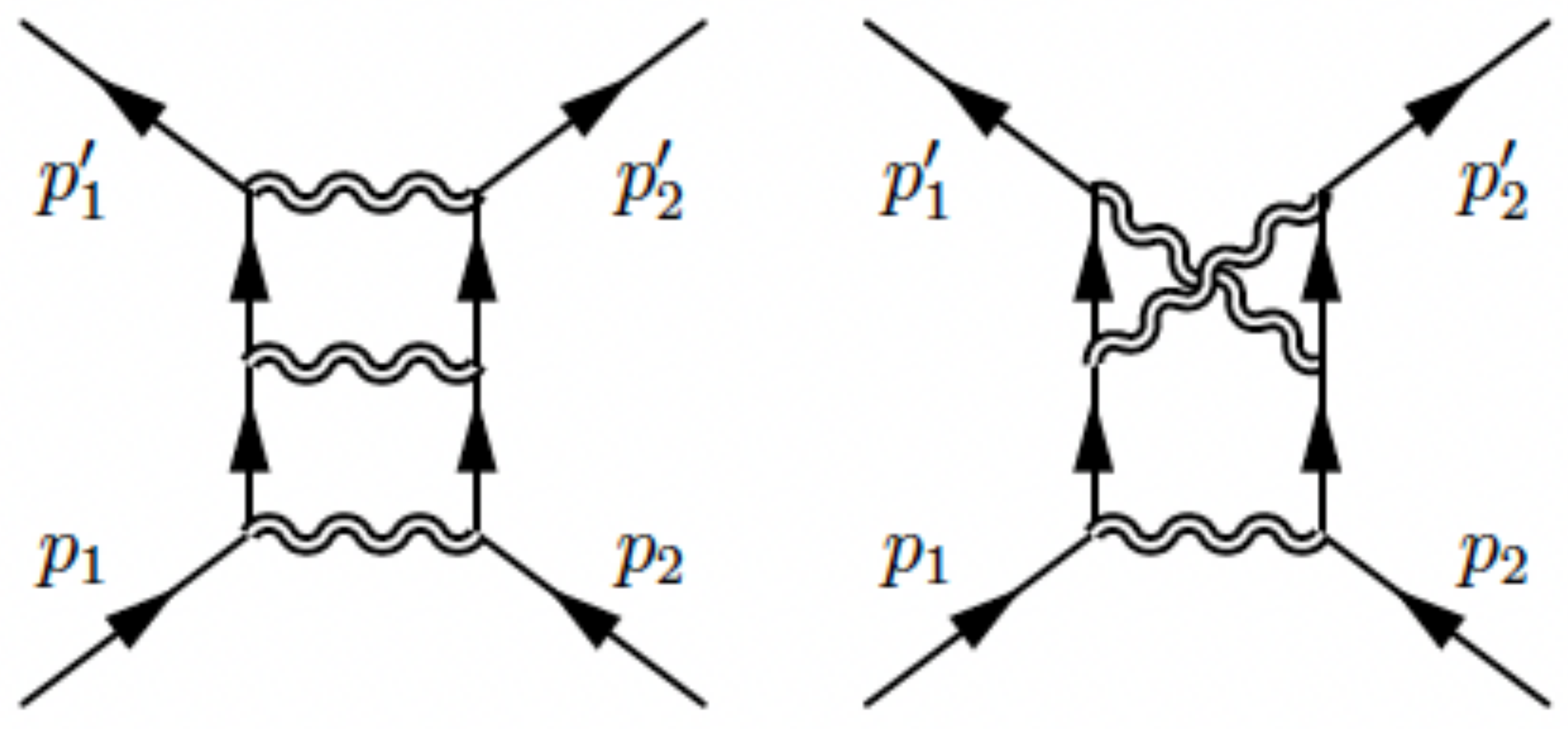

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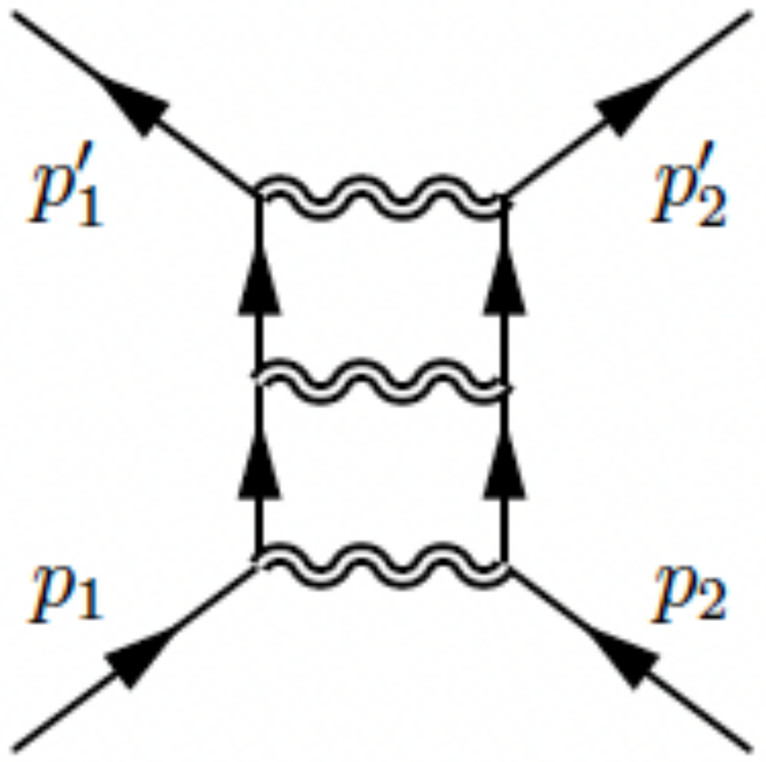
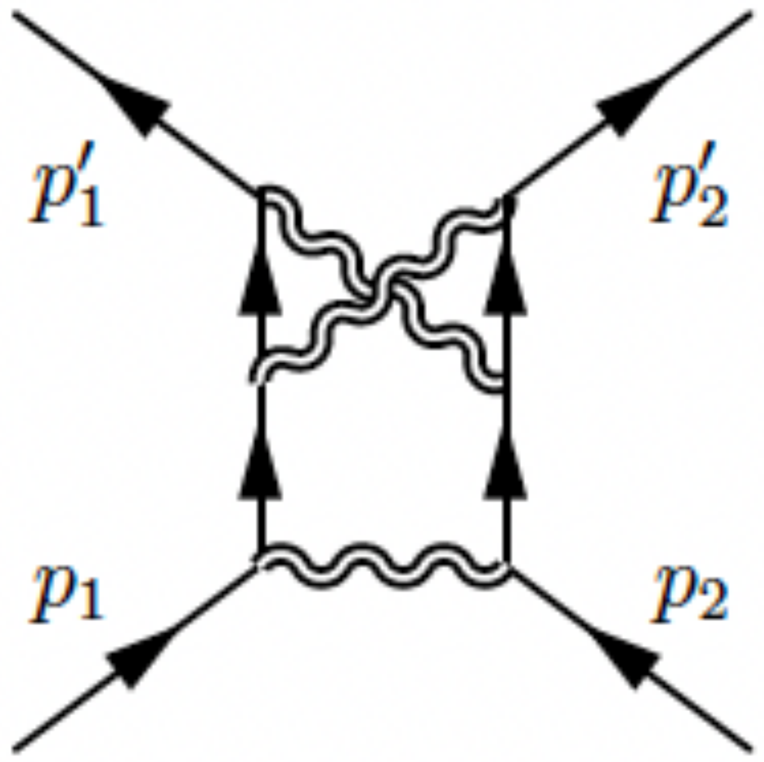
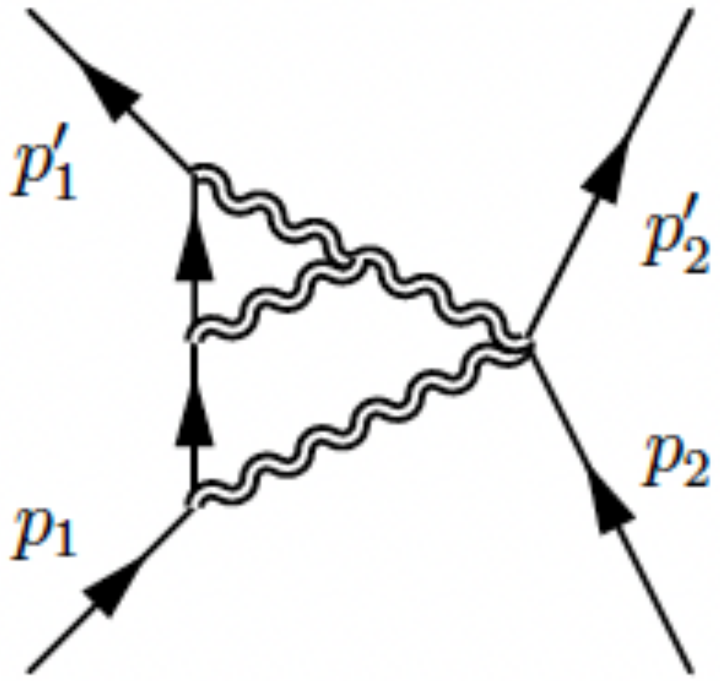
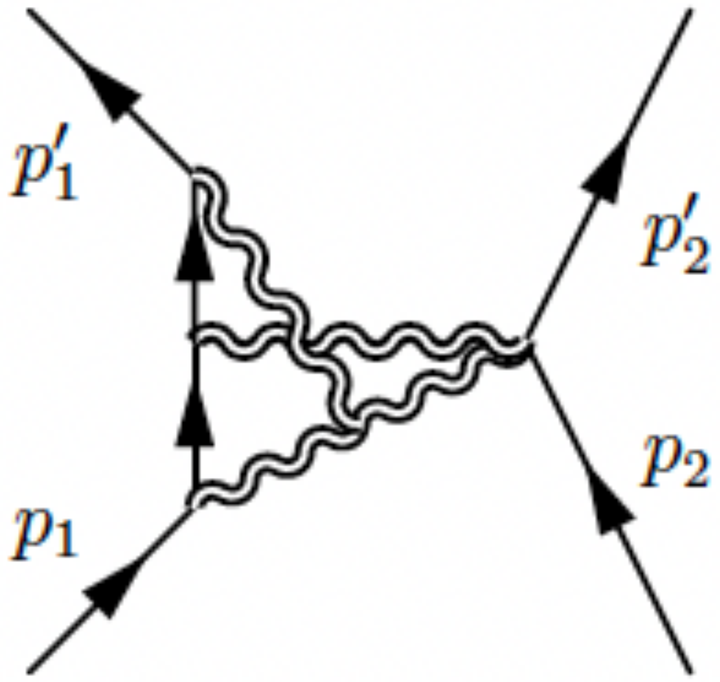
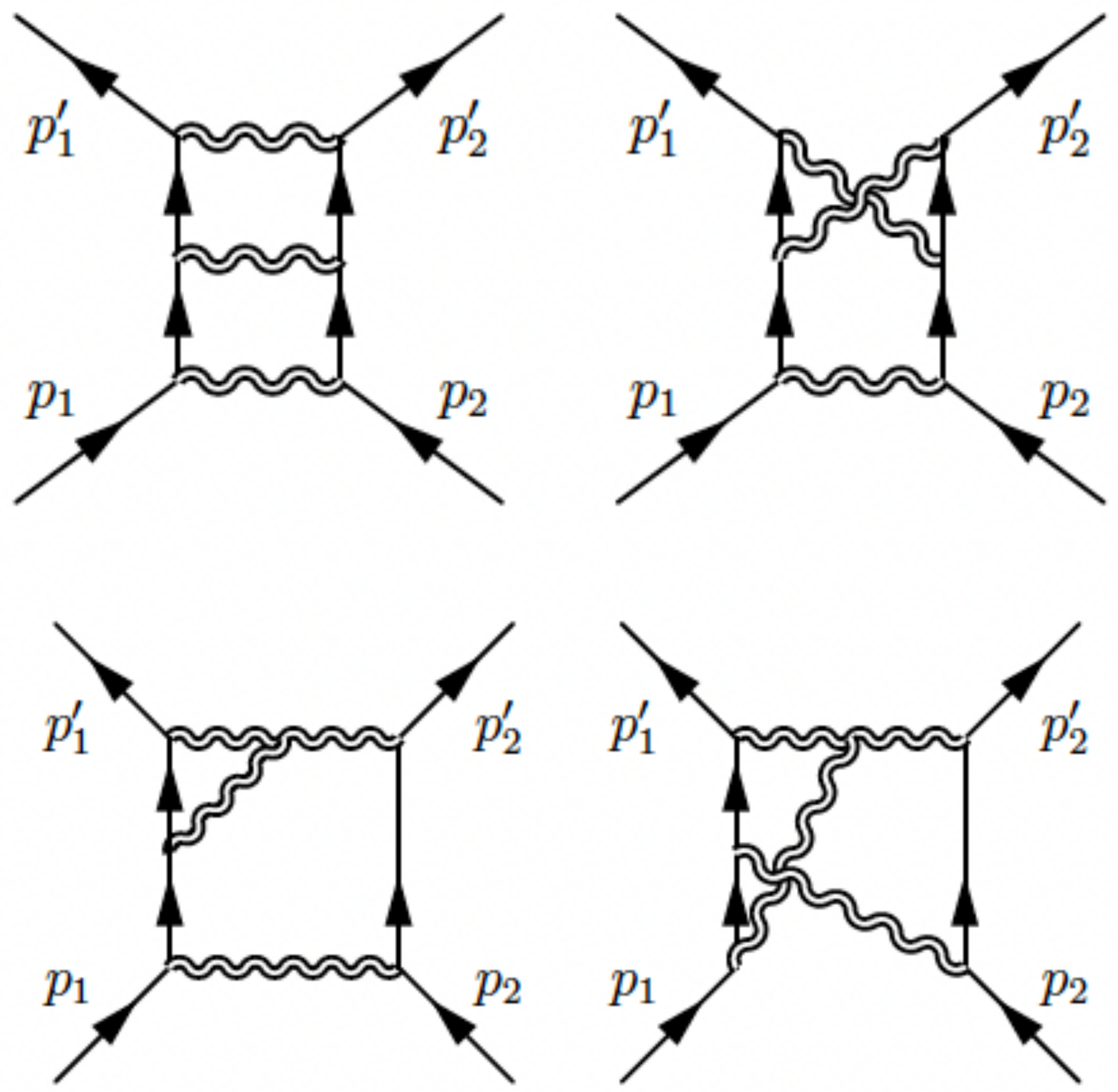


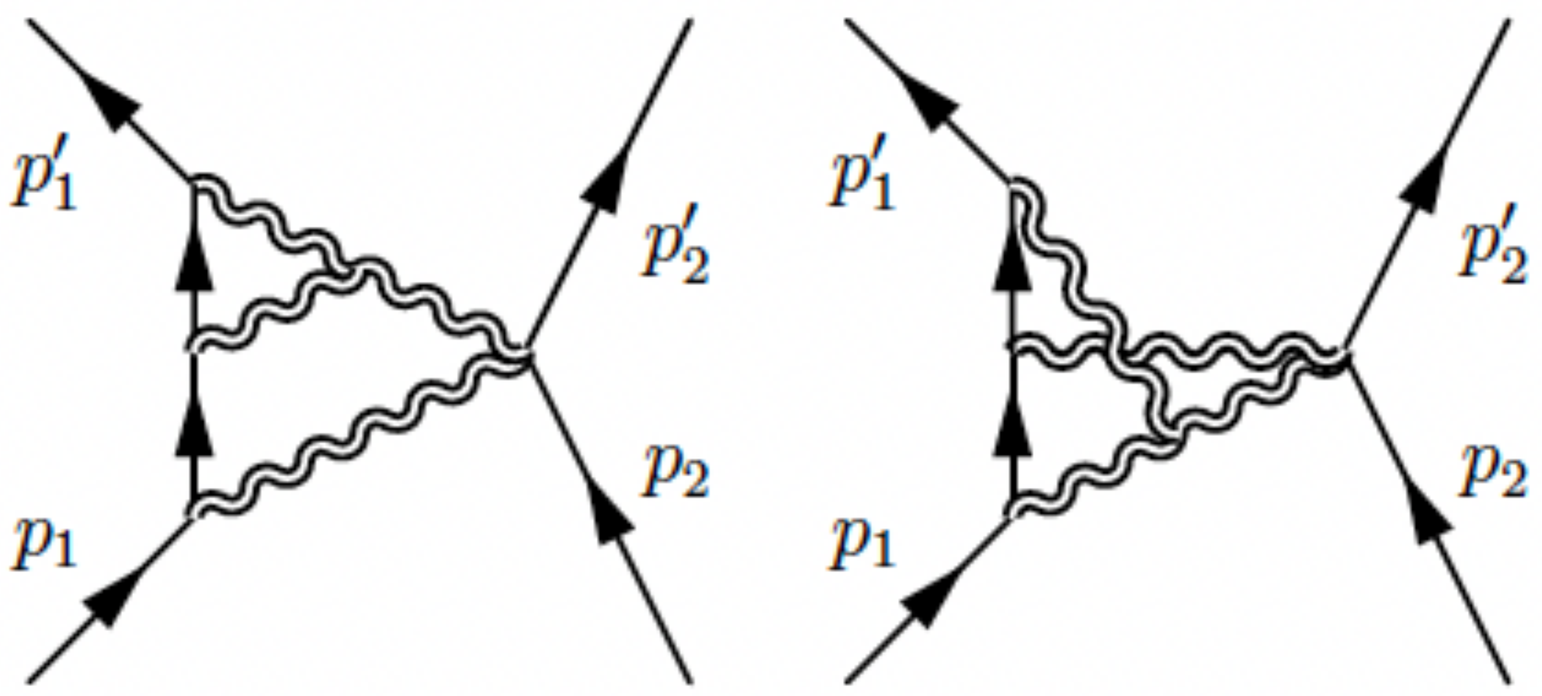




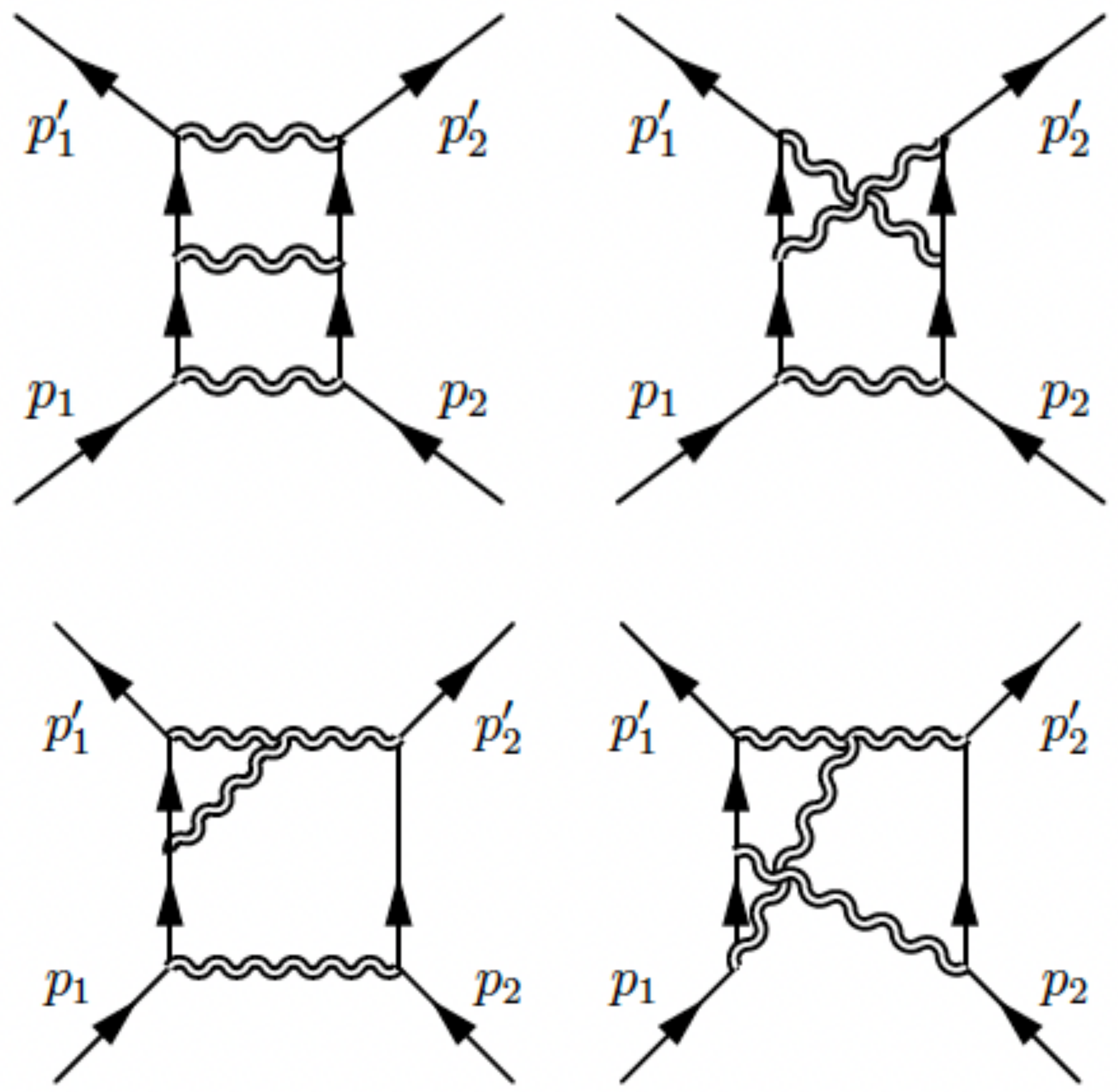


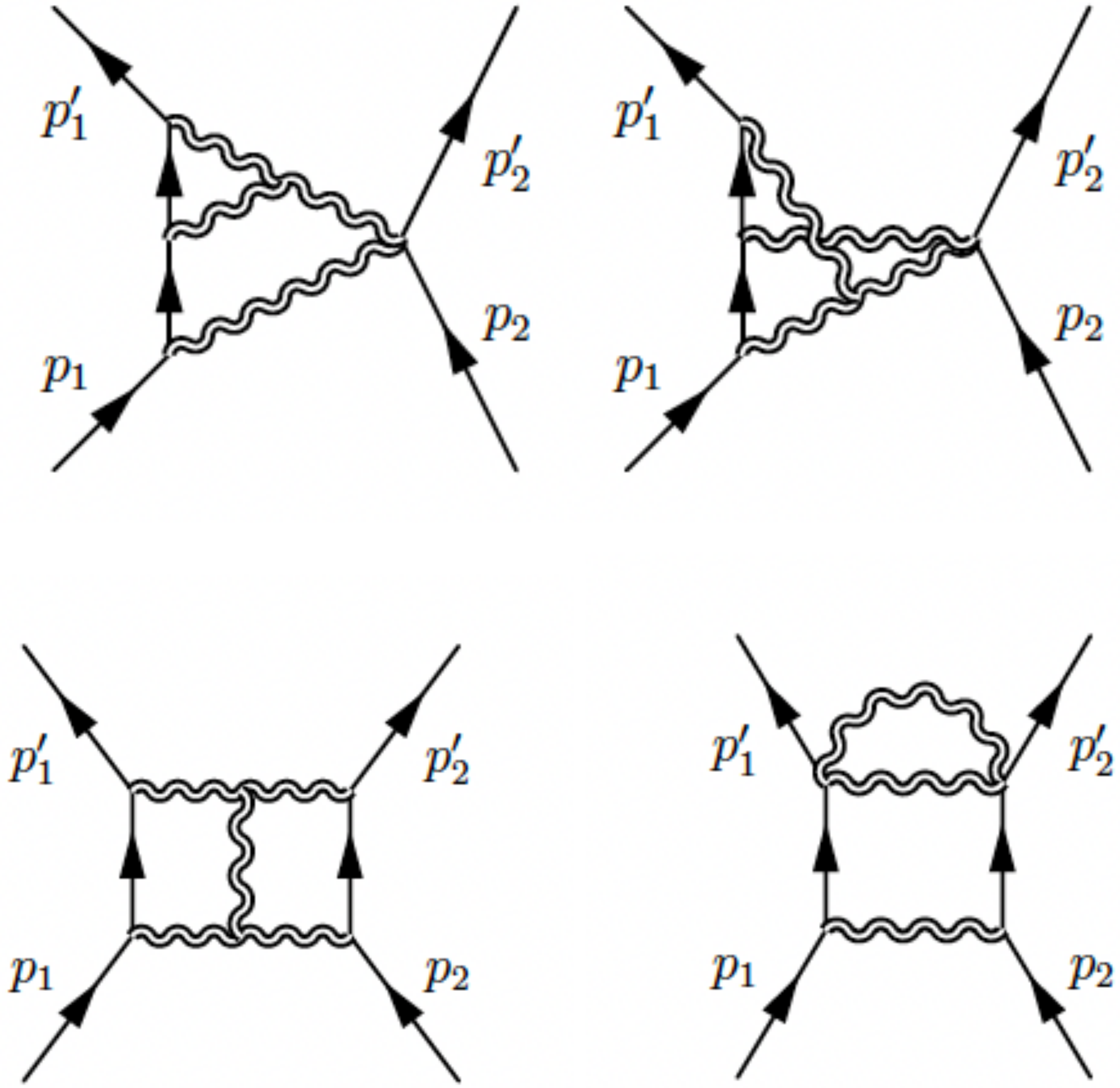
Einstein gravity at two-loop order





Einstein gravity at two-loop order





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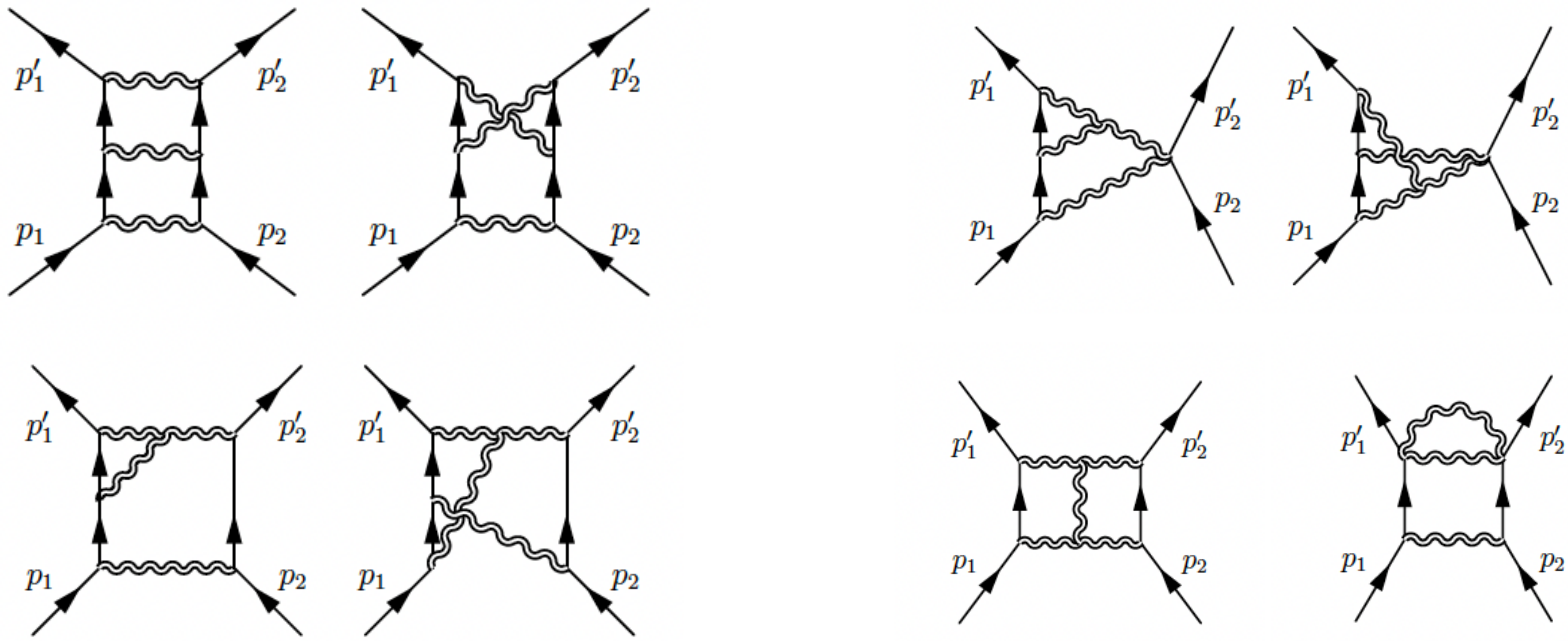

Needed master integrals at two-loops for the conservative part of the amplitude - determined by LiteRed/FIRE6/KIRA etc.

Some examples of numerators

Some examples of numerators

$$\mathcal{N}^{(s)}_{\square\square} = 512\pi^3 G_N^3 (m_1^4 + m_2^4 - 2(m_1^2 + m_2^2)s + s^2)^3 = 2^{12}\pi^3 G_N^3 m_1^6 m_2^6 (2\sigma^2 - 1)^3$$

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$$\mathcal{N}_{\square\square}^{(cross,s)} = 2^{13}\pi^3 G_N^3 \left(96 m_1^6 m_2^6 (2\sigma^2 - 1)^3 + 8 m_1^5 m_2^5 \sigma (2\sigma^2 - 1)^2 (\hbar \vec{\underline{q}})^2 (l_2 \cdot l_3) + \mathcal{O}((\hbar \vec{\underline{q}})^4)\right)$$

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$$\begin{aligned}\mathcal{N}_{\square\square}^{(u)} &= 512\pi^3 G_N^3 (m_1^4 + m_2^4 - 2(m_1^2 + m_2^2)u + u^2)^3 \\ &= 2^{12}\pi^3 G_N^3 \left(96 m_1^6 m_2^6 (2\sigma^2 - 1)^3 - 6 m_1^5 m_2^5 \sigma (2\sigma^2 - 1)^2 (\hbar \vec{\underline{q}})^2 + \mathcal{O}((\hbar \vec{\underline{q}})^4)\right)\end{aligned}$$

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$$\begin{aligned} \mathcal{N}_H = \frac{128\pi^3 G_N^3}{3} \Big(&- 48(-4 m_1^2 m_2^4 ((l_2 + l_3)^2 - (l_1 + l_3)^2 + 4\sigma^2)(\bar{p}_1 \cdot l_2)^2 \\ &- 8 m_2^4 (\bar{p}_1 \cdot l_2)^4 + 16 m_1^3 m_2^3 \sigma (\bar{p}_1 \cdot l_2)(\bar{p}_2 \cdot l_1) \\ &+ m_1^4 \Big(m_2^4 (-1 - 2(l_2 + l_3)^2 (1 + (l_2 + l_3)^2) - 2(l_1 + l_3)^2 (1 + (l_1 + l_3)^2) \\ &+ 4\sigma^2 + 4((l_2 + l_3)^2 + (l_2 + l_3)^4 + (l_1 + l_3)^2 - 2(l_2 + l_3)^2 (l_1 + l_3)^2 + (l_1 + l_3)^4 \Big) \sigma^2 \\ &- 4\sigma^4) + 4 m_2^2 ((l_2 + l_3)^2 - (l_1 + l_3)^2 - 4\sigma^2)(\bar{p}_2 \cdot l_1)^2 - 8(\bar{p}_2 \cdot l_1)^4))(\hbar \underline{\vec{q}})^4 + \mathcal{O}((\hbar \underline{\vec{q}})^5) \Big). \end{aligned}$$

Einstein gravity at two-loop order

$$\mathcal{M}_2^{3-\mathrm{cut}(-1)}(\sigma,q^2) = \frac{2(4\pi e^{-\gamma_E})^{2\epsilon}\pi G_N^3 m_1^2 m_2^2}{3\epsilon|\underline{q}|^{4\epsilon}\hbar}\left(\frac{3s(2\sigma^2-1)^3}{(\sigma^2-1)^2}\right.$$

$$+\frac{im_1m_2(2\sigma^2-1)}{\pi\epsilon(\sigma^2-1)^{\frac{3}{2}}}\left(\frac{1-49\sigma^2+18\sigma^4}{5}-\frac{6\sigma(2\sigma^2-1)(6\sigma^2-7)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)$$

$$-\frac{9(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)}+\frac{3}{2}(m_1^2+m_2^2)(-1+18\sigma^2)-m_1m_2\sigma(103+2\sigma^2)$$

$$+\frac{12m_1m_2(3+12\sigma^2-4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$

$$\left.-\frac{6im_1m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right).$$

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$$\left.-\frac{6im_1m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right).$$

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$$-\frac{9(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)}+\frac{3}{2}(m_1^2+m_2^2)(-1+18\sigma^2)-m_1m_2\sigma(103+2\sigma^2)$$

$$+\frac{12m_1m_2(3+12\sigma^2-4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$

$$\left.-\frac{6im_1m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right).$$

Imaginary

Gravity amplitude in powers of hbar

Gravity amplitude in powers of hbar

$$\mathcal{M}_2(\sigma,|\underline{q}|) = \frac{1}{|\underline{q}|^{4\epsilon}}\left(\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|) + \mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|) + \mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|) + \mathcal{O}(\hbar^0)\right)$$

Gravity amplitude in powers of hbar

$$\mathcal{M}_2(\sigma, |\underline{q}|) = \frac{1}{|\underline{q}|^{4\epsilon}} \left(\mathcal{M}_2^{(-3)}(\sigma, |\underline{q}|) + \mathcal{M}_2^{(-2)}(\sigma, |\underline{q}|) + \mathcal{M}_2^{(-1)}(\sigma, |\underline{q}|) + \mathcal{O}(\hbar^0) \right)$$

$$\mathcal{M}_2^{(-3)}(\sigma, |\underline{q}|) = -\frac{8\pi G_N^3 m_1^4 m_2^4 (2\sigma^2 - 1)^3 \Gamma(-\epsilon)^3 \Gamma(1 + 2\epsilon)}{3\hbar^3 |\underline{q}|^2 (\sigma^2 - 1)(4\pi)^{-2\epsilon} \Gamma(-3\epsilon)}$$

Gravity amplitude in powers of hbar

$$\mathcal{M}_2(\sigma,|\underline{q}|) = \frac{1}{|\underline{q}|^{4\epsilon}}\left(\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|) + \mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|) + \mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|) + \mathcal{O}(\hbar^0)\right)$$

$$\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|) = -\frac{8\pi G_N^3 m_1^4 m_2^4 (2\sigma^2-1)^3 \Gamma(-\epsilon)^3 \Gamma(1+2\epsilon)}{3\hbar^3 |\underline{q}|^2 (\sigma^2-1)(4\pi)^{-2\epsilon}\Gamma(-3\epsilon)}$$

$$\mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|) = \frac{6i\pi^2 G_N^3 (m_1+m_2) m_1^3 m_2^3 (2\sigma^2-1)(1-5\sigma^2)(4\pi e^{-\gamma_E})^{2\epsilon}}{\epsilon\sqrt{\sigma^2-1}\hbar^2|\underline{q}|} + \mathcal{O}(\epsilon^0)$$

Gravity amplitude in powers of hbar

$$\mathcal{M}_2(\sigma, |\underline{q}|) = \frac{1}{|\underline{q}|^{4\epsilon}} \left(\mathcal{M}_2^{(-3)}(\sigma, |\underline{q}|) + \mathcal{M}_2^{(-2)}(\sigma, |\underline{q}|) + \mathcal{M}_2^{(-1)}(\sigma, |\underline{q}|) + \mathcal{O}(\hbar^0) \right)$$

$$\mathcal{M}_2^{(-3)}(\sigma, |\underline{q}|) = -\frac{8\pi G_N^3 m_1^4 m_2^4 (2\sigma^2-1)^3 \Gamma(-\epsilon)^3 \Gamma(1+2\epsilon)}{3\hbar^3 |\underline{q}|^2 (\sigma^2-1)(4\pi)^{-2\epsilon} \Gamma(-3\epsilon)}$$

$$\mathcal{M}_2^{(-2)}(\sigma, |\underline{q}|) = \frac{6i\pi^2 G_N^3 (m_1+m_2) m_1^3 m_2^3 (2\sigma^2-1)(1-5\sigma^2)(4\pi e^{-\gamma_E})^{2\epsilon}}{\epsilon\sqrt{\sigma^2-1}\,\hbar^2 |\underline{q}|} + \mathcal{O}(\epsilon^0)$$

$$\mathcal{M}_2^{(-1)}(\sigma, |\underline{q}|) = \frac{2\pi G_N^3 (4\pi e^{-\gamma_E})^{2\epsilon} m_1^2 m_2^2}{\hbar\epsilon} \Bigg(\frac{s(2\sigma^2-1)^3}{(\sigma^2-1)^2}$$
$$+ \frac{i m_1 m_2 (2\sigma^2-1)}{\pi\epsilon(\sigma^2-1)^{\frac{3}{2}}} \left(\frac{1-49\sigma^2+18\sigma^4}{15} - \frac{2\sigma(7-20\sigma^2+12\sigma^4)\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \right)$$
$$- \frac{3(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)} + \frac{1}{2}(m_1^2+m_2^2)(18\sigma^2-1) - \frac{1}{3} m_1 m_2 \sigma(103+2\sigma^2)$$
$$+ \frac{4m_1 m_2 (3+12\sigma^2-4\sigma^4)\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$
$$- \frac{2i m_1 m_2 (2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}} \left(\frac{-1}{4(\sigma^2-1)} \right)^{\epsilon} \left(-\frac{11}{3} + \frac{d}{d\sigma} \left(\frac{(2\sigma^2-1)\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \right) \right) \Bigg).$$

Gravity amplitude in powers of hbar

$$\mathcal{M}_2(\sigma,|\underline{q}|)=\frac{1}{|\underline{q}|^{4\epsilon}}\left(\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|)+\mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|)+\mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|)+\mathcal{O}(\hbar^0)\right)$$

$$\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|)=-\frac{8\pi G_N^3 m_1^4 m_2^4(2\sigma^2-1)^3\Gamma(-\epsilon)^3\Gamma(1+2\epsilon)}{3\hbar^3|\underline{q}|^2(\sigma^2-1)(4\pi)^{-2\epsilon}\Gamma(-3\epsilon)}$$

$$\mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|)=\frac{6i\pi^2 G_N^3(m_1+m_2)m_1^3m_2^3(2\sigma^2-1)(1-5\sigma^2)(4\pi e^{-\gamma_E})^{2\epsilon}}{\epsilon\sqrt{\sigma^2-1}\hbar^2|\underline{q}|}+\mathcal{O}(\epsilon^0)$$

$$\mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|)=\frac{2\pi G_N^3(4\pi e^{-\gamma_E})^{2\epsilon}m_1^2m_2^2}{\hbar\epsilon}\Bigg(\frac{s(2\sigma^2-1)^3}{(\sigma^2-1)^2}$$
$$+\frac{im_1m_2(2\sigma^2-1)}{\pi\epsilon(\sigma^2-1)^{\frac{3}{2}}}\Big(\frac{1-49\sigma^2+18\sigma^4}{15}-\frac{2\sigma(7-20\sigma^2+12\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\Big)$$
$$-\frac{3(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)}+\frac{1}{2}(m_1^2+m_2^2)(18\sigma^2-1)-\frac{1}{3}m_1m_2\sigma(103+2\sigma^2)$$
$$+\frac{4m_1m_2(3+12\sigma^2-4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$
$$-\frac{2im_1m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\left(-\frac{11}{3}+\frac{d}{d\sigma}\Big(\frac{(2\sigma^2-1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\Big)\right)\Bigg).$$

Laurant expansion in Planck's constant

- imaginary contribution cancelled by radiative contributions

Gravity amplitude in powers of hbar

$$\mathcal{M}_2(\sigma,|\underline{q}|)=\frac{1}{|\underline{q}|^{4\epsilon}}\left(\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|)+\mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|)+\mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|)+\mathcal{O}(\hbar^0)\right)$$

$$\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|)=-\frac{8\pi G_N^3 m_1^4 m_2^4(2\sigma^2-1)^3\Gamma(-\epsilon)^3\Gamma(1+2\epsilon)}{3\hbar^3|\underline{q}|^2(\sigma^2-1)(4\pi)^{-2\epsilon}\Gamma(-3\epsilon)}$$

$$\mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|)=\frac{6i\pi^2 G_N^3(m_1+m_2)m_1^3m_2^3(2\sigma^2-1)(1-5\sigma^2)(4\pi e^{-\gamma_E})^{2\epsilon}}{\epsilon\sqrt{\sigma^2-1}\hbar^2|\underline{q}|}+\mathcal{O}(\epsilon^0)$$

$$\begin{aligned}\mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|)=\frac{2\pi G_N^3(4\pi e^{-\gamma_E})^{2\epsilon}m_1^2m_2^2}{\hbar\epsilon}\Bigg(&\frac{s(2\sigma^2-1)^3}{(\sigma^2-1)^2}\\&+\frac{im_1m_2(2\sigma^2-1)}{\pi\epsilon(\sigma^2-1)^{\frac{3}{2}}}\left(\frac{1-49\sigma^2+18\sigma^4}{15}-\frac{2\sigma(7-20\sigma^2+12\sigma^4)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\\&-\frac{3(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)}+\frac{1}{2}(m_1^2+m_2^2)(18\sigma^2-1)-\frac{1}{3}m_1m_2\sigma(103+2\sigma^2)\\&+\frac{4m_1m_2(3+12\sigma^2-4\sigma^4)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\\&-\frac{2im_1m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\left(-\frac{11}{3}+\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right)\Bigg).\end{aligned}$$

Laurant expansion in Planck's constant

- imaginary contribution cancelled by radiative contributions

(Di Vecchia, Heissenberg, Russo, Veneziano)

Gravity amplitude in powers of hbar

$$\mathcal{M}_2(\sigma,|\underline{q}|) = \frac{1}{|\underline{q}|^{4\epsilon}}\Big(\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|) + \mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|) + \mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|) + \mathcal{O}(\hbar^0)\Big)$$

$$\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|) = -\frac{8\pi G_N^3 m_1^4 m_2^4 (2\sigma^2-1)^3\Gamma(-\epsilon)^3\Gamma(1+2\epsilon)}{3\hbar^3|\underline{q}|^2(\sigma^2-1)(4\pi)^{-2\epsilon}\Gamma(-3\epsilon)}$$

(Bern et al, Parra-Martinez et al)

$$\mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|) = \frac{6i\pi^2 G_N^3(m_1+m_2)m_1^3m_2^3(2\sigma^2-1)(1-5\sigma^2)(4\pi e^{-\gamma_E})^{2\epsilon}}{\epsilon\sqrt{\sigma^2-1}\hbar^2|\underline{q}|} + \mathcal{O}(\epsilon^0)$$

$$\mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|) = \frac{2\pi G_N^3(4\pi e^{-\gamma_E})^{2\epsilon}m_1^2m_2^2}{\hbar\epsilon}\Bigg(\frac{s(2\sigma^2-1)^3}{(\sigma^2-1)^2} + \frac{im_1m_2(2\sigma^2-1)}{\pi\epsilon(\sigma^2-1)^{\frac{3}{2}}}\Big(\frac{1-49\sigma^2+18\sigma^4}{15} - \frac{2\sigma(7-20\sigma^2+12\sigma^4)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\Big) - \frac{3(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)} + \frac{1}{2}(m_1^2+m_2^2)(18\sigma^2-1) - \frac{1}{3}m_1m_2\sigma(103+2\sigma^2) + \frac{4m_1m_2(3+12\sigma^2-4\sigma^4)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} - \frac{2im_1m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\Big(\frac{-1}{4(\sigma^2-1)}\Big)^{\epsilon}\Big(-\frac{11}{3} + \frac{d}{d\sigma}\Big(\frac{(2\sigma^2-1)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\Big)\Big)\Bigg).$$

Laurant expansion in Planck's constant

- imaginary contribution cancelled by radiative contributions

(Di Vecchia, Heissenberg, Russo, Veneziano)

Gravity amplitude in b-space

Gravity amplitude in b-space

$$\widetilde{\mathcal{M}}_2(\sigma, b) = \frac{1}{4E_{\text{c.m.}}P} \int_{\mathbb{R}^{D-2}} \frac{d^{D-2}\vec{\underline{q}}}{(2\pi)^{D-2}} \mathcal{M}_2(p_1, p_2, p_1', p_2') e^{i\vec{\underline{q}}\cdot\vec{b}}$$

Gravity amplitude in b-space

$$\widetilde{\mathcal{M}}_2(\sigma, b) = \frac{1}{4E_{\rm c.m.}P}\int_{\mathbb{R}^{D-2}}\frac{d^{D-2}\vec{\underline{q}}}{(2\pi)^{D-2}}\mathcal{M}_2(p_1, p_2, p_1', p_2')e^{i\vec{\underline{q}}\cdot\vec{b}}$$

$$\widetilde{\mathcal{M}}_2(\sigma, b) = -\frac{1}{6}\left(\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\right)^3 + i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\rm Cl.}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\rm Qt.}(\sigma, b)\right) + \widetilde{\mathcal{M}}_2^{\rm Cl.}(\sigma, b) + \mathcal{O}(\hbar^0).$$

Gravity amplitude in b-space

$$\widetilde{\mathcal{M}}_2(\sigma, b) = \frac{1}{4E_{\text{c.m.}}P}\int_{\mathbb{R}^{D-2}} \frac{d^{D-2}\underline{\vec{q}}}{(2\pi)^{D-2}} \mathcal{M}_2(p_1, p_2, p_1', p_2') e^{i\underline{\vec{q}}\cdot\vec{b}}$$

$$\widetilde{\mathcal{M}}_2(\sigma, b) = -\frac{1}{6}\left(\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\right)^3 + i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\text{Cl.}}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\text{Qt.}}(\sigma, b)\right) + \widetilde{\mathcal{M}}_2^{\text{Cl.}}(\sigma, b) + \mathcal{O}(\hbar^0).$$

$$\widetilde{\mathcal{M}}_2^{\Box\Box(-3)}(\sigma, b) = -\frac{1}{6}\left(\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\right)^3,$$

$$\widetilde{\mathcal{M}}_2^{\Box\Box(-2)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\Box(-1)}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\lhd\Box(-2)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\Box\rhd(-2)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\triangleleft(-1)}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\triangleright(-1)}(\sigma, b)\right)$$

$$\widetilde{\mathcal{M}}_2^{\Box\Box(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\Box(0)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\Box\Box\ \text{Cl.}}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\lhd\Box(-1)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\Box\rhd(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\triangleleft(0)}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\triangleright(0)}(\sigma, b)\right) + \widetilde{\mathcal{M}}_2^{\lhd\Box\ \text{Cl.}}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\Box\rhd\ \text{Cl.}}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\Box\circ(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\circ(0)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\Box\circ\ \text{Cl.}}(\sigma, b),$$

Gravity amplitude in b-space

$$\widetilde{\mathcal{M}}_2(\sigma, b) = \frac{1}{4E_{\mathrm{c.m.}}P} \int_{\mathbb{R}^{D-2}} \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}} \mathcal{M}_2(p_1, p_2, p_1', p_2') e^{i\vec{q}\cdot\vec{b}}$$

$$\widetilde{\mathcal{M}}_2(\sigma, b) = -\frac{1}{6}\left(\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\right)^3 + i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\mathrm{Cl.}}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\mathrm{Qt.}}(\sigma, b)\right) + \widetilde{\mathcal{M}}_2^{\mathrm{Cl.}}(\sigma, b) + \mathcal{O}(\hbar^0).$$

$$\widetilde{\mathcal{M}}_2^{\square\square(-3)}(\sigma, b) = -\frac{1}{6}\left(\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\right)^3,$$

$$\widetilde{\mathcal{M}}_2^{\square\square(-2)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\square(-1)}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\triangleleft\square(-2)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\triangleright(-2)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\triangleleft(-1)}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\triangleright(-1)}(\sigma, b)\right)$$

$$\widetilde{\mathcal{M}}_2^{\square\square(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\square(0)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\square\ \mathrm{Cl.}}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\triangleleft\square(-1)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\triangleright(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\triangleleft(0)}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\triangleright(0)}(\sigma, b)\right) + \widetilde{\mathcal{M}}_2^{\triangleleft\square\ \mathrm{Cl.}}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\triangleright\ \mathrm{Cl.}}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\square\circ(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\circ(0)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\circ\ \mathrm{Cl.}}(\sigma, b),$$

Again iterative structure like one-loop, part of a bigger scheme..Seen after Fourier transform to b space

Scattering angle from amplitudes

Scattering angle from amplitudes

$$1 + i \sum_{L \geq 0} \widetilde{\mathcal{M}}_L(\sigma, b) = (1 + 2i\Delta(\sigma, b)) \exp\left(\frac{2i}{\hbar} \sum_{L \geq 0} \delta_L(\sigma, b)\right)$$

Scattering angle from amplitudes

$$1 + i \sum_{L \geq 0} \widetilde{\mathcal{M}}_L(\sigma, b) = (1 + 2i\Delta(\sigma, b)) \exp\left(\frac{2i}{\hbar} \sum_{L \geq 0} \delta_L(\sigma, b)\right)$$

Gravity eikonal

Scattering angle from amplitudes

$$1 + i \sum_{L \geq 0} \widetilde{\mathcal{M}}_L(\sigma, b) = (1 + 2i\Delta(\sigma, b)) \exp\left(\frac{2i}{\hbar} \sum_{L \geq 0} \delta_L(\sigma, b) \right)$$

Gravity eikonal

$$\delta_0(\sigma, b) = -\frac{G_N m_1 m_2 (2\sigma^2 - 1)}{2\epsilon\sqrt{\sigma^2 - 1}} (\pi b^2 e^{\gamma_E})^{\epsilon} + \mathcal{O}(\epsilon),$$

Scattering angle from amplitudes

$$1 + i\sum_{L\geq 0}\widetilde{\mathcal{M}}_L(\sigma,b) = (1+2i\Delta(\sigma,b))\exp\left(\frac{2i}{\hbar}\sum_{L\geq 0}\delta_L(\sigma,b)\right)$$

Gravity eikonal

$$\delta_0(\sigma,b) = -\frac{G_N m_1 m_2(2\sigma^2-1)}{2\epsilon\sqrt{\sigma^2-1}}(\pi b^2 e^{\gamma_E})^{\epsilon} + \mathcal{O}(\epsilon),$$

$$\delta_1(\sigma,b) = \frac{3\pi G_N^2(m_1+m_2)m_1 m_2(5\sigma^2-1)}{8b\sqrt{\sigma^2-1}}(\pi b^2 e^{\gamma_E})^{2\epsilon}$$

Scattering angle from amplitudes

$$1 + i\sum_{L\geq 0} \widetilde{\mathcal{M}}_L(\sigma, b) = (1 + 2i\Delta(\sigma, b)) \exp\left(\frac{2i}{\hbar}\sum_{L\geq 0}\delta_L(\sigma, b)\right)$$

Gravity eikonal

$$2\Delta_1 = \widetilde{\mathcal{M}}_1^{\mathrm{Qt.}}(\sigma, b)$$

$$\delta_2(\sigma, b) = \frac{G_N^3 m_1 m_2 (\pi b^2 e^{\gamma_E})^{3\epsilon}}{2b^2\sqrt{\sigma^2 - 1}}\Bigg(\frac{2s(12\sigma^4 - 10\sigma^2 + 1)}{\sigma^2 - 1}$$
$$- \frac{4m_1 m_2 \sigma}{3}(25 + 14\sigma^2) + \frac{4m_1 m_2(3 + 12\sigma^2 - 4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}$$
$$+ \frac{2m_1 m_2 (2\sigma^2 - 1)^2}{\sqrt{\sigma^2 - 1}}\frac{1}{(4(\sigma^2 - 1))^\epsilon}\left(-\frac{11}{3} + \frac{d}{d\sigma}\left(\frac{(2\sigma^2 - 1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}\right)\right)\Bigg).$$

Scattering angle from amplitudes

Scattering angle from amplitudes

$$\sin\left(\frac{\chi}{2}\right)\Big|_{3PM} = -\frac{\sqrt{s}}{m_1 m_2\sqrt{\sigma^2-1}}\frac{\partial\delta_2(\sigma,b)}{\partial b}$$

Scattering angle from amplitudes

$$\sin\left(\frac{\chi}{2}\right)\Big|_{3PM} = -\frac{\sqrt{s}}{m_1 m_2 \sqrt{\sigma^2 - 1}} \frac{\partial \delta_2(\sigma, b)}{\partial b}$$

$$J = \frac{m_1 m_2 \sqrt{\sigma^2 - 1}}{\sqrt{s}} b \cos\left(\frac{\chi}{2}\right)$$

Scattering angle from amplitudes

$$\sin\left(\frac{\chi}{2}\right)\Big|_{3PM} = -\frac{\sqrt{s}}{m_1 m_2\sqrt{\sigma^2-1}}\frac{\partial\delta_2(\sigma,b)}{\partial b}$$

$$J = \frac{m_1 m_2\sqrt{\sigma^2-1}}{\sqrt{s}} b\cos\left(\frac{\chi}{2}\right)$$

$$\chi_{1PM} = \frac{2G_N m_1 m_2(2\sigma^2-1)}{J\sqrt{\sigma^2-1}},$$

$$\chi_{2PM} = \frac{3\pi G_N^2 m_1^2 m_2^2(m_1+m_2)(5\sigma^2-1)}{4J^2\sqrt{s}};$$

Scattering angle from amplitudes

Scattering angle from amplitudes

$$\widehat{\chi}_{3PM} = \frac{2G_N^3 m_1^3 m_2^3 \left(64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5\right)}{3J^3 \left(\sigma^2 - 1\right)^{\frac{3}{2}}}$$

$$+ \frac{8G_N^3 m_1^4 m_2^4 \sqrt{\sigma^2 - 1}}{3J^3 s} \left(\sigma(-25 - 14\sigma^2) + \frac{3(3 + 12\sigma^2 - 4\sigma^4)\, \mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \right)$$

Scattering angle from amplitudes

$$\widehat{\chi}_{3PM} = \frac{2G_N^3 m_1^3 m_2^3 \left(64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5\right)}{3J^3 \left(\sigma^2 - 1\right)^{\frac{3}{2}}} + \frac{8G_N^3 m_1^4 m_2^4 \sqrt{\sigma^2 - 1}}{3J^3 s} \left(\sigma(-25 - 14\sigma^2) + \frac{3(3 + 12\sigma^2 - 4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \right)$$

$$\chi_{3PM}^{\mathrm{Rad.}} = \frac{4G_N^3 m_1^4 m_2^4 (2\sigma^2 - 1)^2}{J^3 s} \frac{1}{(4(\sigma^2 - 1))^\epsilon} \left(-\frac{11}{3} + \frac{d}{d\sigma} \left(\frac{(2\sigma^2 - 1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \right) \right)$$

Scattering angle from amplitudes

$$\widehat{\chi}_{3PM} = \frac{2G_N^3 m_1^3 m_2^3 \left(64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5\right)}{3J^3 \left(\sigma^2 - 1\right)^{\frac{3}{2}}}$$
$$+ \frac{8G_N^3 m_1^4 m_2^4 \sqrt{\sigma^2 - 1}}{3J^3 s} \left(\sigma(-25 - 14\sigma^2) + \frac{3(3 + 12\sigma^2 - 4\sigma^4)\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \right)$$

$$\chi_{3PM}^{\text{Rad.}} = \frac{4G_N^3 m_1^4 m_2^4 (2\sigma^2 - 1)^2}{J^3 s} \frac{1}{(4(\sigma^2 - 1))^{\epsilon}} \left(-\frac{11}{3} + \frac{d}{d\sigma} \left(\frac{(2\sigma^2 - 1)\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \right) \right)$$

Match with expectations

(Damour; Di Vecchia et al; Hermann et al)

Scattering angle from amplitudes

$$\widehat{\chi}_{3PM} = \frac{2G_N^3 m_1^3 m_2^3 \left(64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5\right)}{3J^3\left(\sigma^2-1\right)^{\frac{3}{2}}}$$
$$+ \frac{8G_N^3 m_1^4 m_2^4 \sqrt{\sigma^2-1}}{3J^3 s}\left(\sigma(-25-14\sigma^2) + \frac{3(3+12\sigma^2-4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)$$

$$\chi_{3PM}^{\mathrm{Rad.}} = \frac{4G_N^3 m_1^4 m_2^4 (2\sigma^2-1)^2}{J^3 s}\frac{1}{(4(\sigma^2-1))^{\epsilon}}\left(-\frac{11}{3} + \frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right)$$

Match with expectations
(Damour; Di Vecchia et al; Hermann et al)

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Scattering angle from amplitudes

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What is nice to see is the fact that everything matches up!
- the cancellation of terms that is demonstrated explicitly gives important consistency of computations.

Even simpler organisation of results — velocity cuts, exponentiation and soft expansion

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An example of this is the ‘velocity cuts’ is a clever to organise the integrand for simpler computations. The basic observation is that the combination of linear propagators

$$\left(\frac{1}{(p_A\cdot\ell_A+i\varepsilon)(p_A\cdot\ell_B-i\varepsilon)}-\frac{1}{(p_A\cdot\ell_B+i\varepsilon)(p_A\cdot\ell_A-i\varepsilon)}\right)\times$$
$$\left(\frac{1}{(p_B\cdot\ell_A-i\varepsilon)(p_B\cdot\ell_C+i\varepsilon)}-\frac{1}{(p_B\cdot\ell_C-i\varepsilon)(p_B\cdot\ell_A+i\varepsilon)}\right)$$

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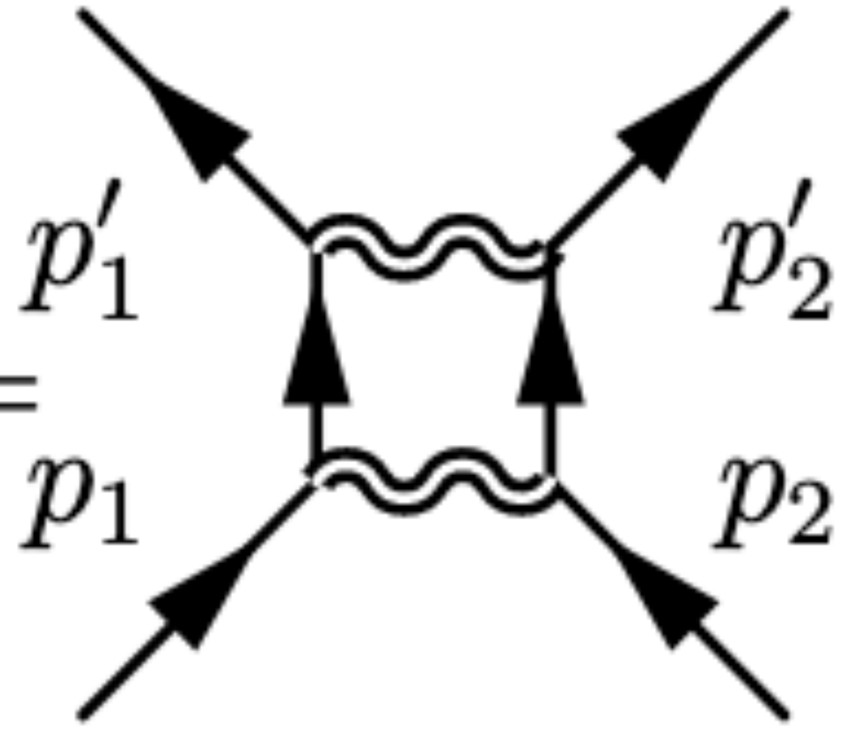
using

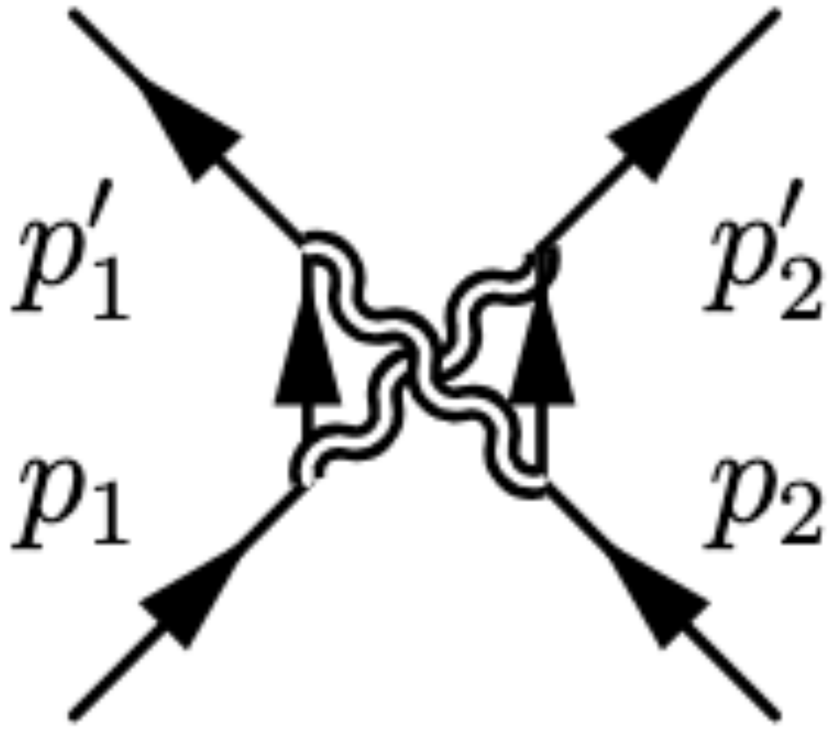
$$\frac{1}{x+i\varepsilon}-\frac{1}{x-i\varepsilon}=-2i\pi\delta(x)$$

We can see this in the organisation of the one-loop

$I_{\square} =$



$+$



$$= \int \frac{d^D\ell}{(2\pi\hbar)^D} \frac{1}{\ell^2(\ell+q)^2} \left(\frac{1}{(-p_1+\ell)^2 - m_1^2 + i\varepsilon} + \frac{1}{(p_1'+\ell)^2 - m_1^2 + i\varepsilon} \right)$$

$$\times \left(\frac{1}{(-p_2+\ell)^2 - m_2^2 + i\varepsilon} + \frac{1}{(p_2'+\ell)^2 - m_2^2 + i\varepsilon} \right) .$$

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Another is ‘stringy’ inspiration for efficient trees

Different form for amplitude

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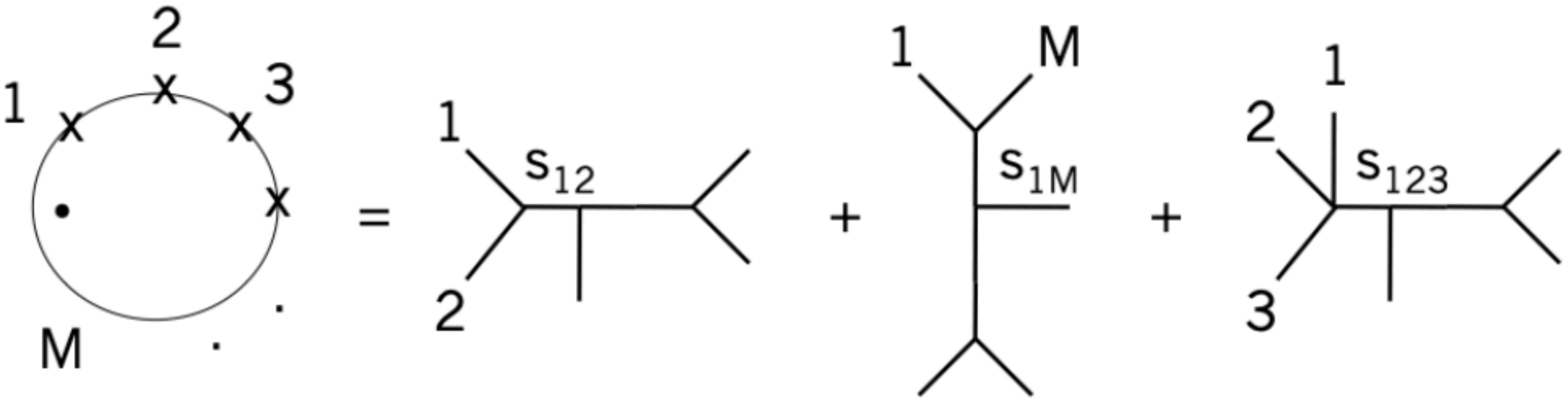

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Different form for amplitude

String theory
add channels up..

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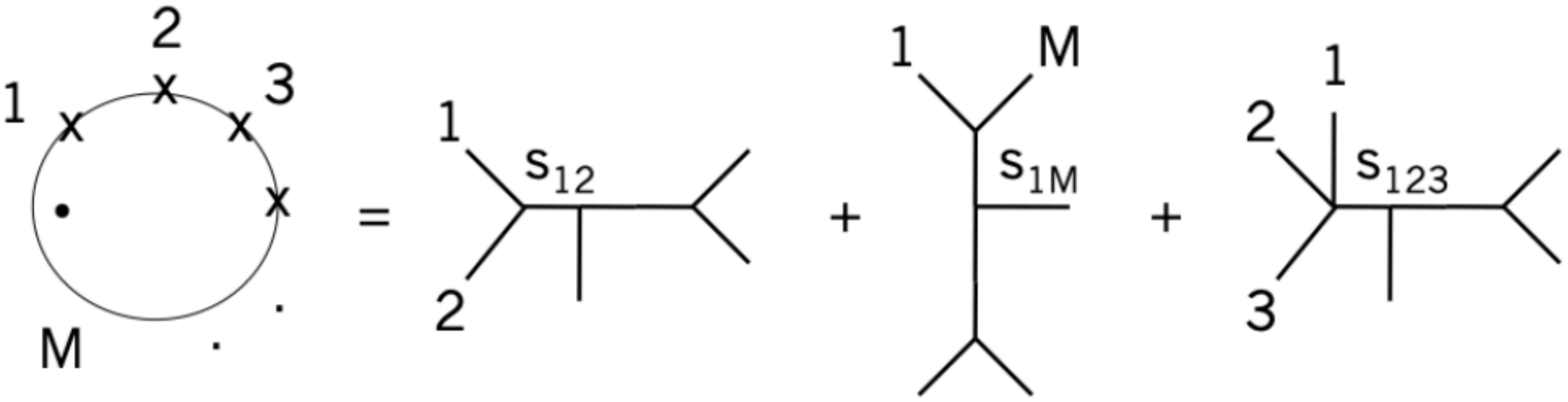

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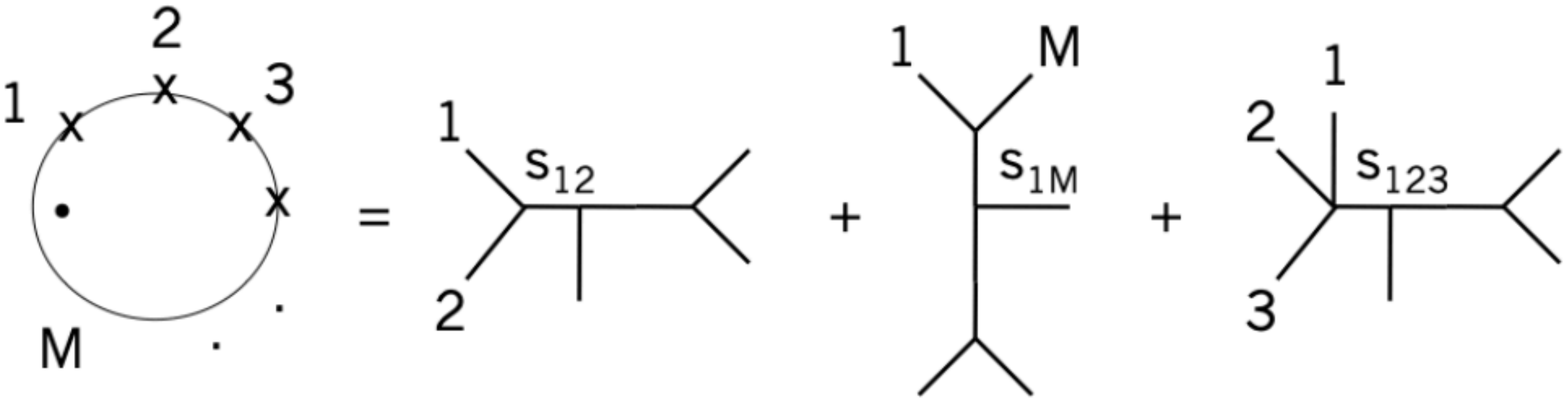

Compact massive tree amplitudes

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Find ‘stringy’ structure in the scattering equation prescription (CHY)

(NEJB, Damgaard, Tourkine, Vanhove)

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$$A_{n-2}(1,\{2,\ldots,n-1\},n) = \int \frac{\prod_{i=1}^n dz_i}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C}))} \prod_{i=1}^{n} \delta'\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{k_i\cdot k_j}{z_{ij}}\right) \frac{1}{z_{12}\cdots z_{n-1\,n}}$$

$$\times \sum_{\beta\in\mathfrak{S}_{n-2}} \frac{N_{n-2}(1,\beta(2,\ldots,n-1),n)}{z_{1\beta(2)}z_{\beta(2)\beta(3)}\cdots z_{\beta(n-1)\,n}},$$

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We can generate gravity amplitudes in the following way

$$M^{\mathrm{tree}}_{n-2}(1,2,\ldots,n) = i \sum_{\beta\in\mathfrak{S}_{n-2}} N_{n-2}(1,\beta(2,\cdots,n-1),n)A_{n-2}(1,\beta(2,\ldots,n-1),n)$$

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CHY formalism leads to the following
very compact amplitudes

$$M_1^{\text{tree}}(p, \ell_2, -p') = i\, N_1(p, \ell_2, -p') A_1(p, \ell_2, -p') = i\, N_1(p, \ell_2, -p')^2,$$

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$$= \frac{iN_2(p,2,3,-p')^2}{(\ell_2+p)^2-m^2+i\varepsilon} + \frac{iN_2(p,3,2,-p')^2}{(\ell_3+p)^2-m^2+i\varepsilon} + \frac{i(N_2{}^{[2,3]})^2}{(\ell_2+\ell_3)^2+i\varepsilon}$$

$$N_1(p,\ell_2,-p') = i\sqrt{2}\,\zeta_2\cdot p, \qquad A_1(p,\ell_2,-p') = N_1(p,\ell_2,-p').$$

$$N_2(p,\ell_2,\ell_3,-p') = \frac{i}{2}\Big(s_{2p}(\zeta_2\cdot\zeta_3) - 4(\zeta_2\cdot p)\zeta_3\cdot(p+\ell_2)\Big)$$

Straightforward to compute any tree order needed with manifest color-kinematic numerators

- no double poles (from KLT)
- Spin-0, spin-1/2 .. easy to derive

(NEJB, Brown, Gomez)

Velocity cuts and relation to world lines

Can open up massive propagators - direct connection to world-line formulation -
direct computation of probe amplitude to four-loop order
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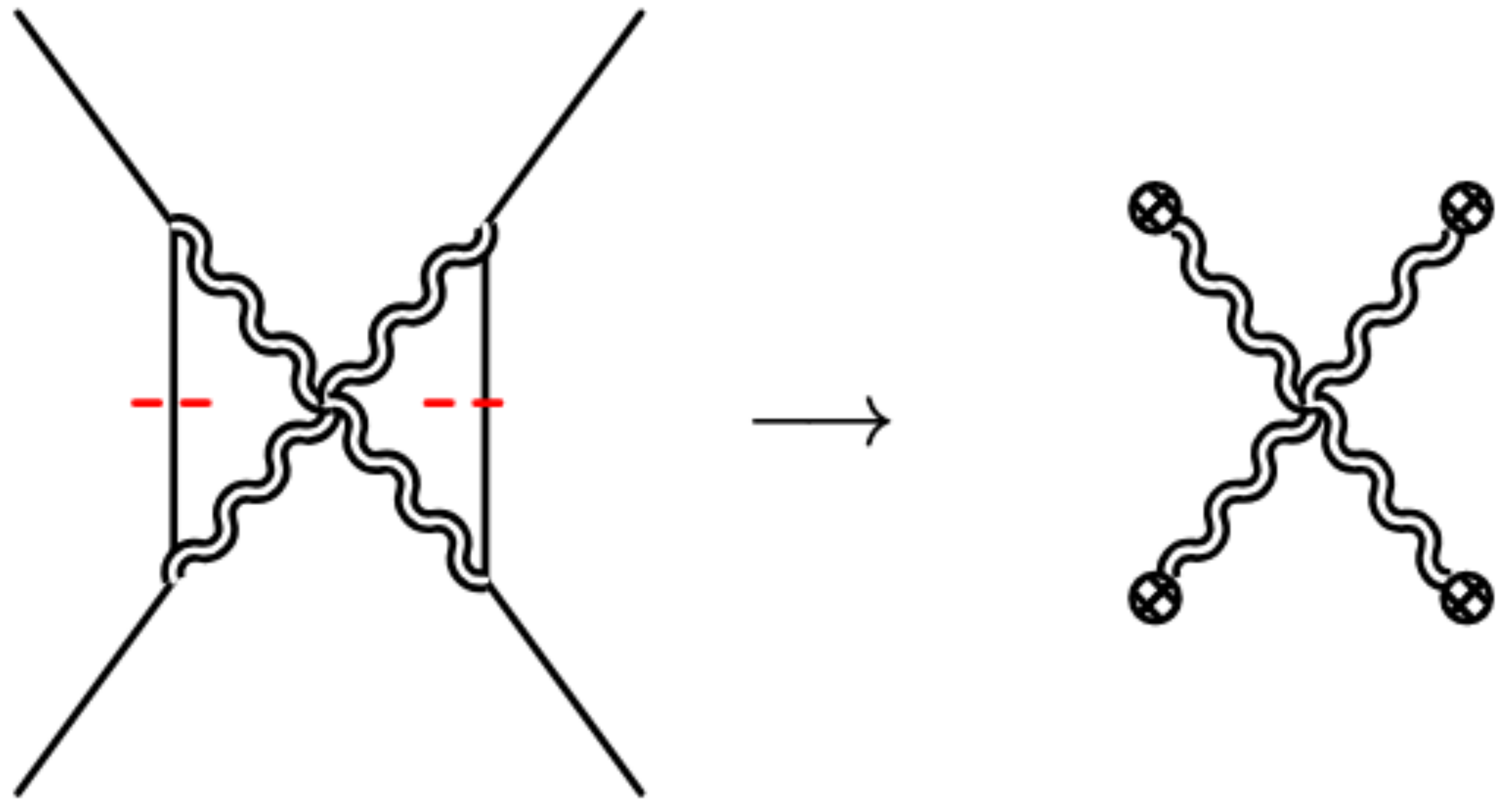

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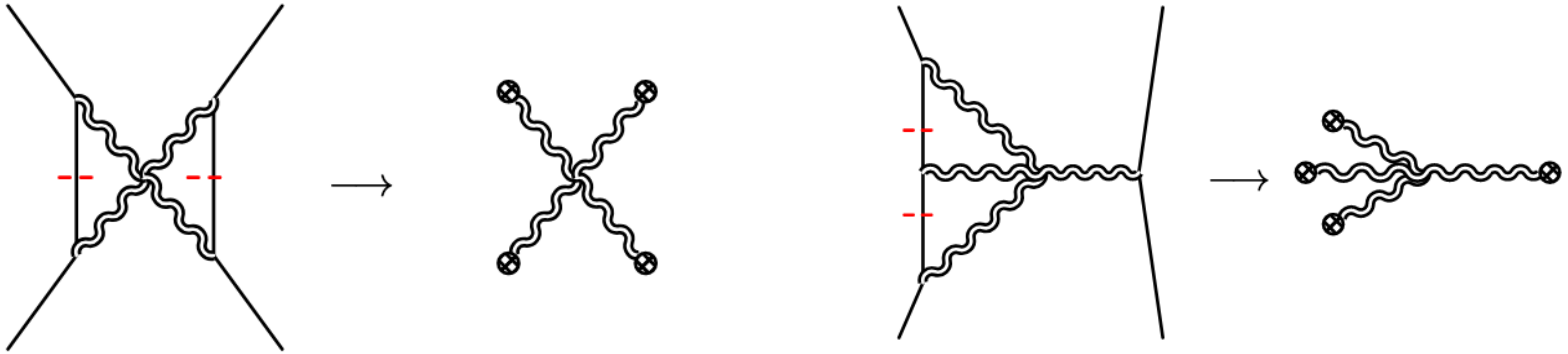

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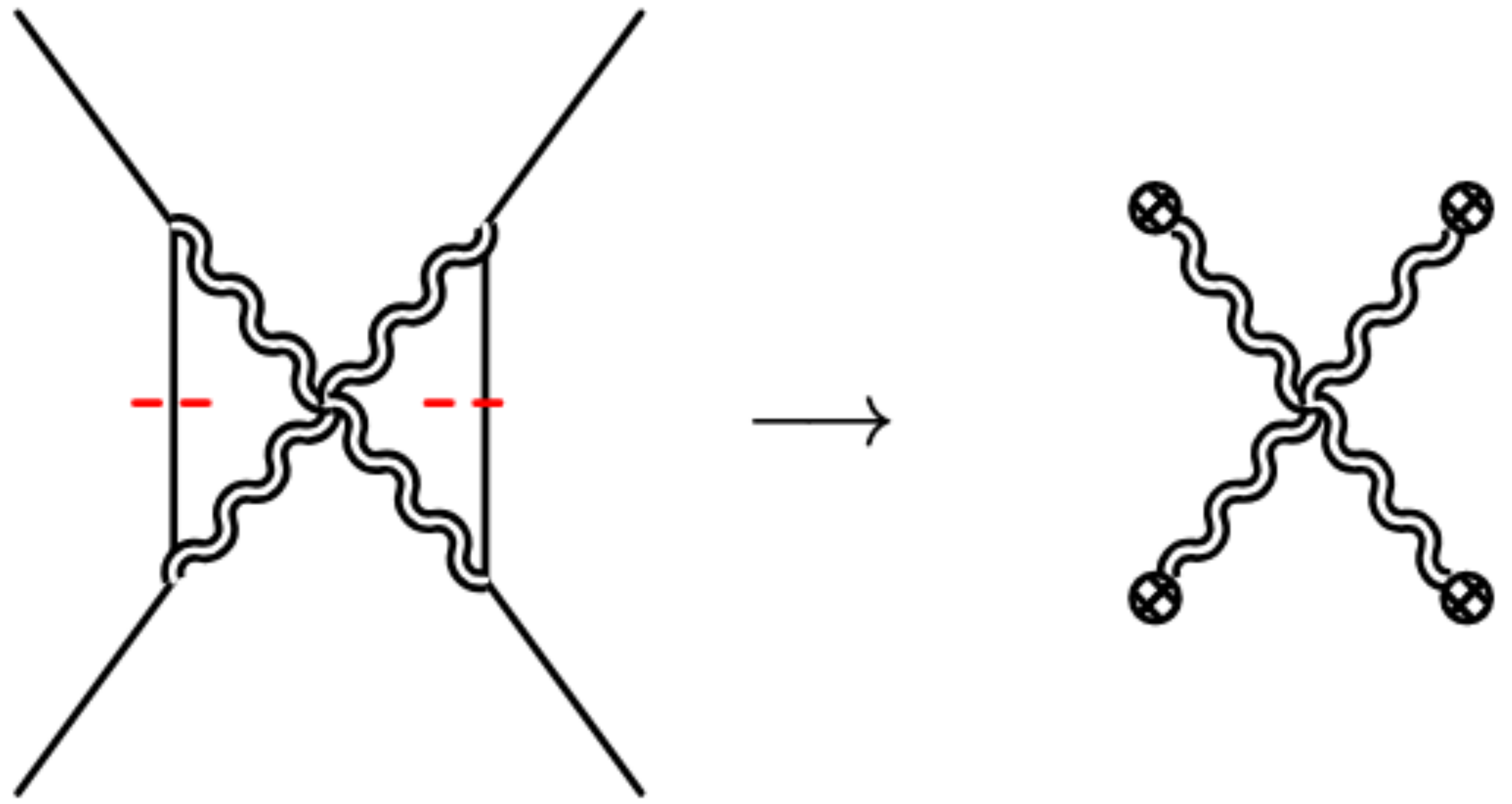

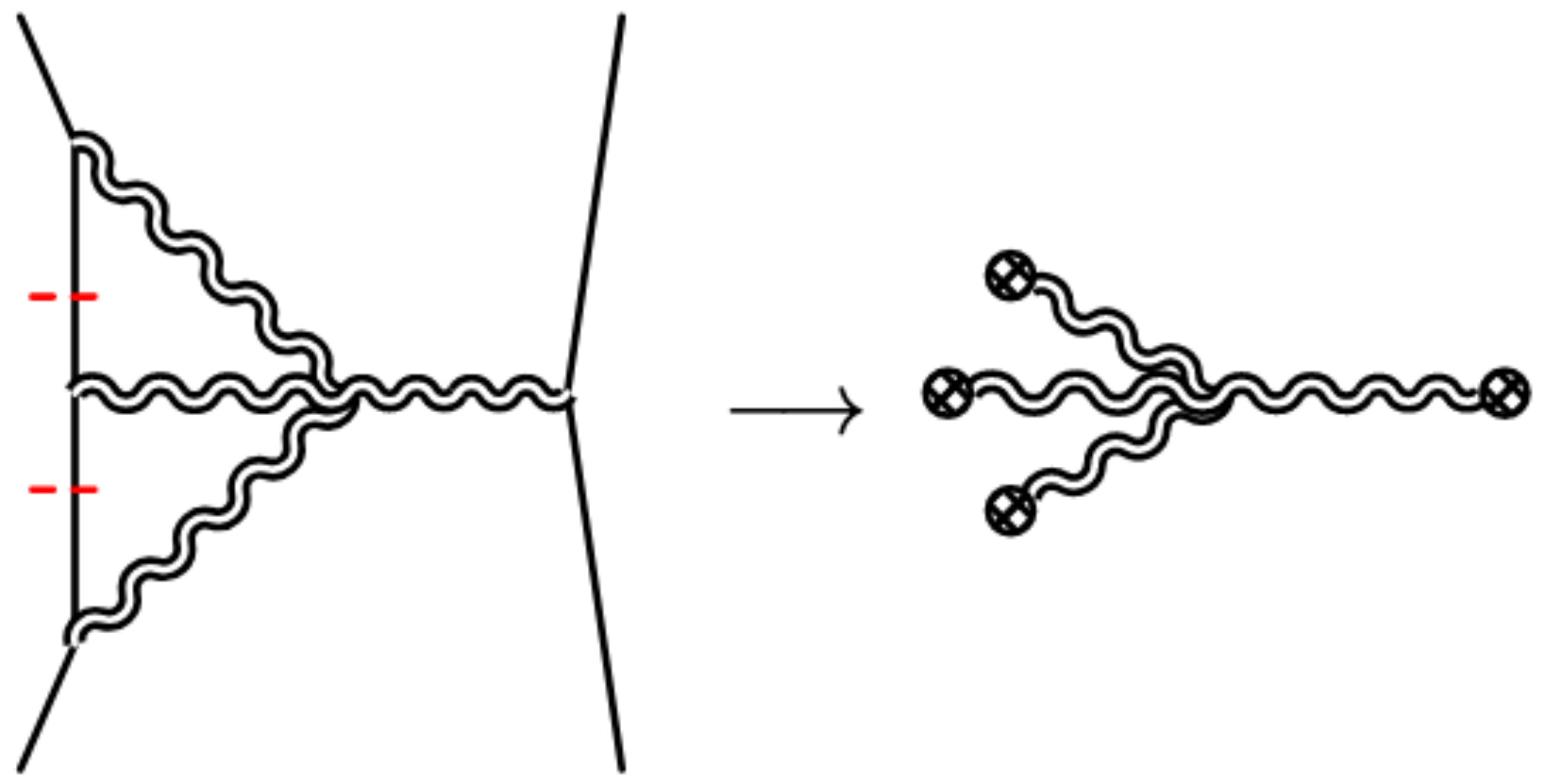

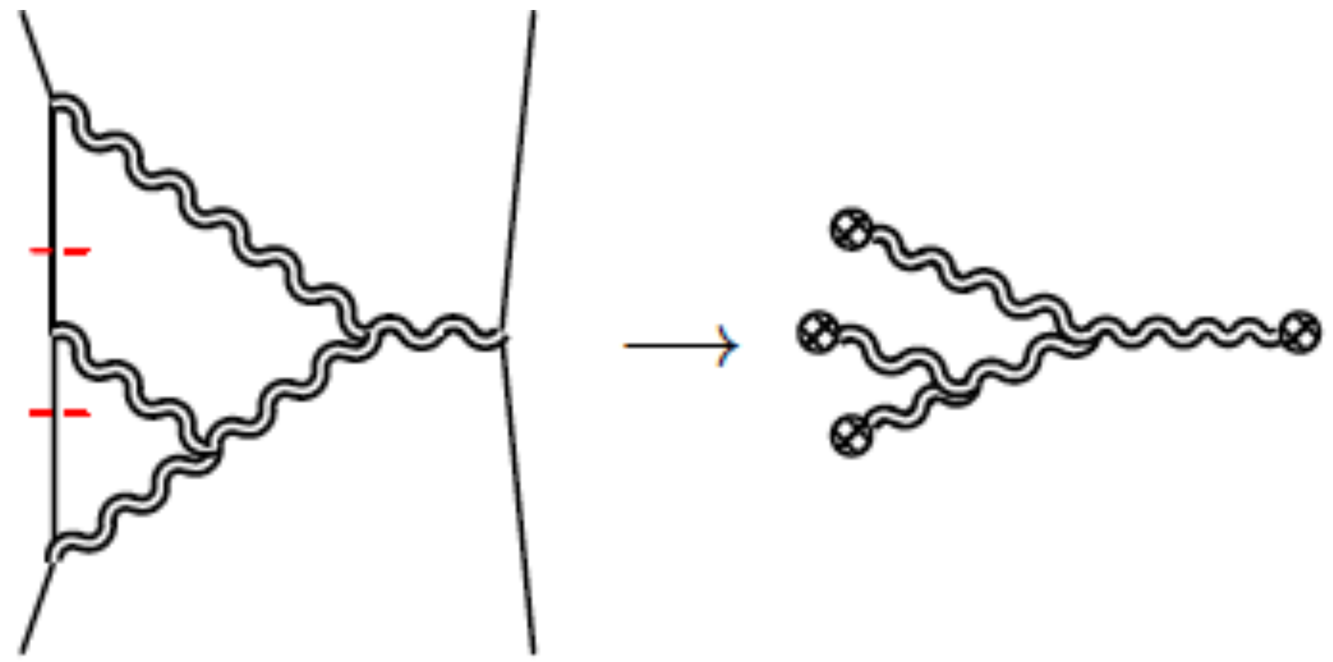

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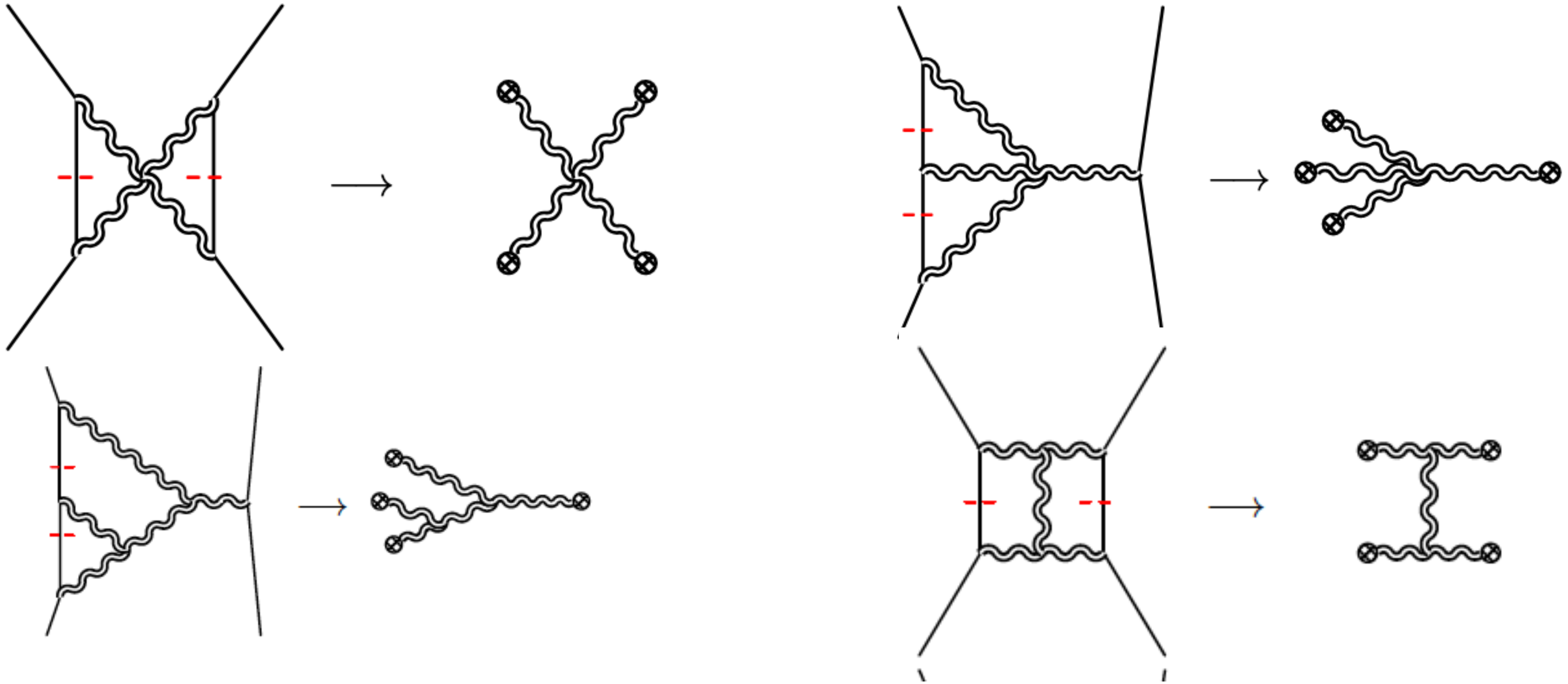

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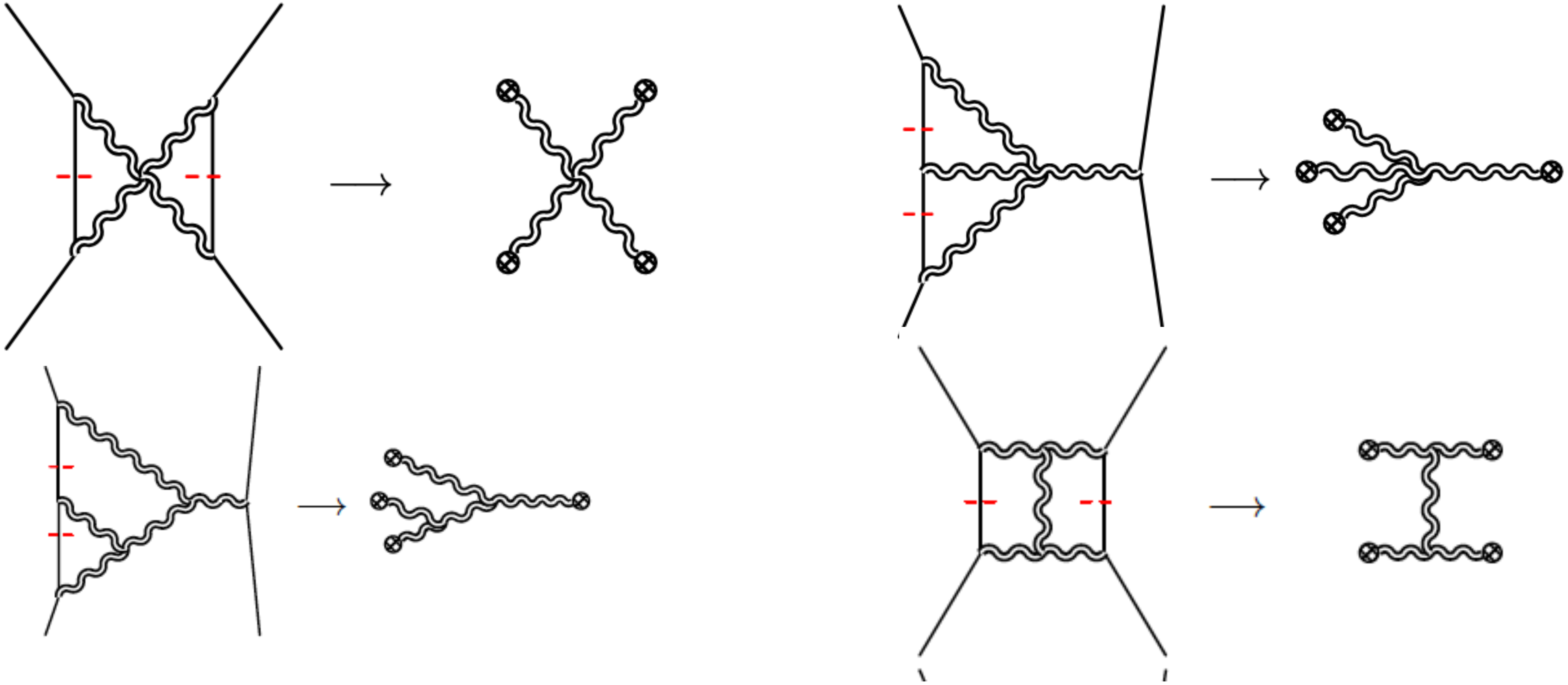

(NEJBB, Damgaard, Plante, Vanhove; NEJBB, Plante, Vanhove)

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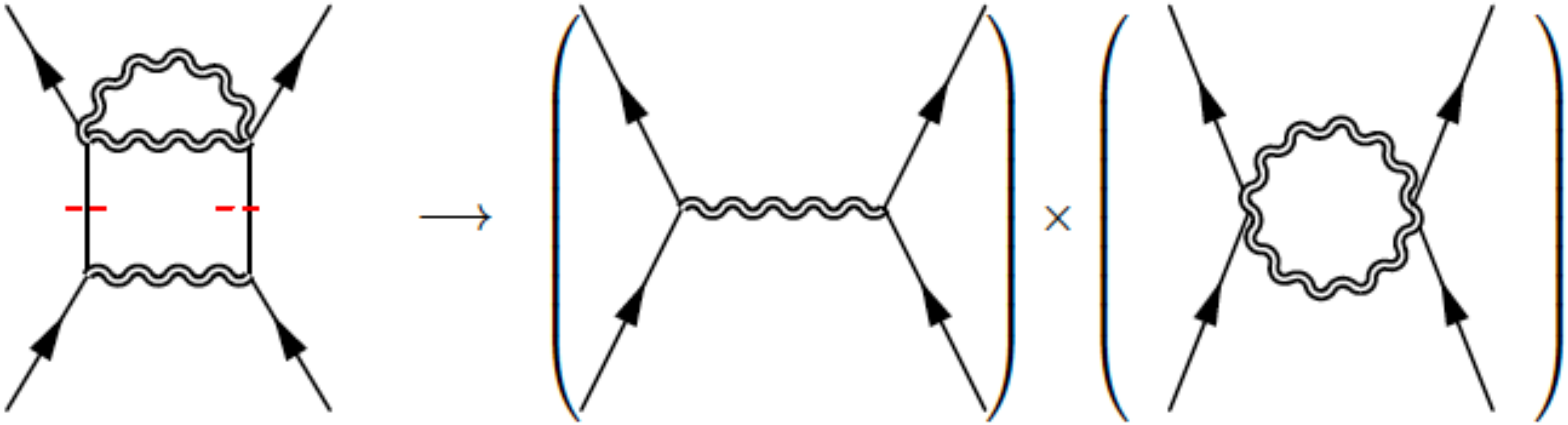

Lessons from exponentiation of the S-matrix

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Considering the following representation of the exponentiated amplitude, one has

Damgaard, Plante, Vanhove

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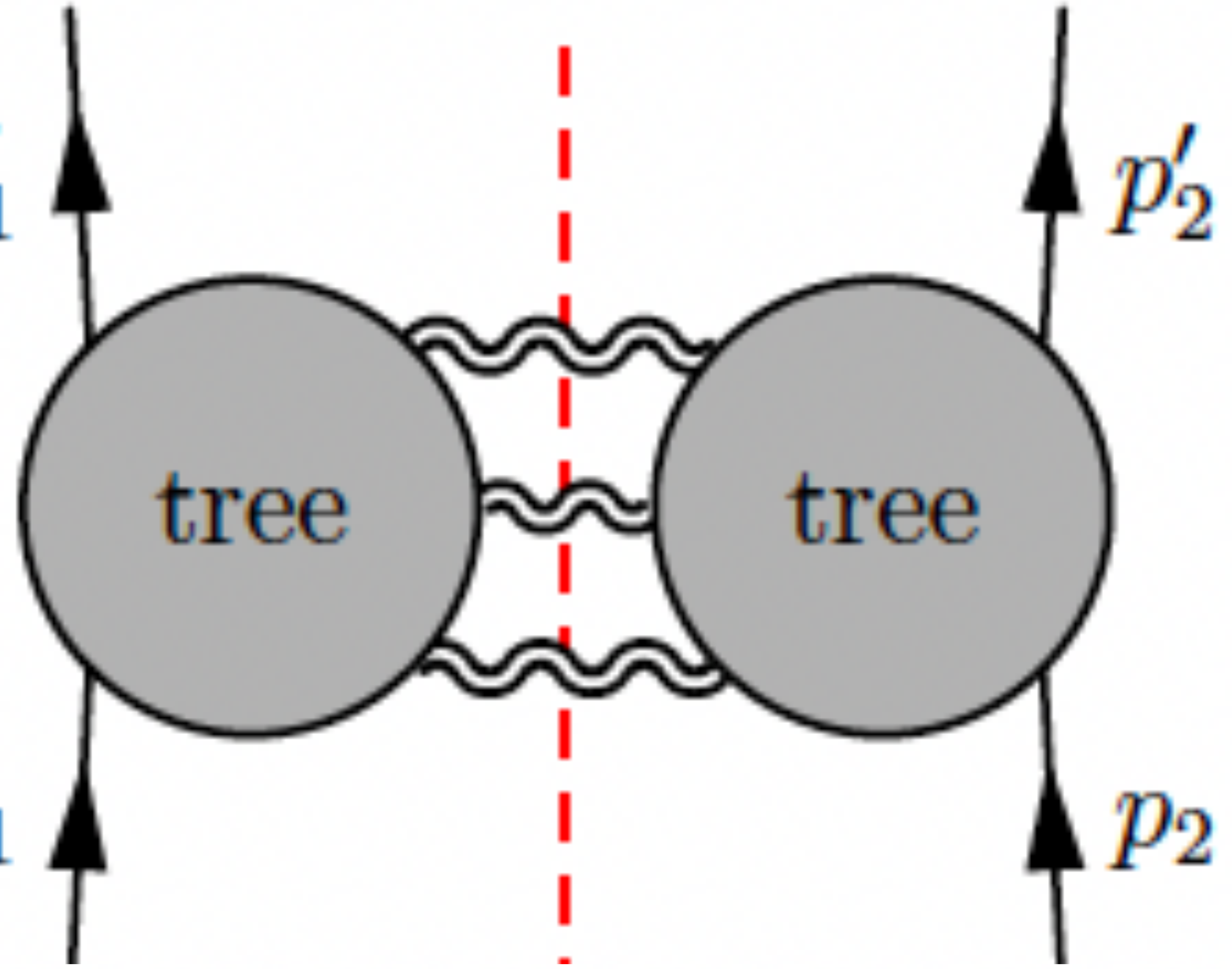
Bern et al   Damgaard, Plante, Vanhove

- It is easy to see which terms needs to be computed and identify the classical contributions to the radial action
- new radiation terms allow ‘radiation reaction’ to be automatically correctly accounted for

Example: Einstein gravity at two-loop order

$\mathcal{M}^2_{3-\mathrm{cut}}(\sigma, q^2) =$



$$\mathcal{M}_2^{3-\mathrm{cut}}(\sigma, q^2) = \int \frac{d^D l_1 d^D l_2 d^D l_3}{(2\pi)^{3D}} (2\pi)^D \delta^{(D)}(l_1 + l_2 + l_3 + q) \frac{i^3}{l_1^2 l_2^2 l_3^2}$$
$$\times \frac{1}{3!} \sum_{\substack{\mathrm{Perm}(l_1,l_2,l_3) \\ \lambda_1=\pm,\lambda_2=\pm,\lambda_3=\pm}} \mathcal{M}_0(p_1, p_1', l_1^{\lambda_1}, l_2^{\lambda_2}, l_3^{\lambda_3})(\mathcal{M}_0(p_2, p_2', -l_1^{\lambda_1}, -l_2^{\lambda_2}, -l_3^{\lambda_3}))^*$$

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$$N_1(|\underline{\vec{q}}|, \gamma) = \frac{3\pi^2 G_N^2 m_1^2 m_2^2 (m_1+m_2)(5\gamma^2-1)(4\pi e^{-\gamma_E})^{\frac{4-D}{2}}}{|\underline{\vec{q}}|^{5-D}}$$
$$- \frac{8G_N^2 m_1^2 m_2^2 (4\pi e^{-\gamma_E})^{\frac{4-D}{2}}\hbar}{(4-D)|\underline{\vec{q}}|^{4-D}}\Big(\frac{2(2\gamma^2-1)(7-6\gamma^2)\,\mathrm{arccosh}(\gamma)}{(\gamma^2-1)^{\frac{3}{2}}} + \frac{1-49\gamma^2+18\gamma^4}{15(\gamma^2-1)}\Big)$$
$$+ \mathcal{O}(|\underline{\vec{q}}|^{5-D}) .$$

Simplifications from the exponentiation of the S-matrix

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Two-loop radial action contribution

Simplifications from the exponentiation of the S-matrix

Two-loop radial action contribution

$$N_2(|\underline{\vec{q}}|,\gamma) = \frac{4\pi G_N^3 (4\pi e^{-\gamma_E})^{4-D} m_1^2 m_2^2}{(4-D)|\underline{\vec{q}}|^{8-2D}} \Bigg(\frac{\mathcal{E}_{\text{C.M.}}^2 \,(64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5)}{3\left(\gamma^2-1\right)^2}$$

$$- \frac{4}{3} m_1 m_2 \gamma \left(14\gamma^2 + 25\right) + \frac{4 m_1 m_2 (3 + 12\gamma^2 - 4\gamma^4)\,\text{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}$$

$$+ \frac{2 m_1 m_2 (2\gamma^2-1)^2}{\sqrt{\gamma^2-1}} \left(- \frac{11}{3} + \frac{d}{d\gamma} \left(\frac{(2\gamma^2-1)\,\text{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} \right) \right) \Bigg) + \mathcal{O}(\hbar)$$

Velocity cuts tree diagrams / soft expansion

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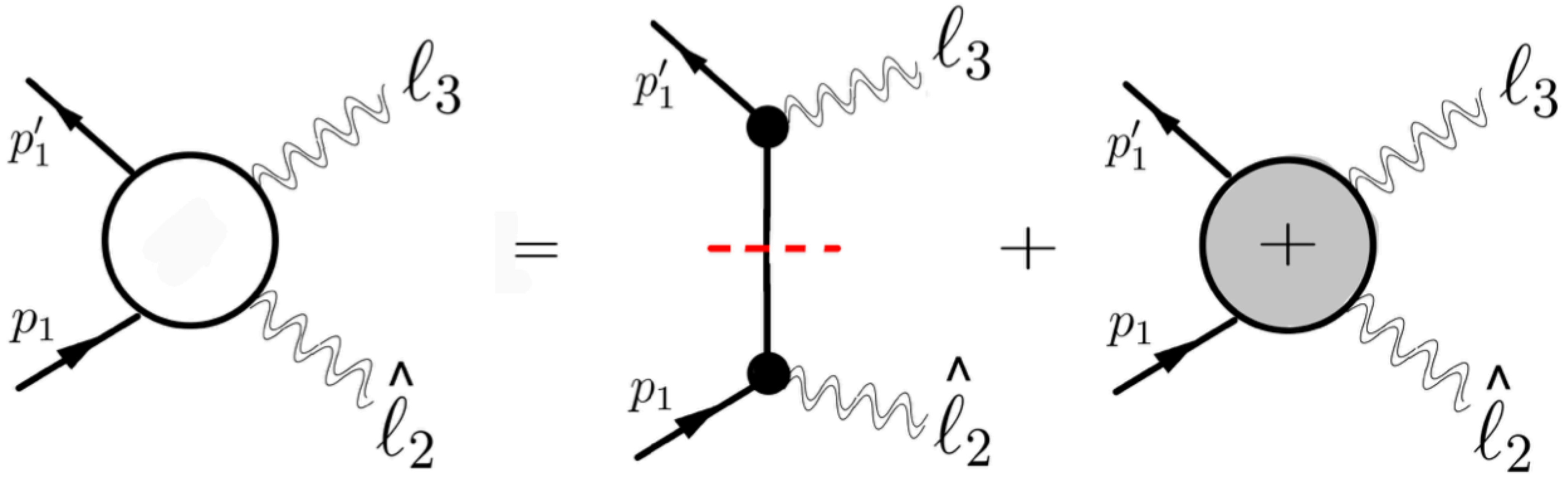

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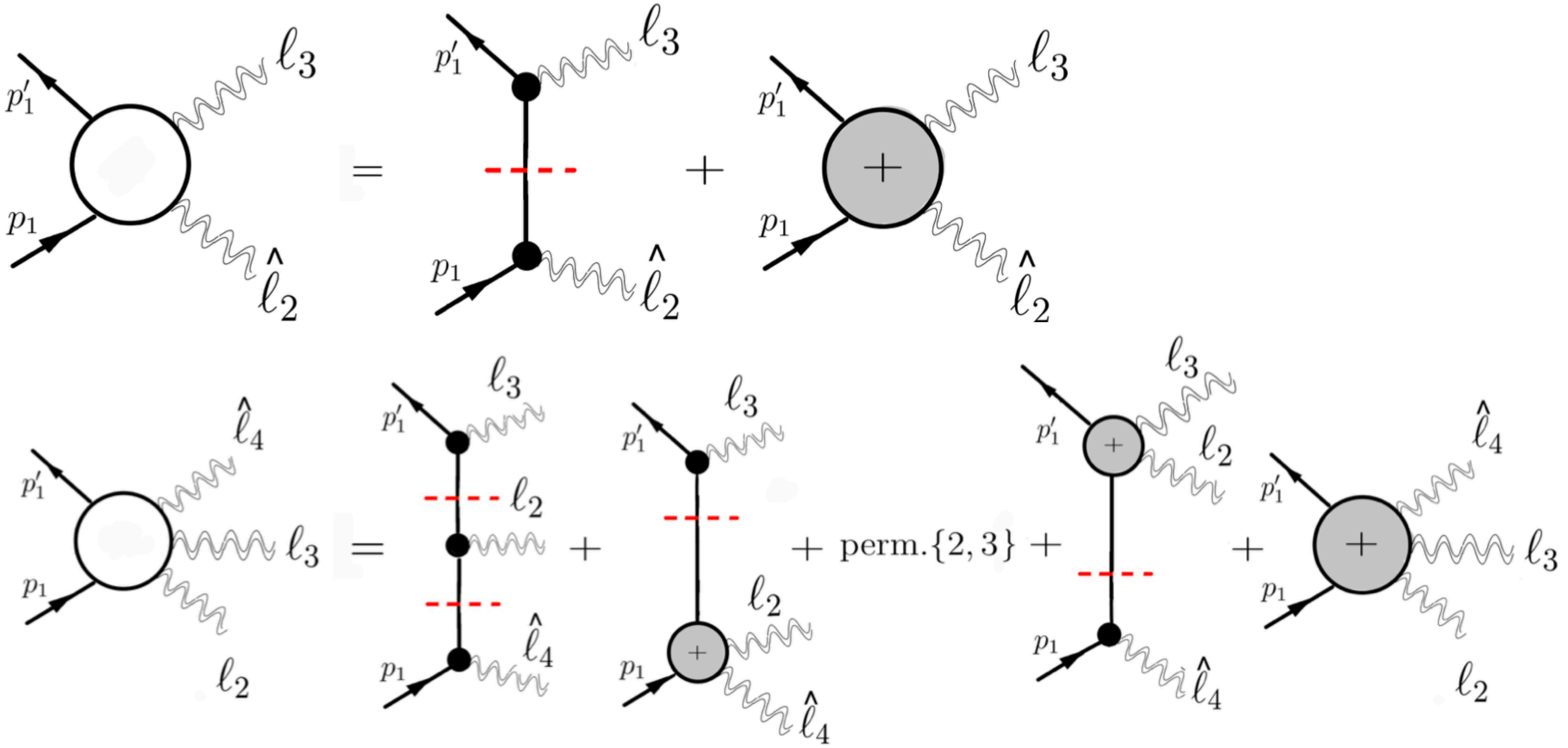

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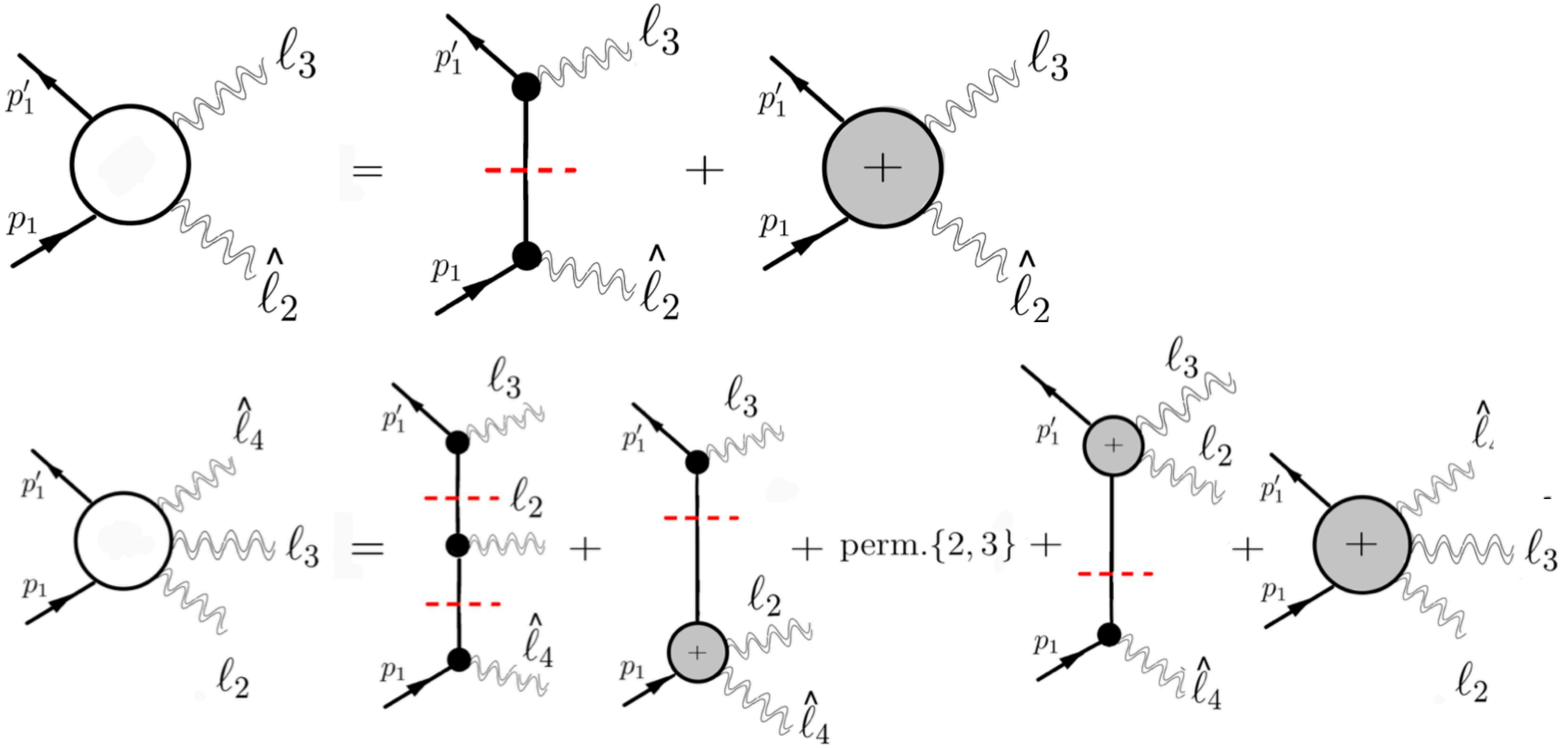
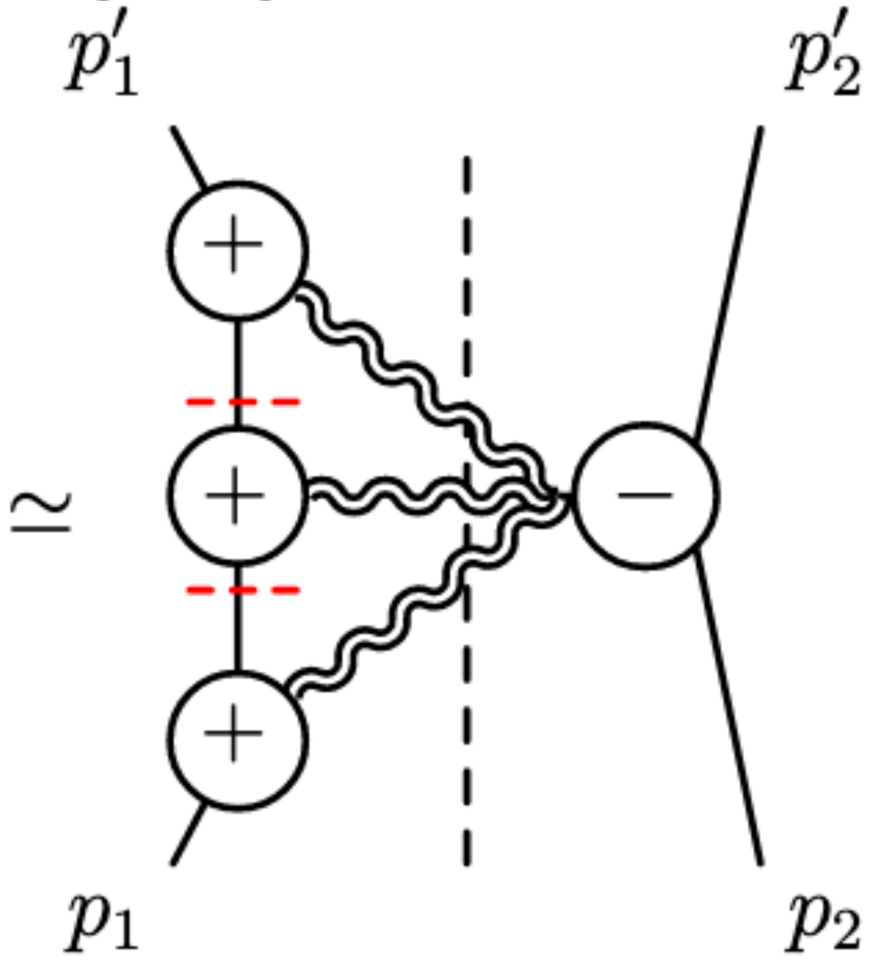


$m_1^4 m_2^2\, c_{3,1}(\gamma, D) \simeq$



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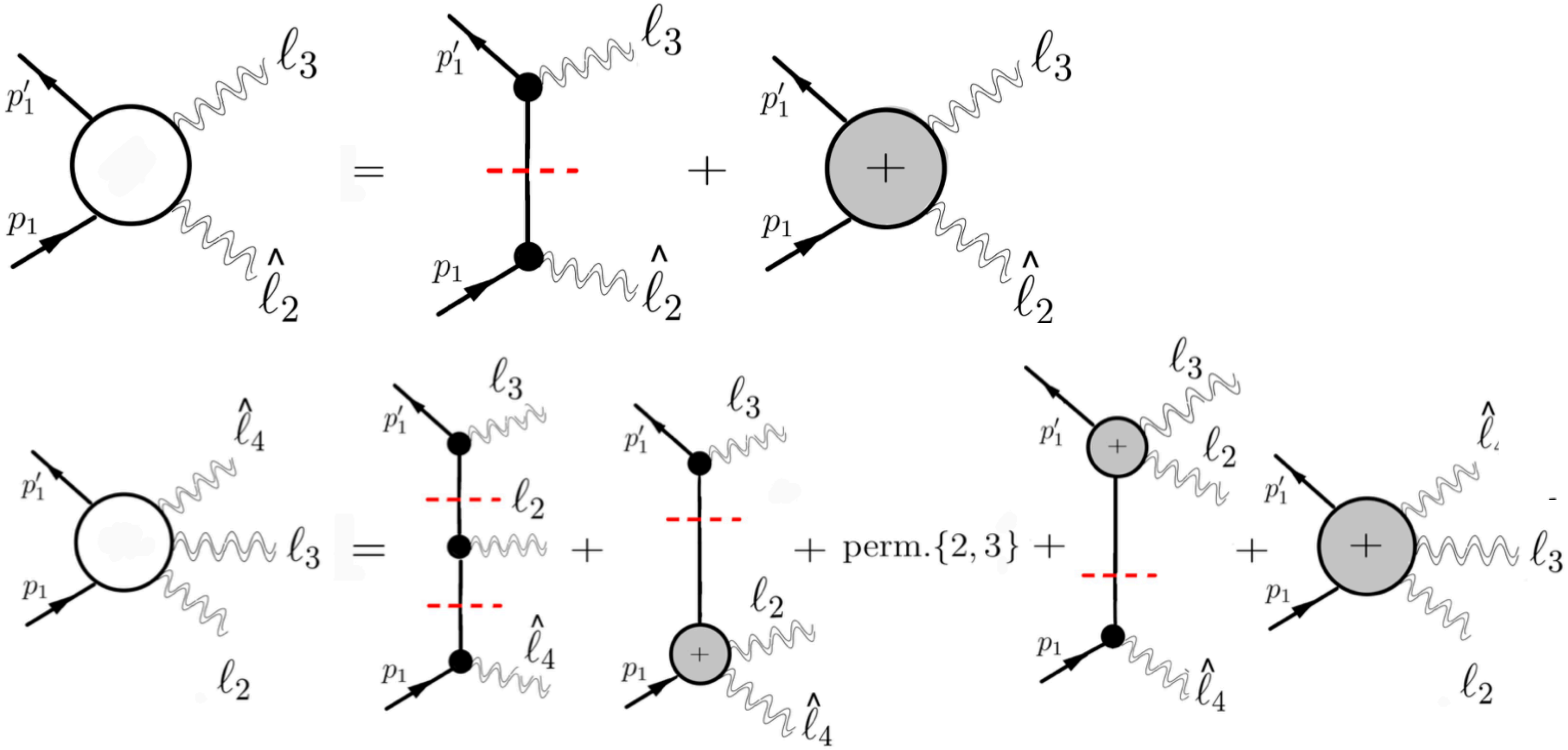

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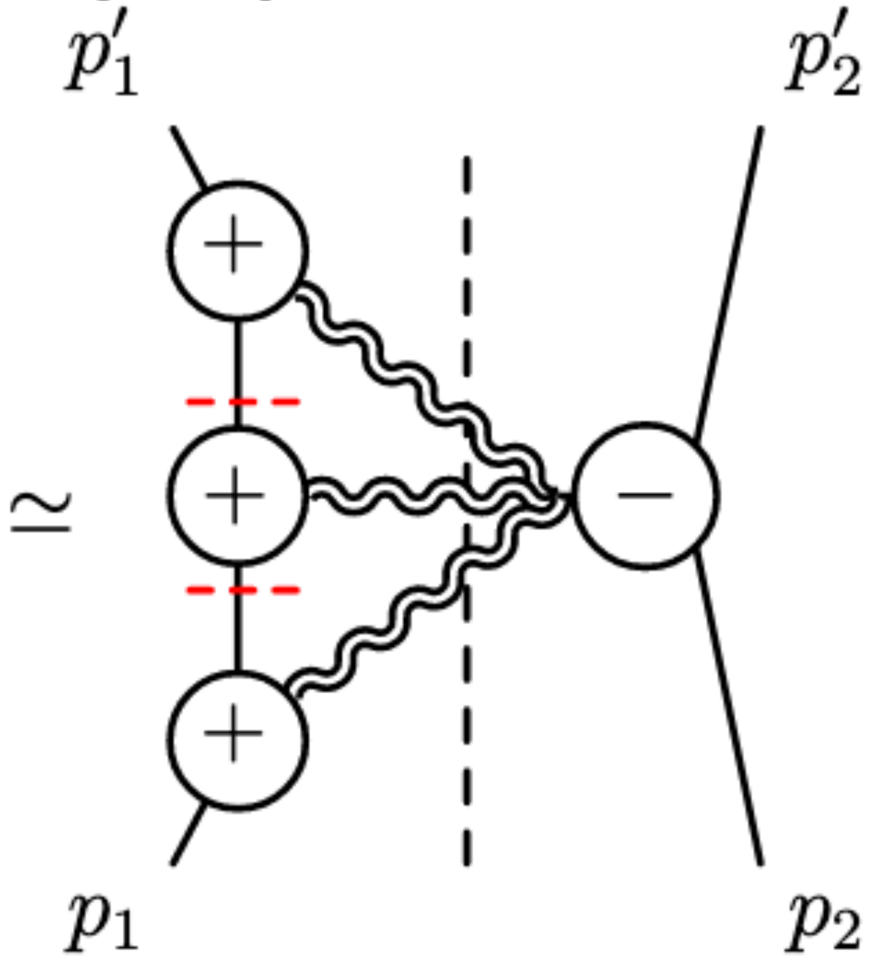

$$\mathcal{M}_{L+1}^{\text{tree}} \sim (\mathcal{M}_1^{\text{tree}(+)})^{L+1} \prod_i^L \delta_i(\ldots) + (\mathcal{M}_1^{\text{tree}(+)})^{L-1} (\mathcal{M}_2^{\text{tree}(+)}) \prod_i^{L-1} \delta_i(\ldots) + \cdots$$

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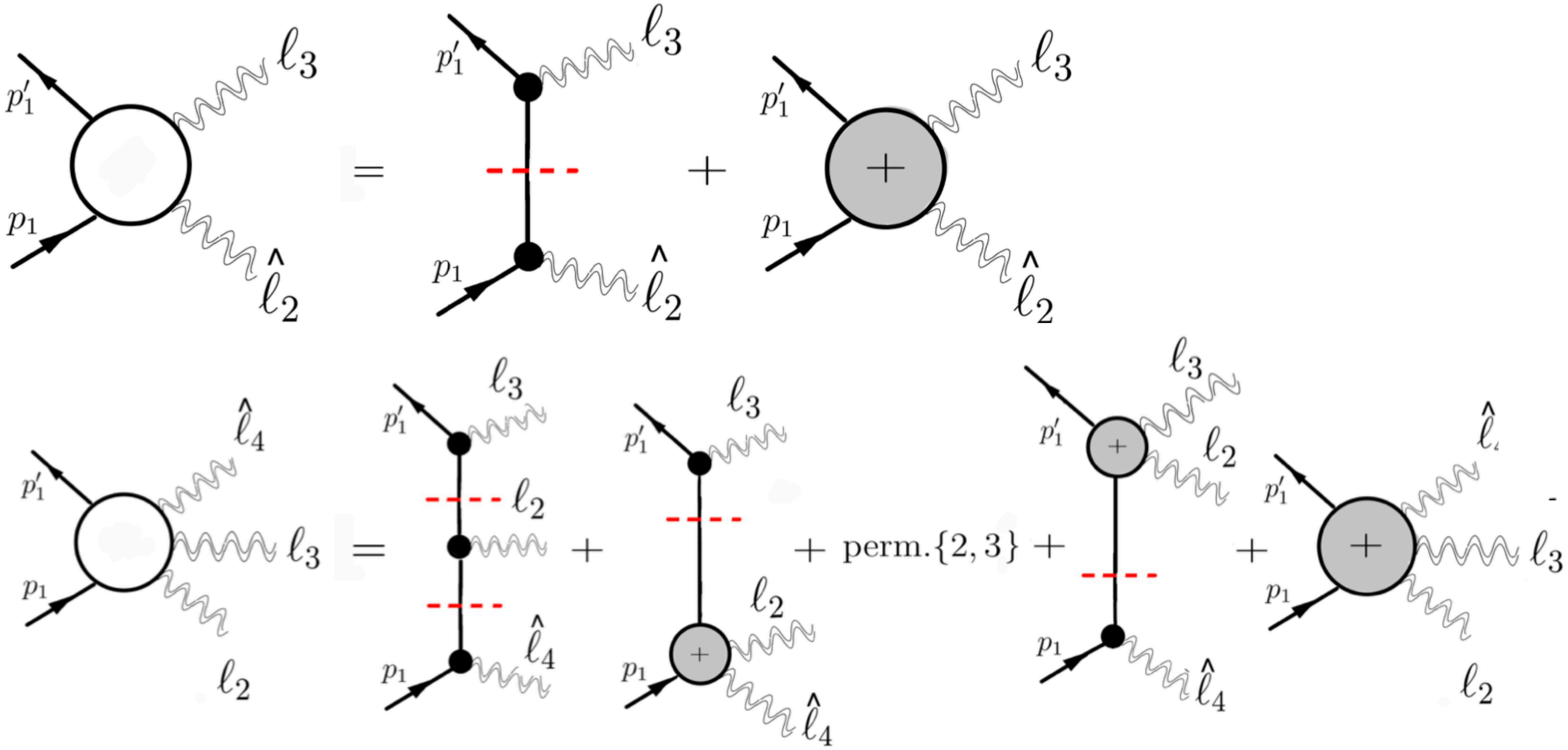

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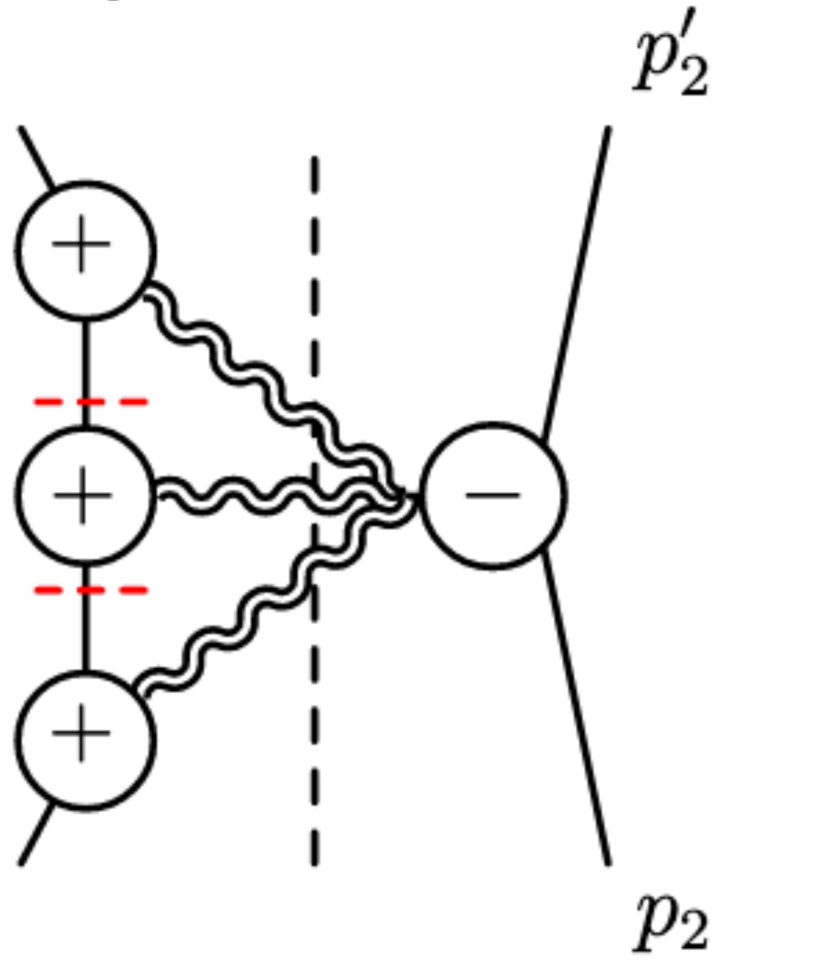

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(NEJBB, Plante, Vanhove)

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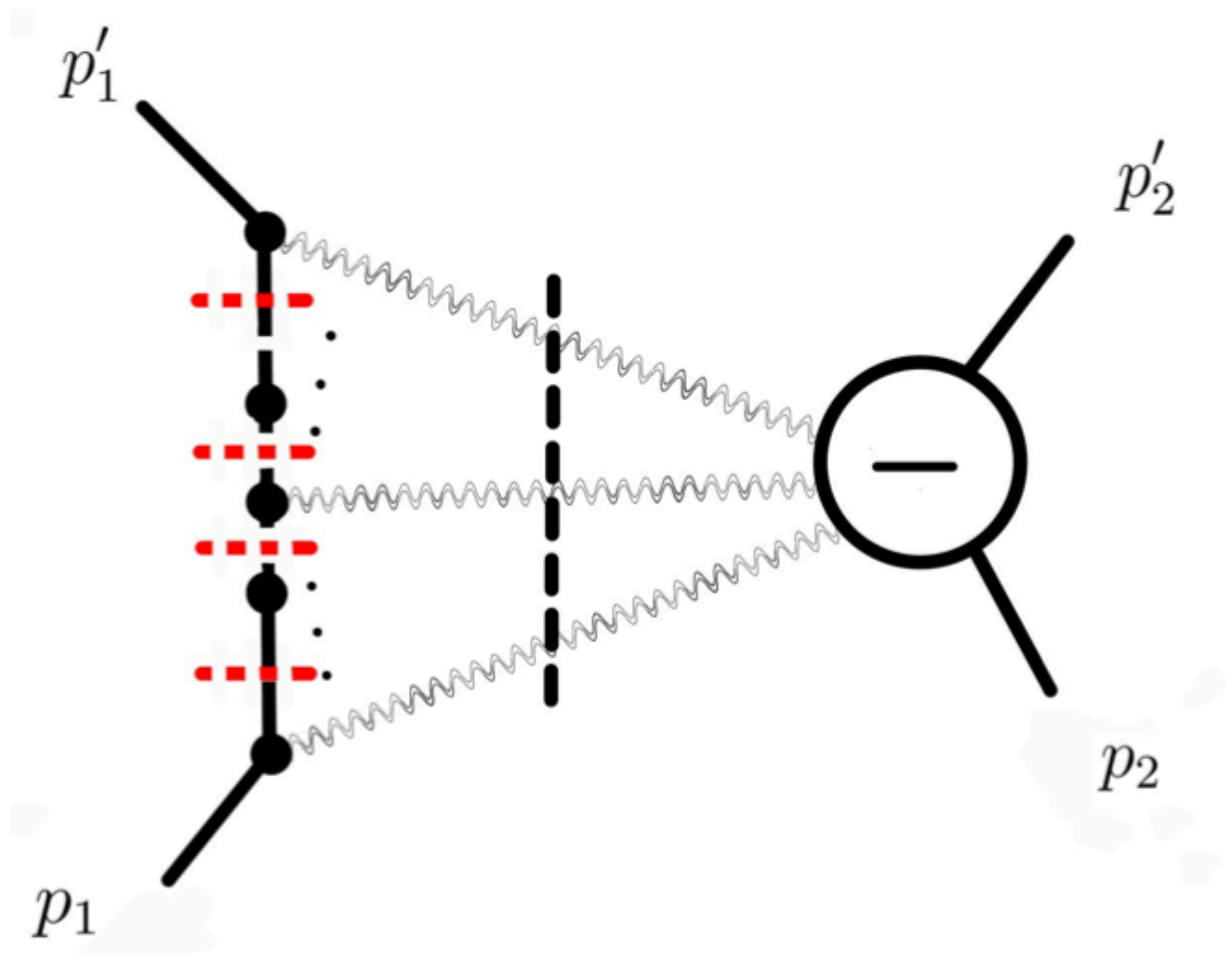

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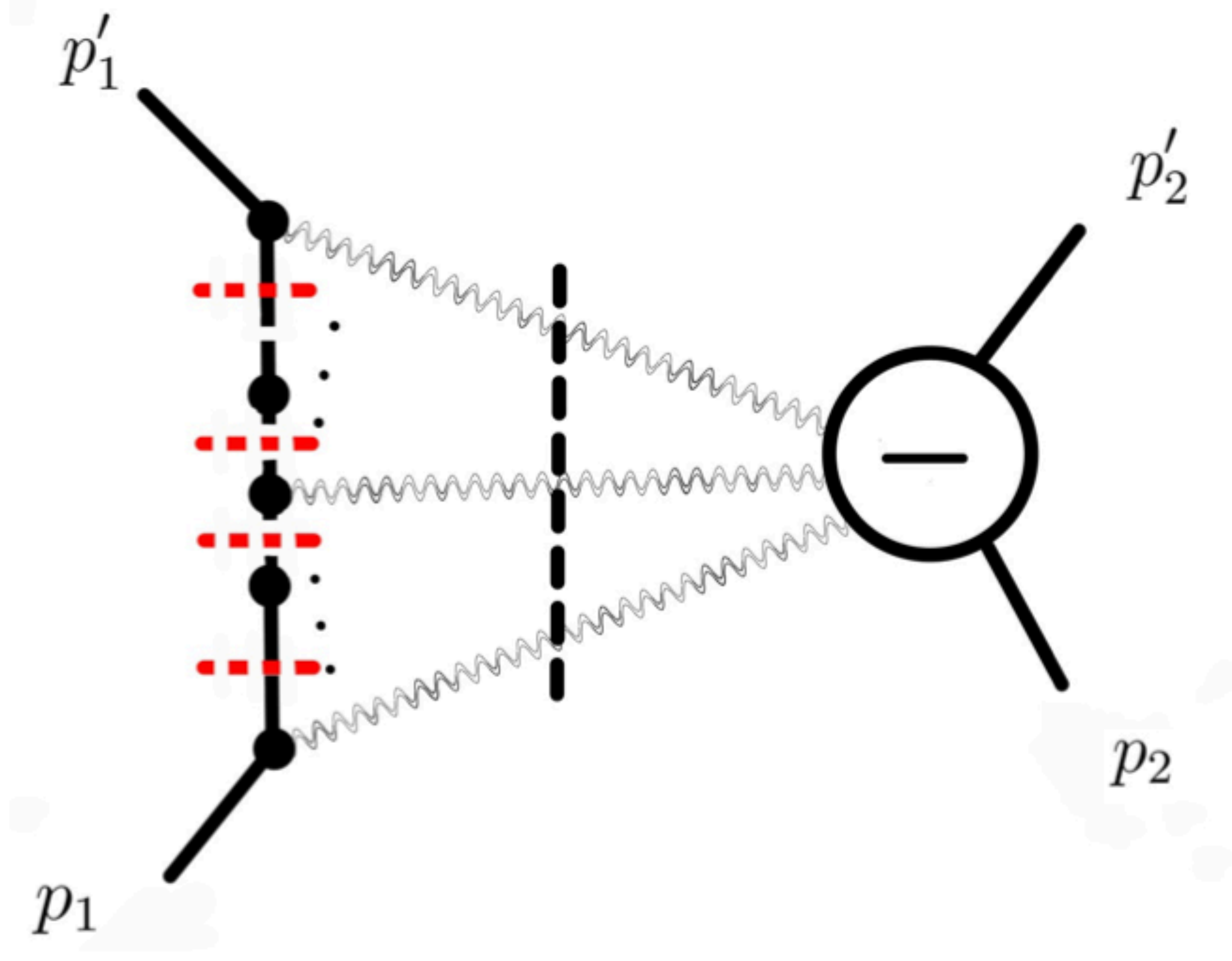

(NEJBB,
Plante,
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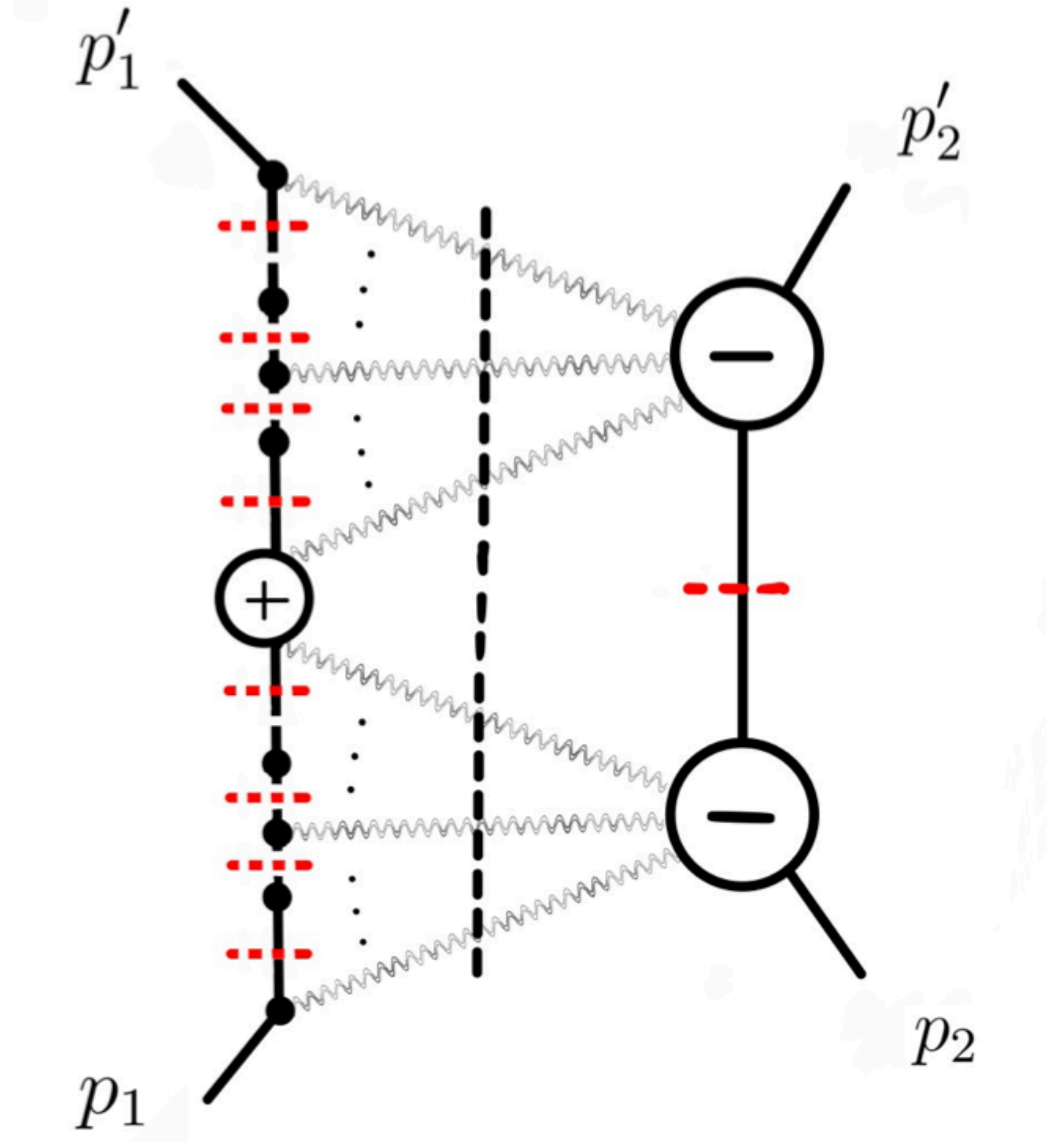

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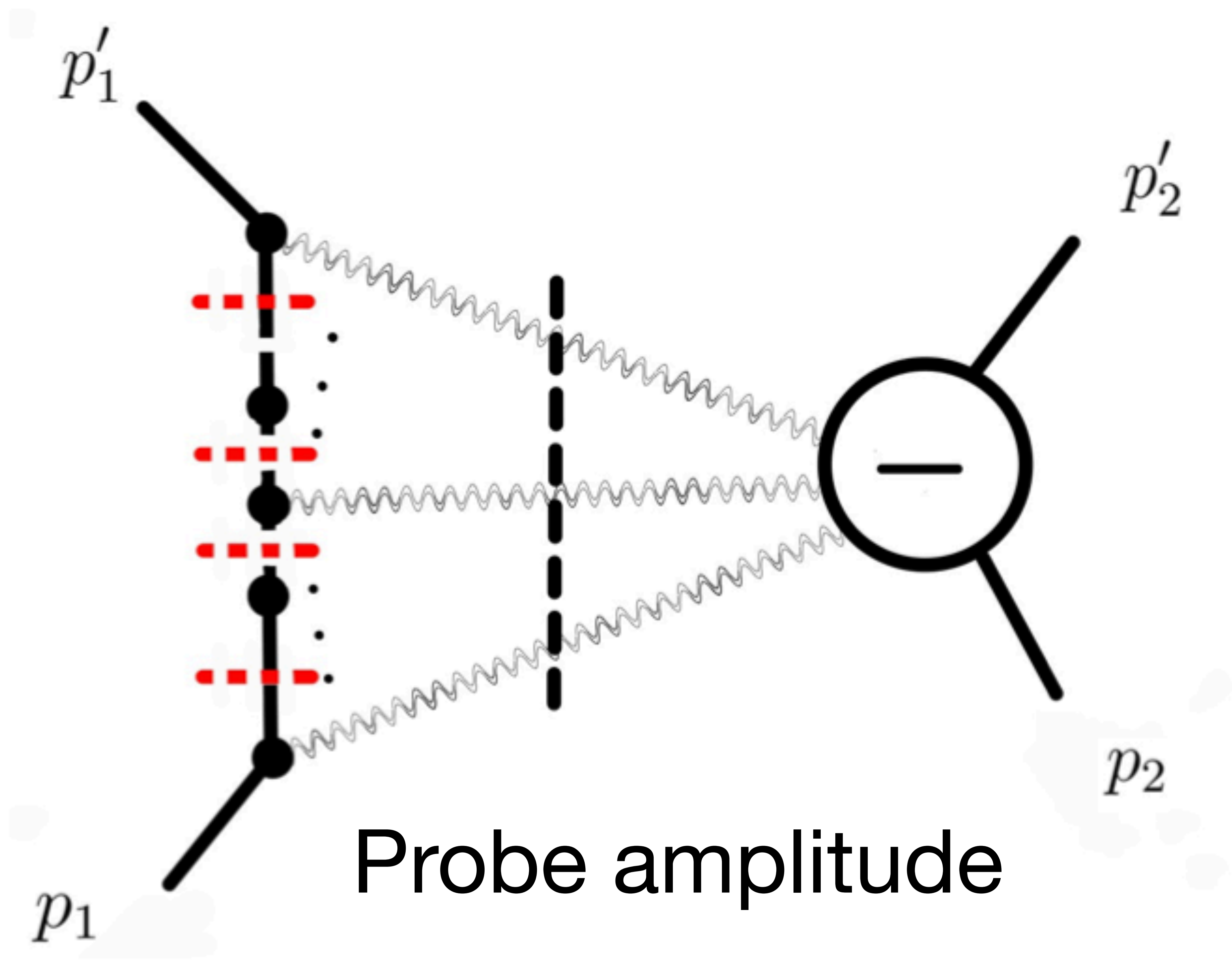

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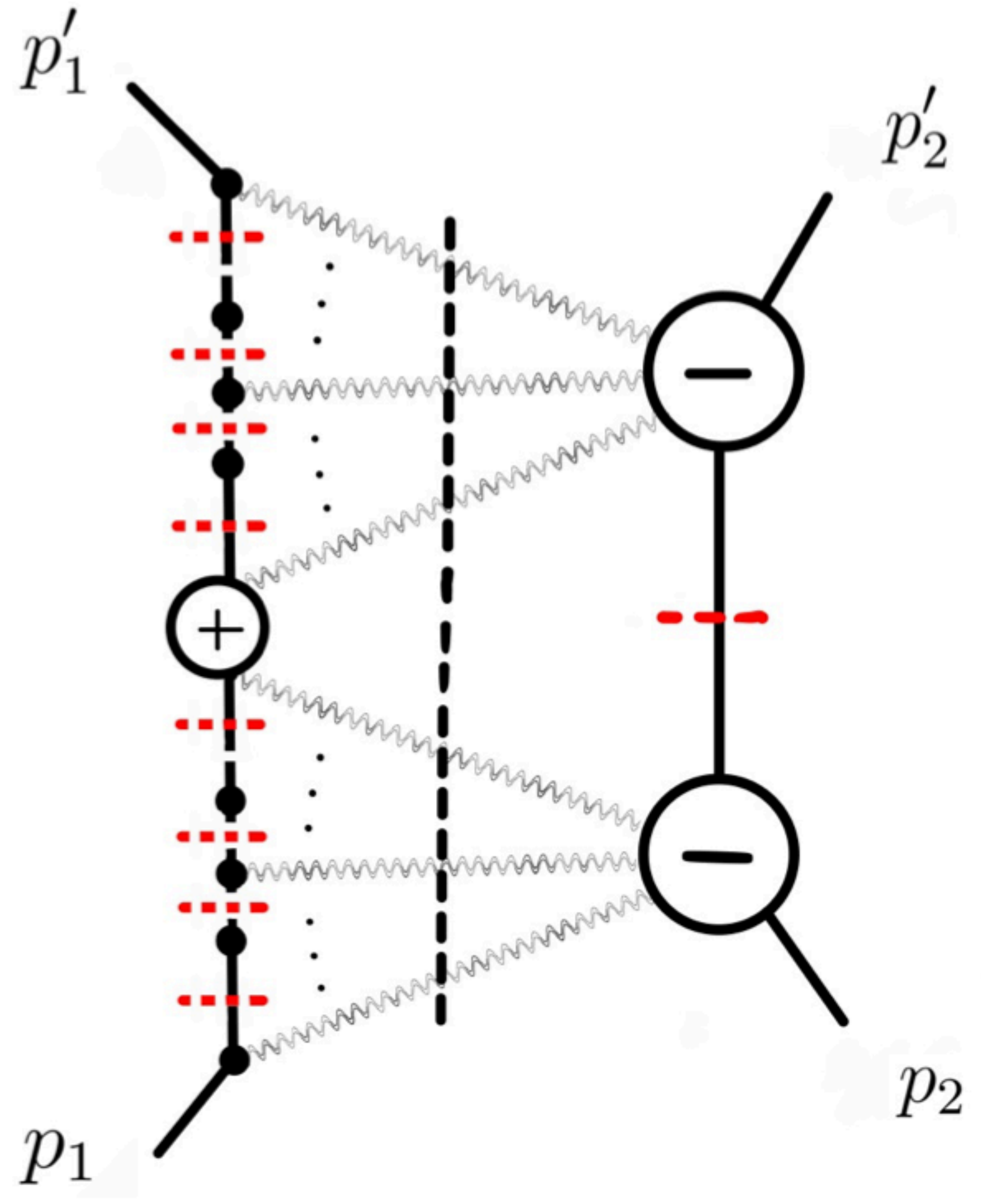

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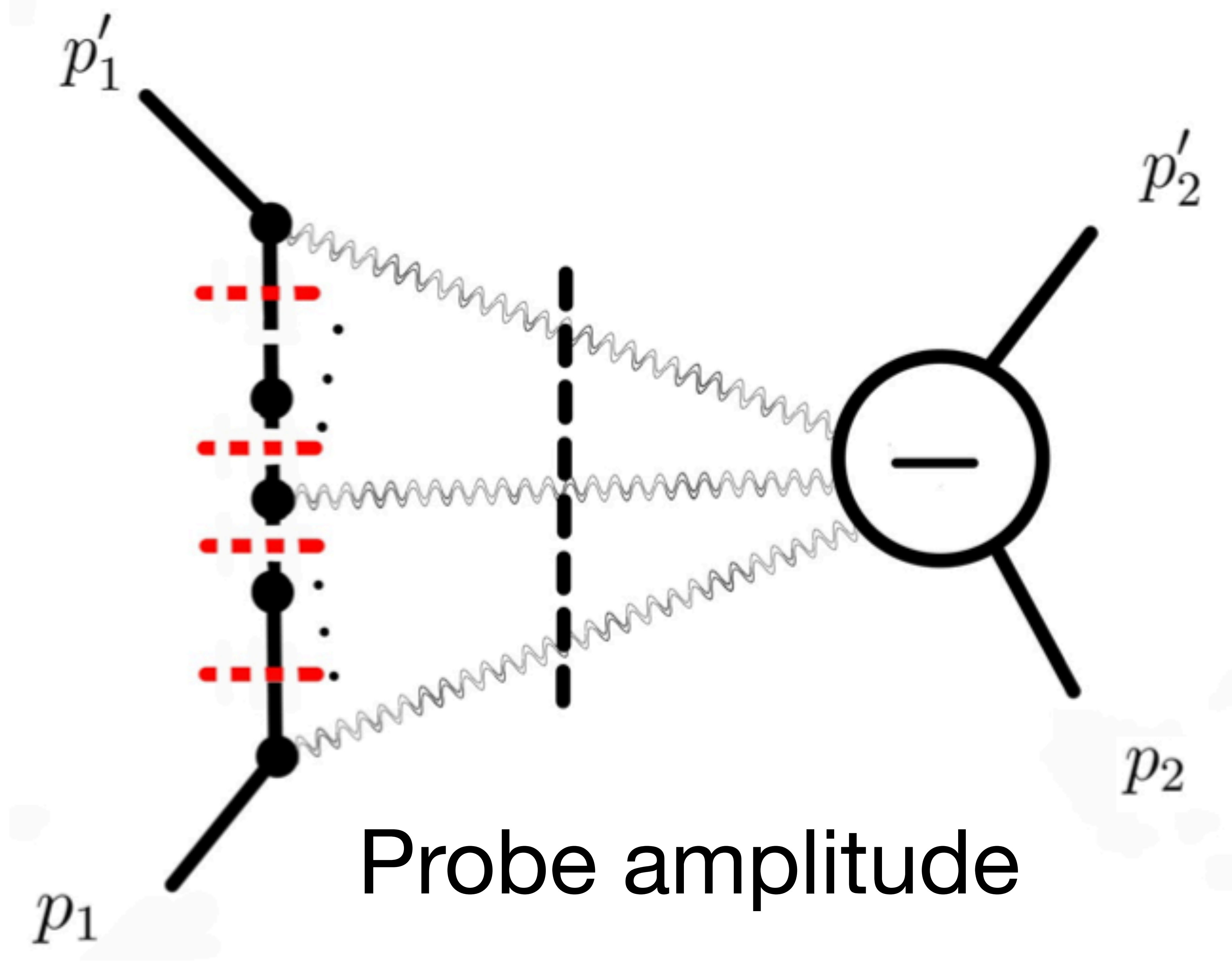
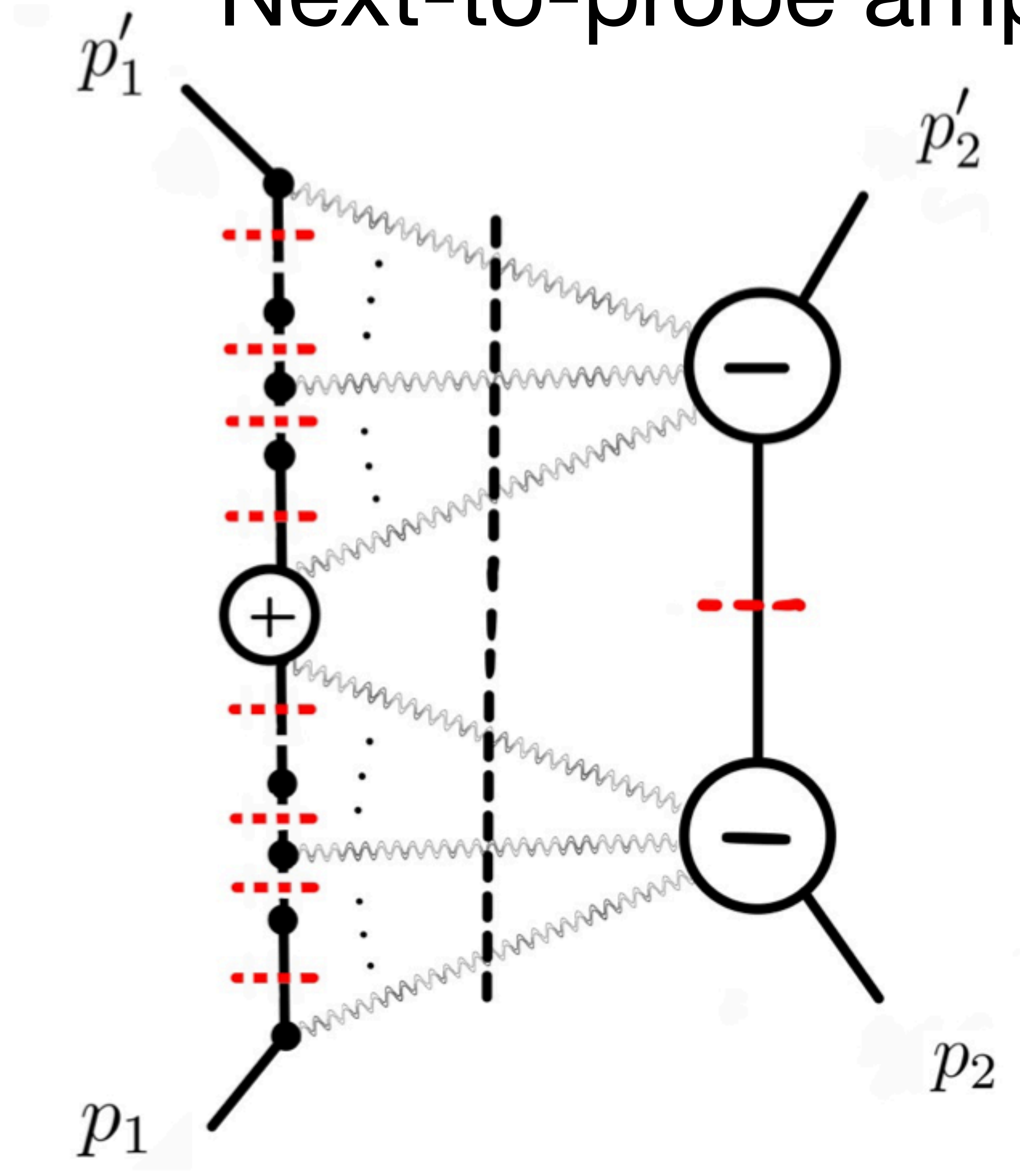


Probe amplitude

(NEJBB, Plante, Vanhove)

Next-to-probe amplitude



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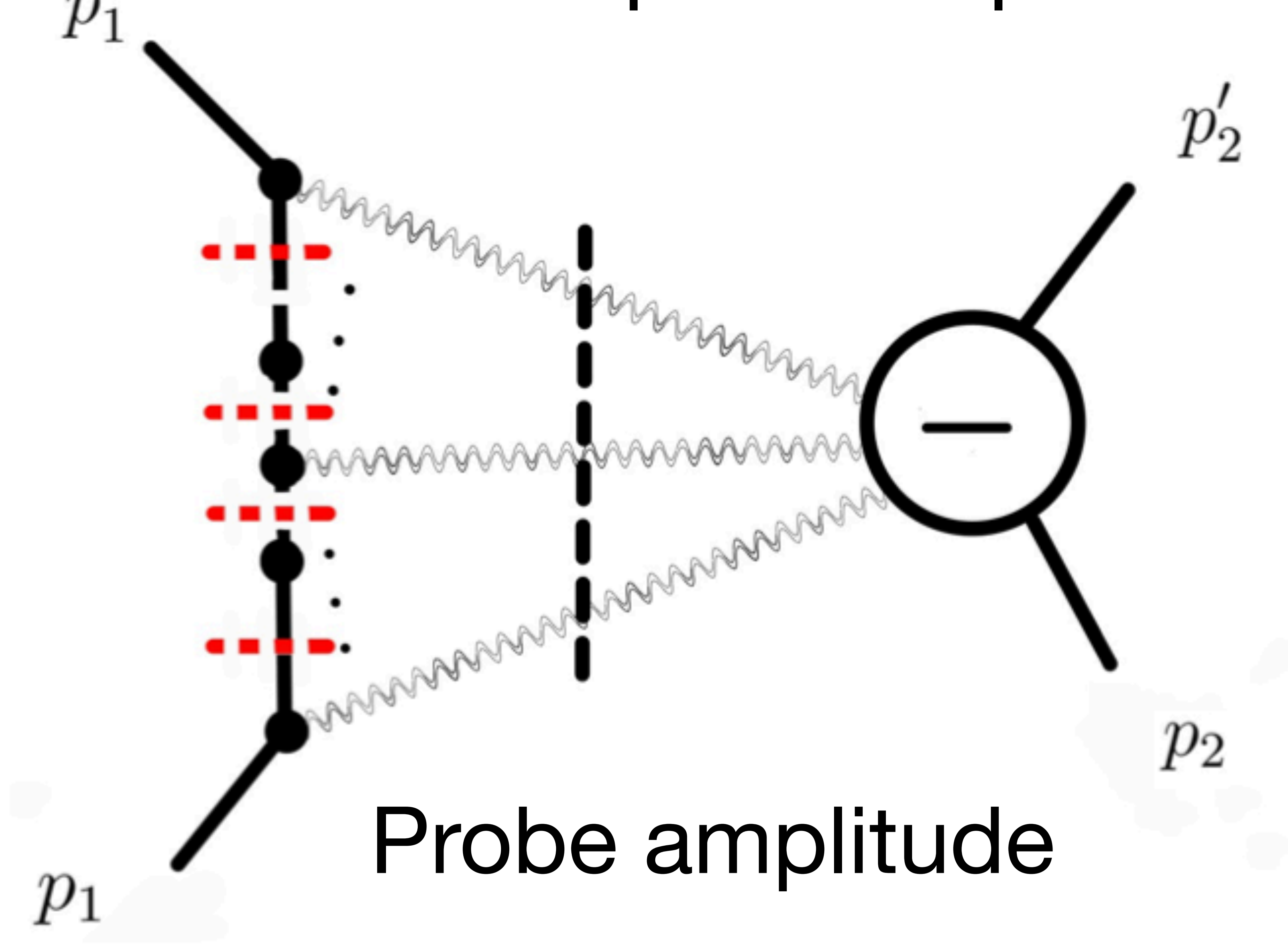

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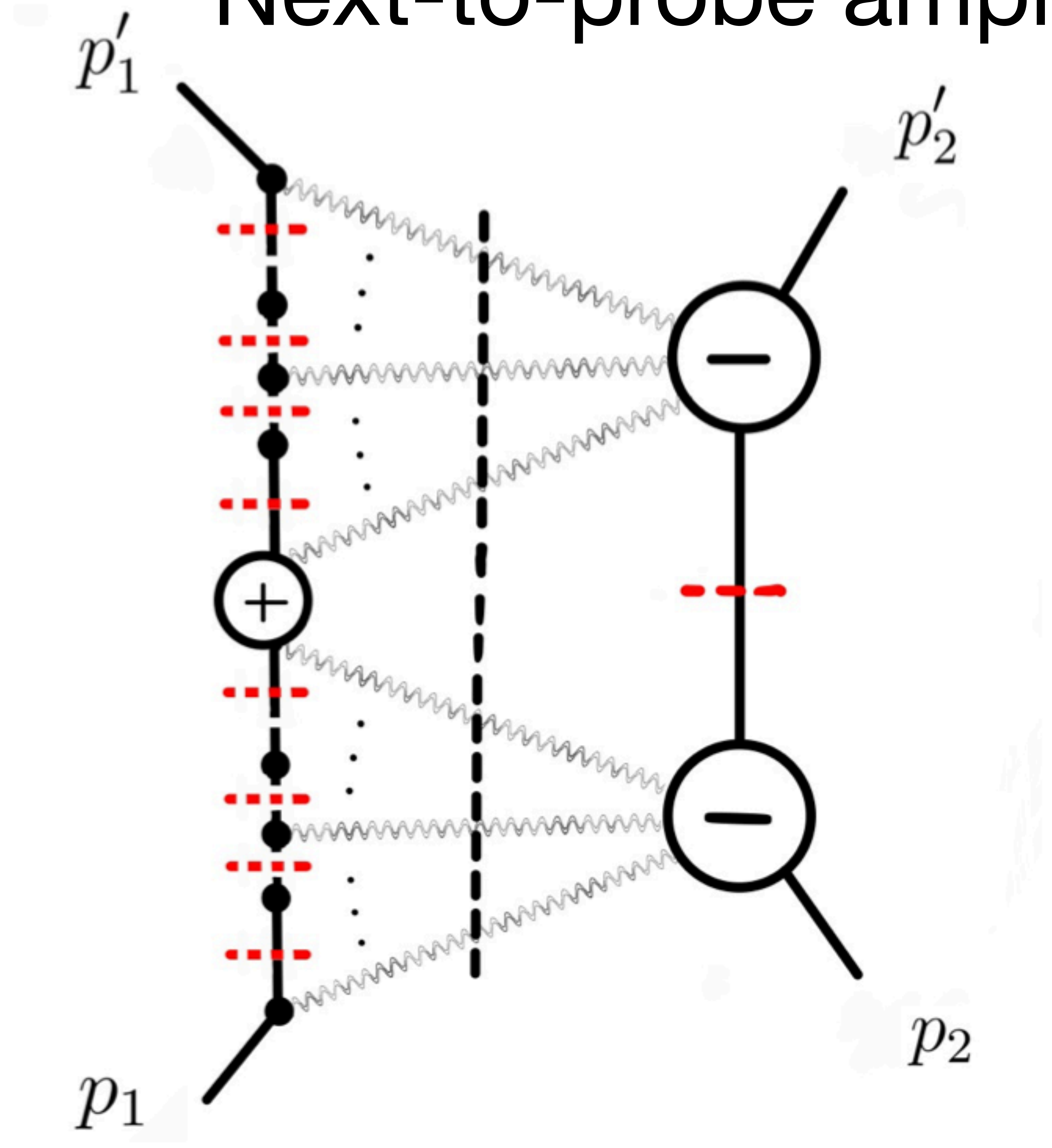

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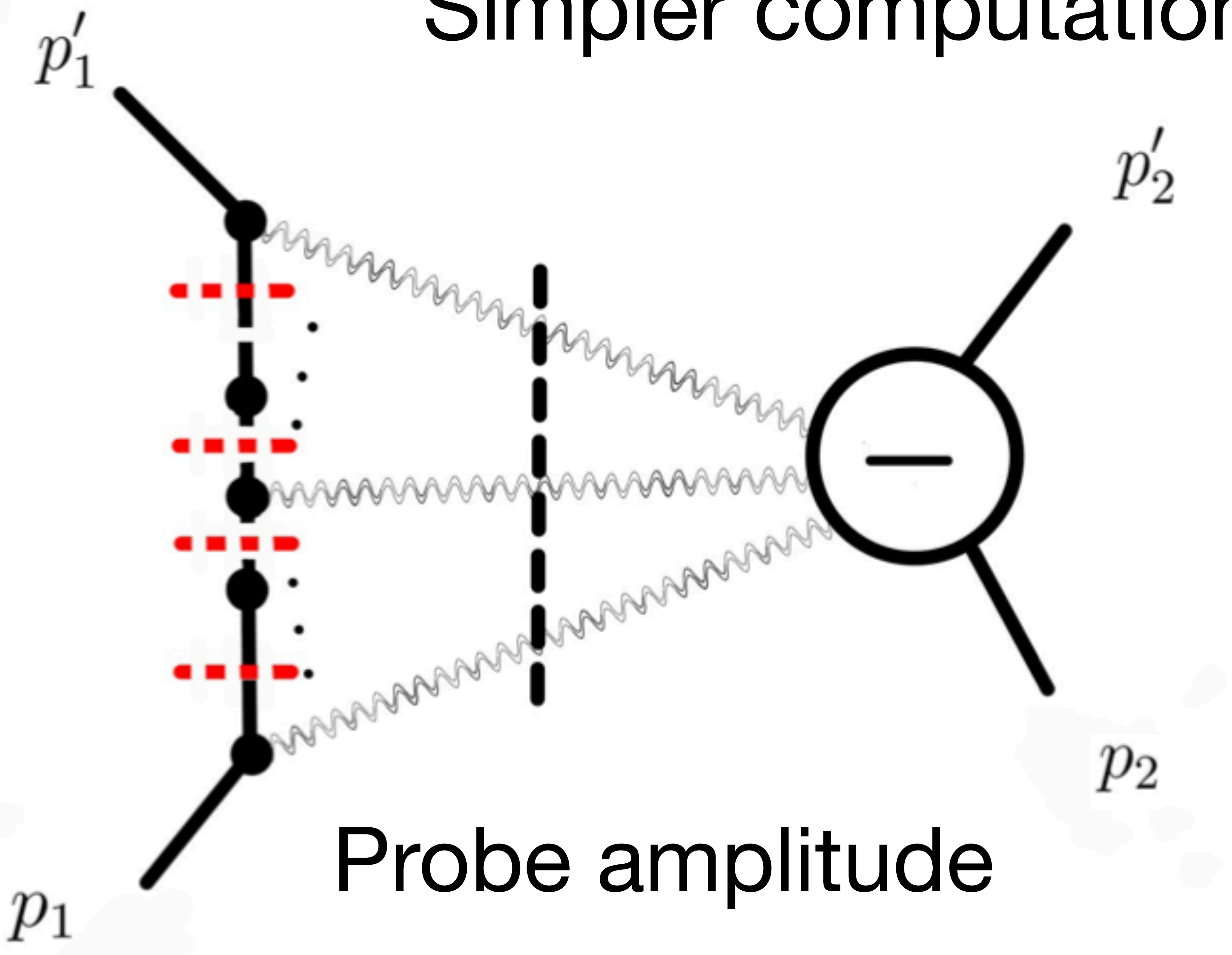

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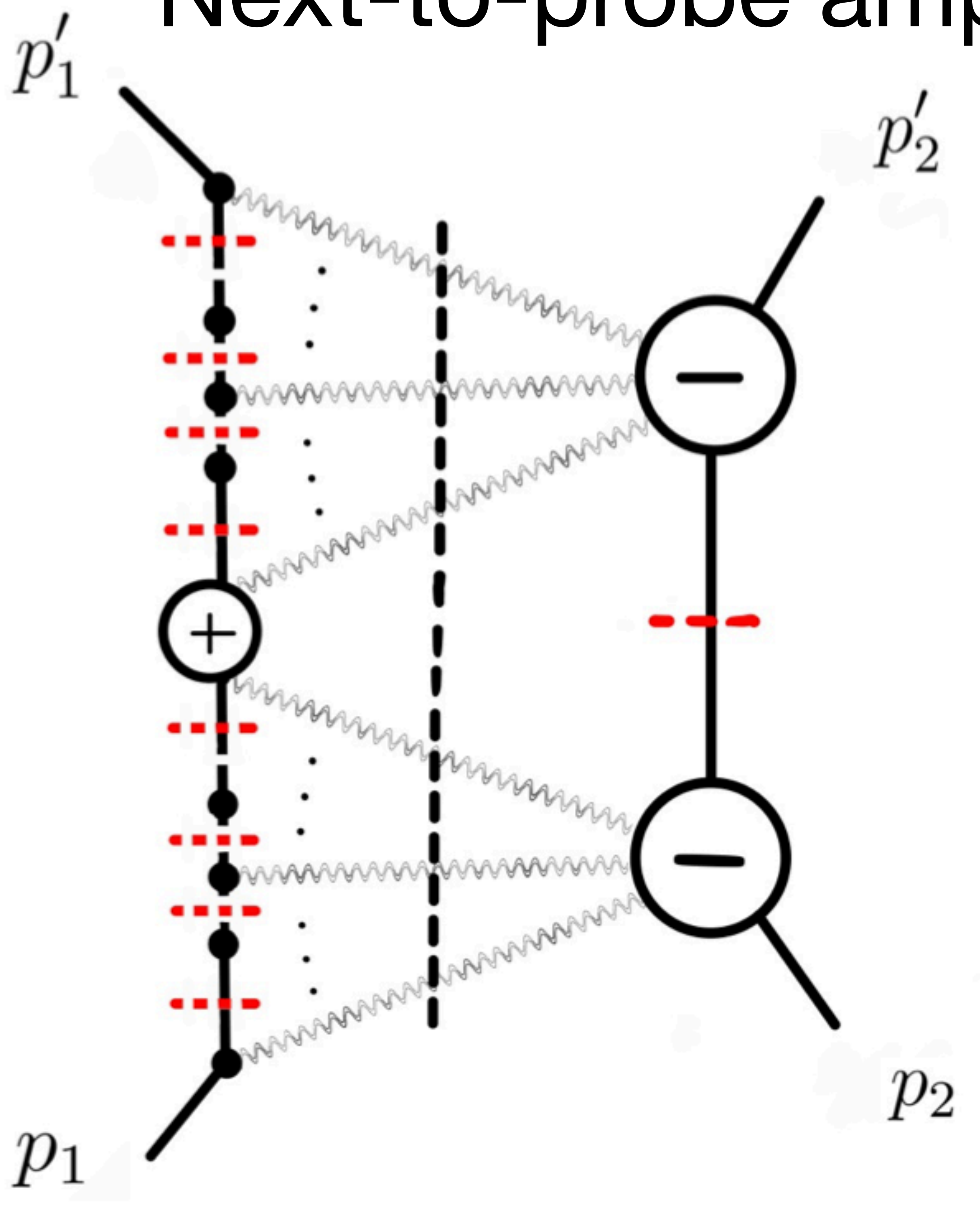

(Brandhuber, Chen, Travaglini, Wen)

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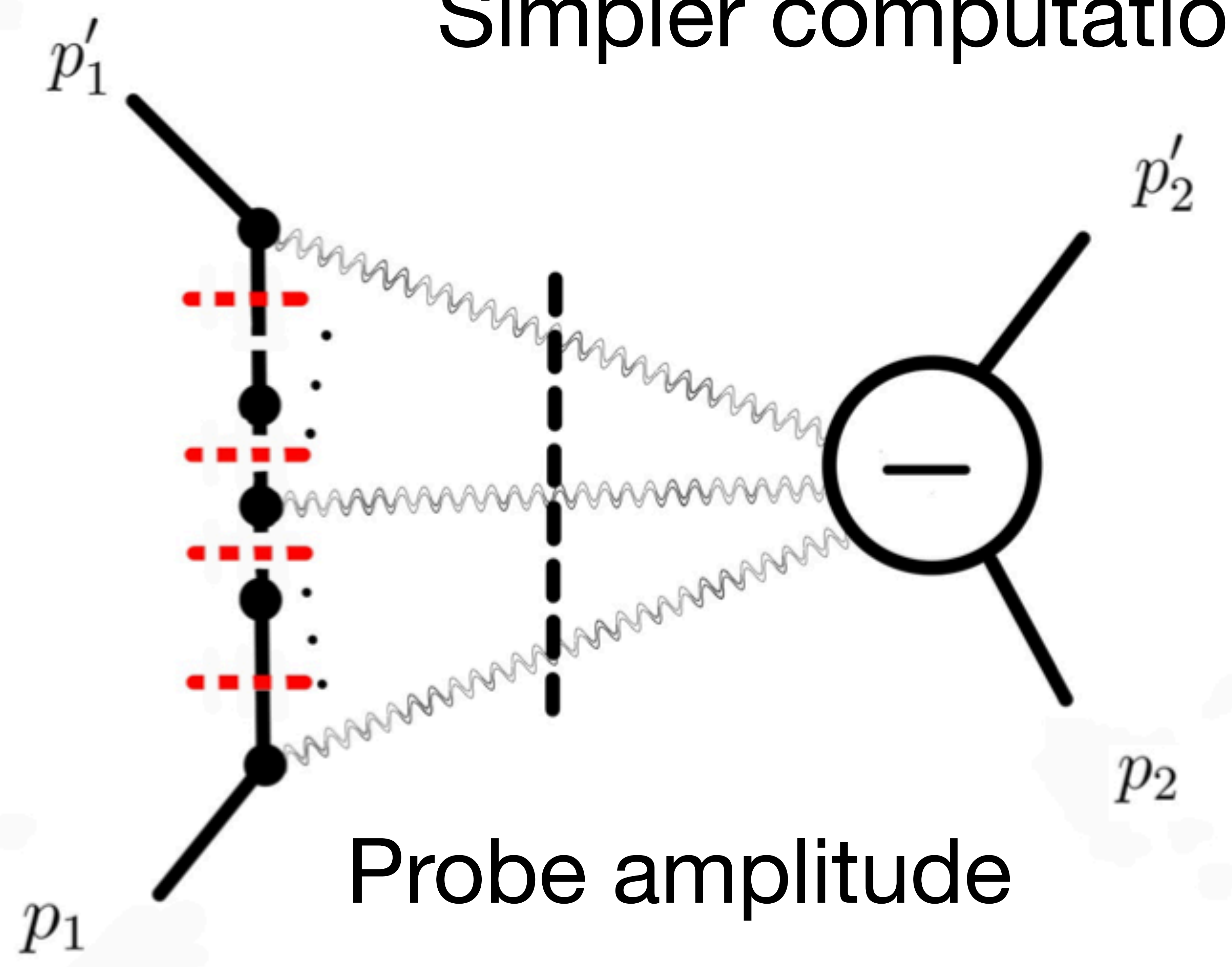

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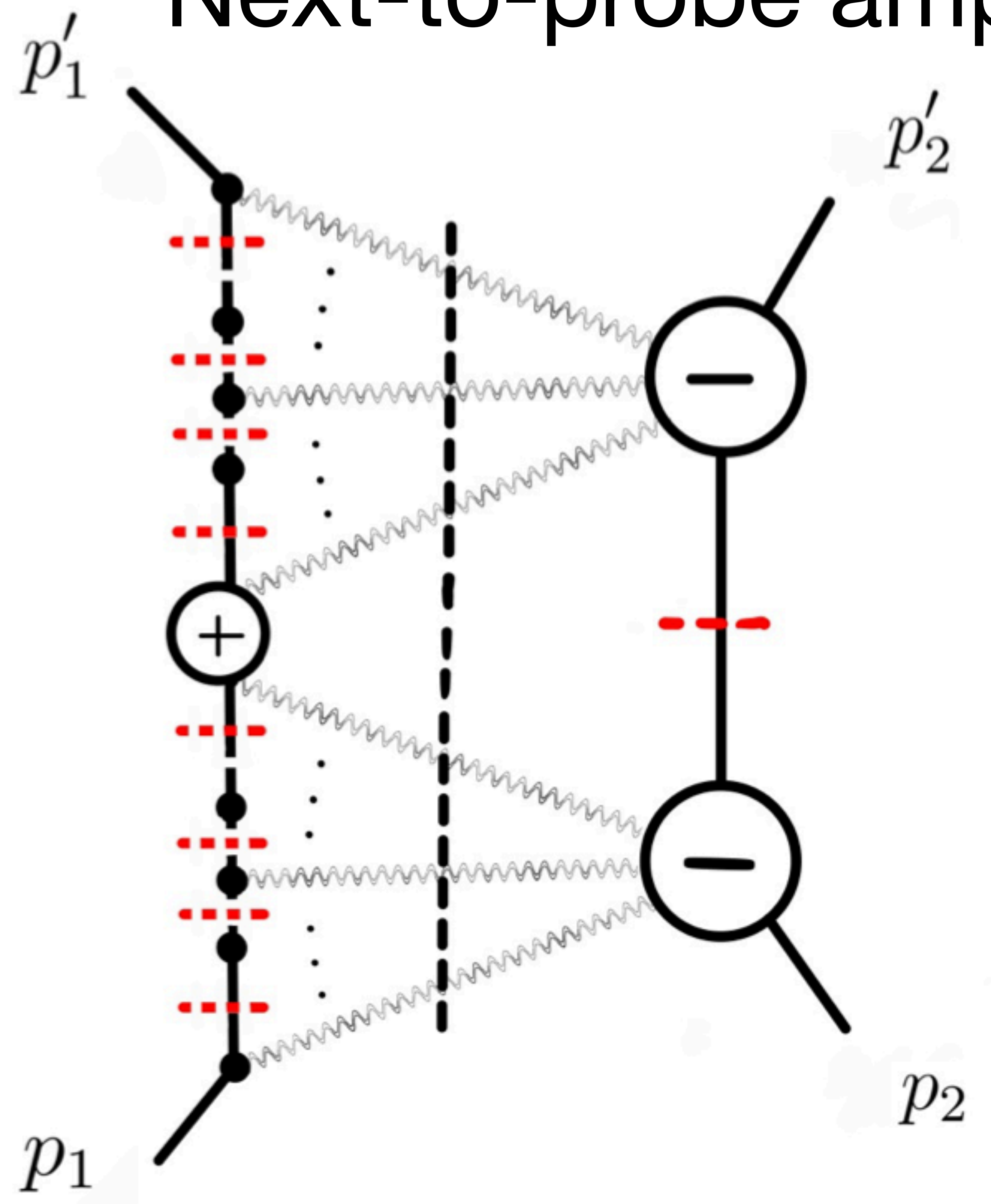

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- some similarities / some differences

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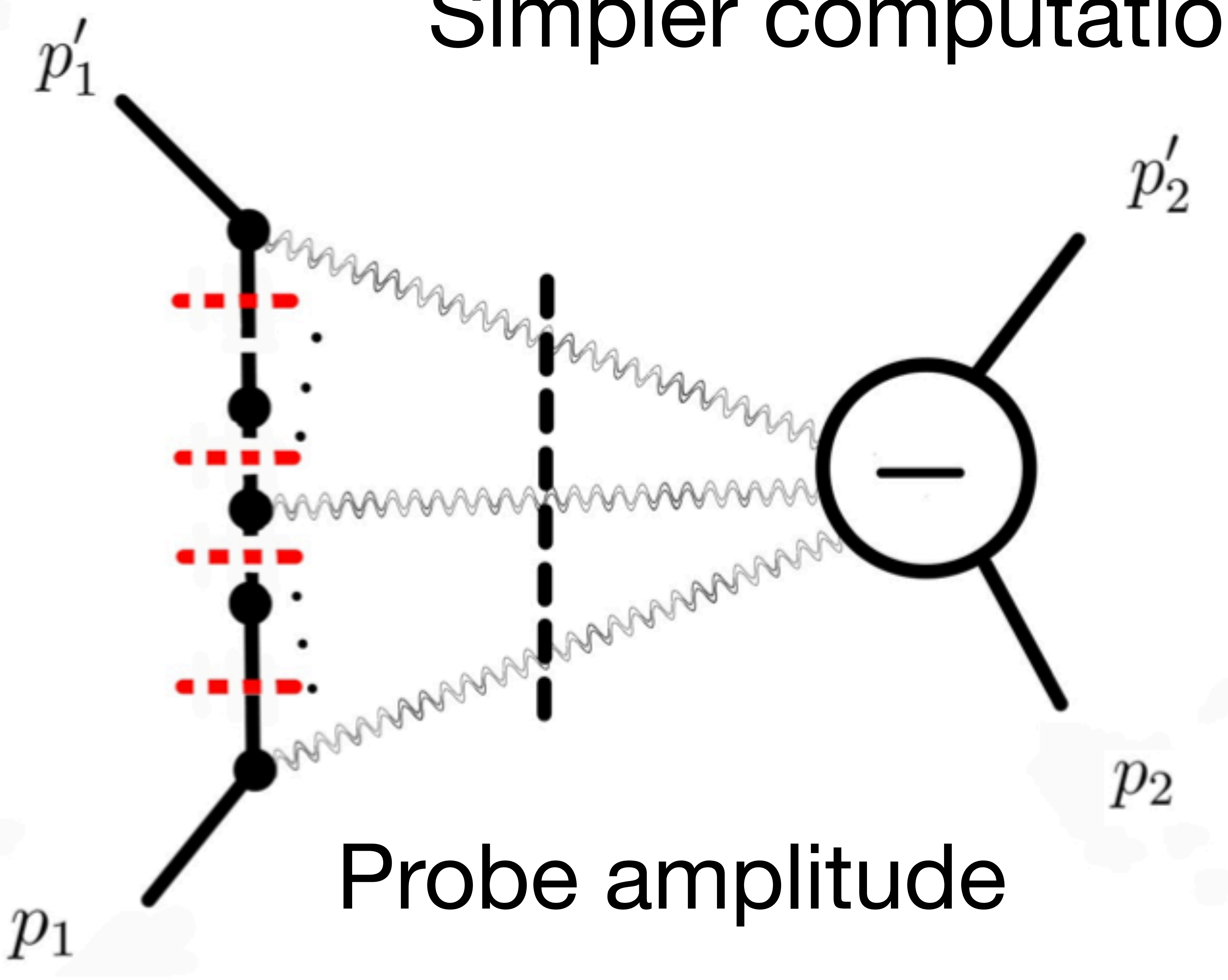

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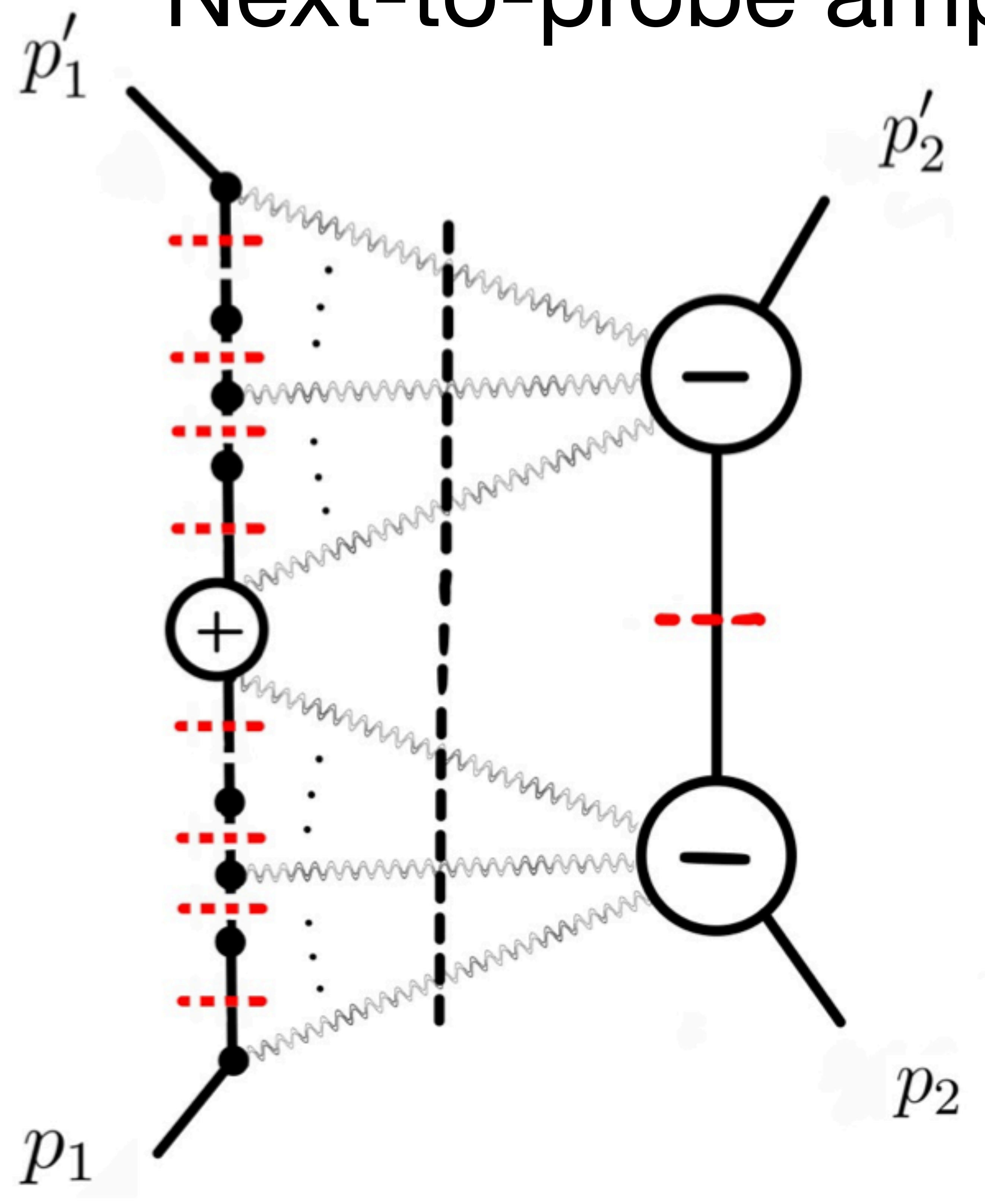

Interesting stuff to investigate

(Brandhuber, Chen, Travaglini, Wen)

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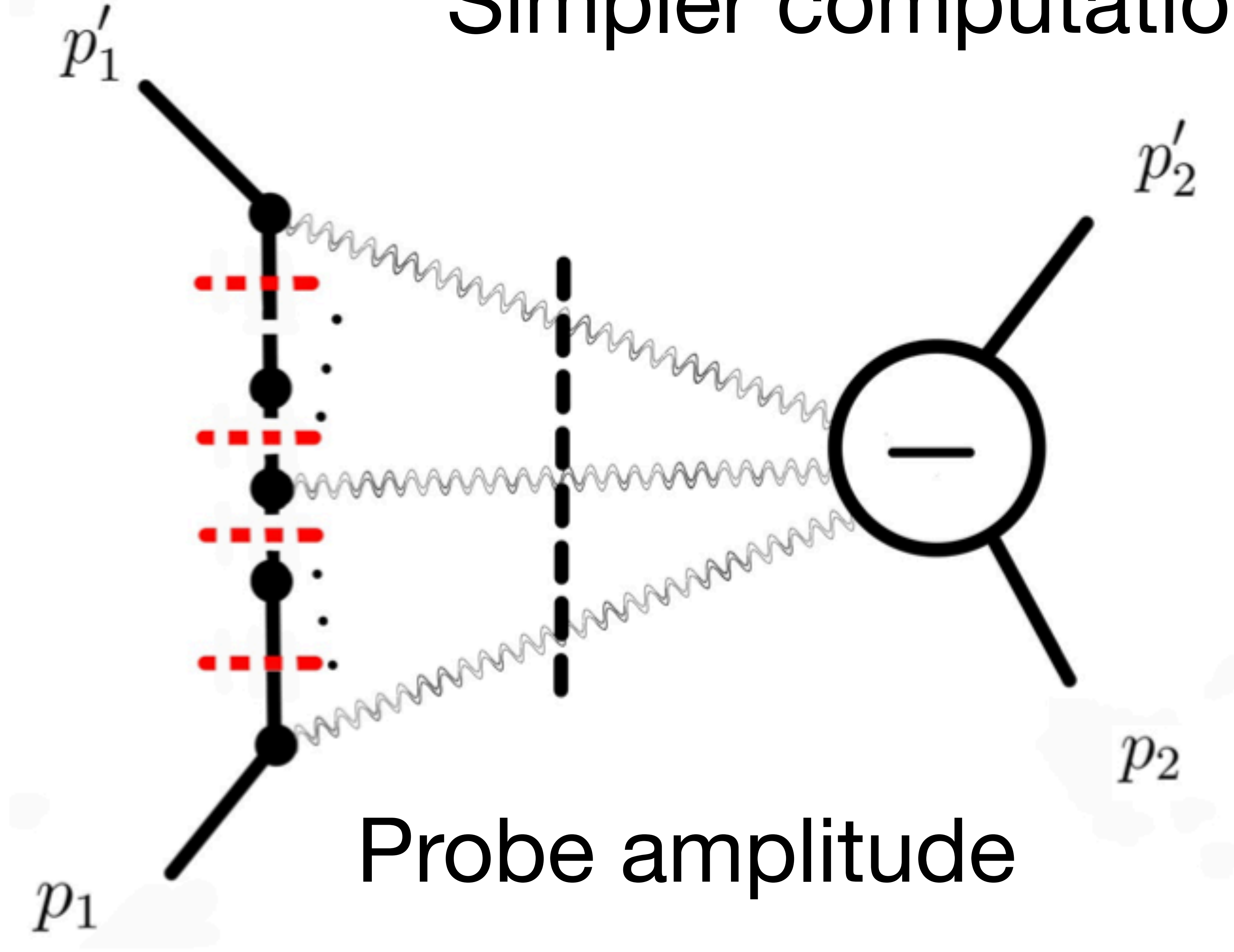

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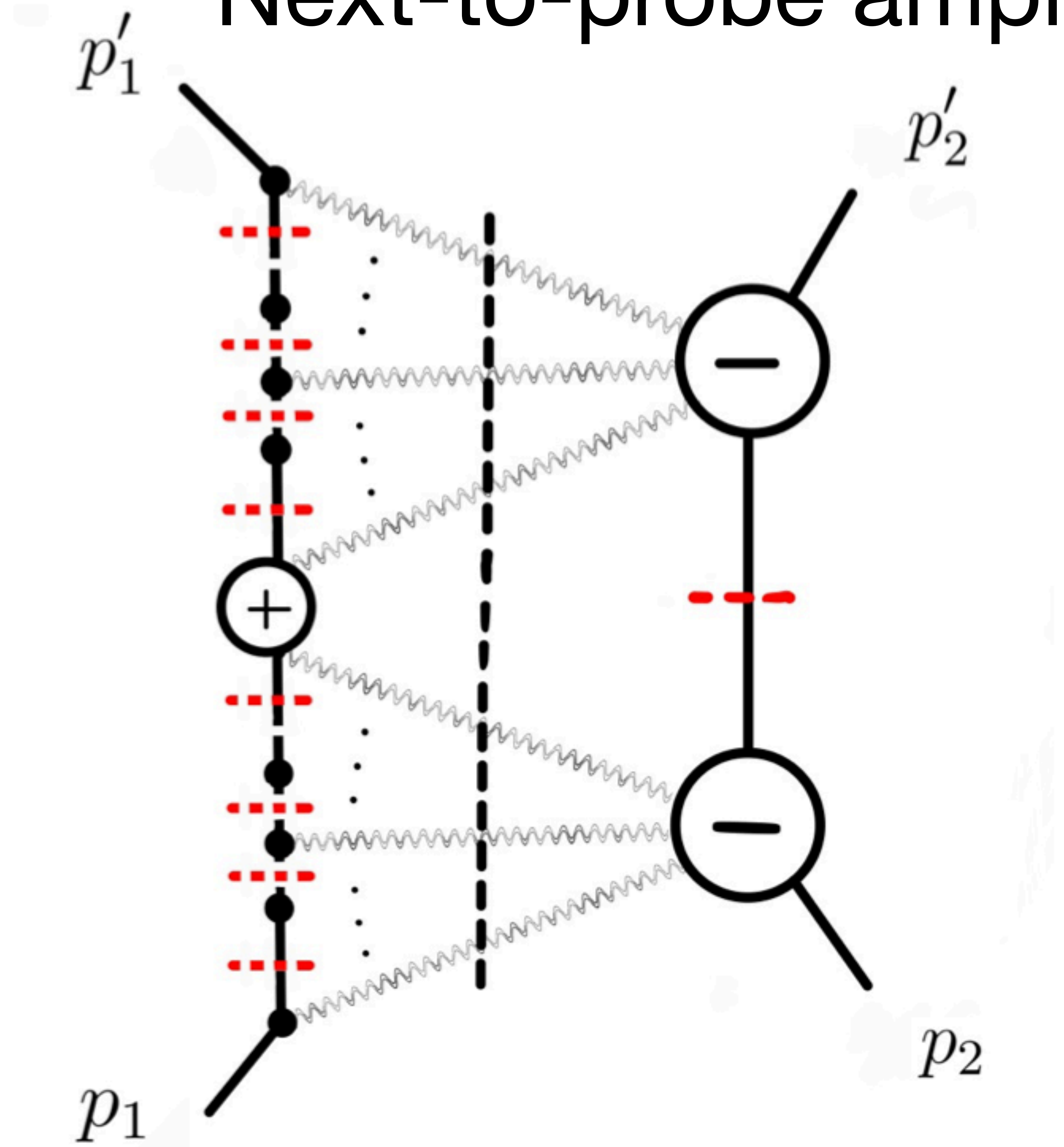


Heavy-quark—EFT inspiration:

(Damgaard, Haddad, Helset)

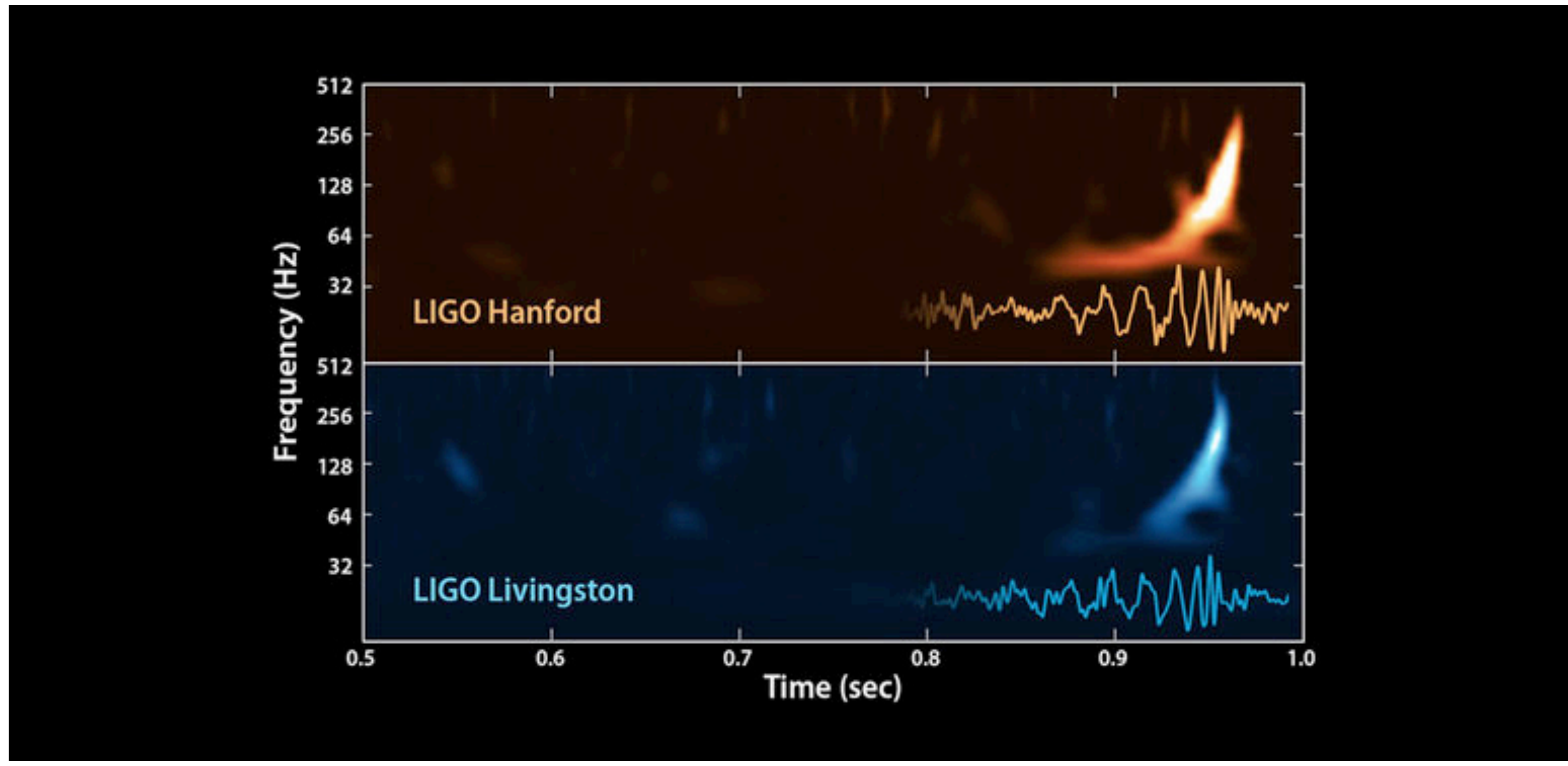
(Brandhuber, Chen, Travaglini, Wen)

- heavy mass vs small |q| expansion?
- some similarities / some differences

Interesting stuff to investigate

Conclusion

Conclusion


Frequency (Hz)
512
256
128
64
32
LIGO Hanford
512
256
128
64
32
LIGO Livingston
0.5
0.6
0.7
0.8
0.9
1.0
Time (sec)

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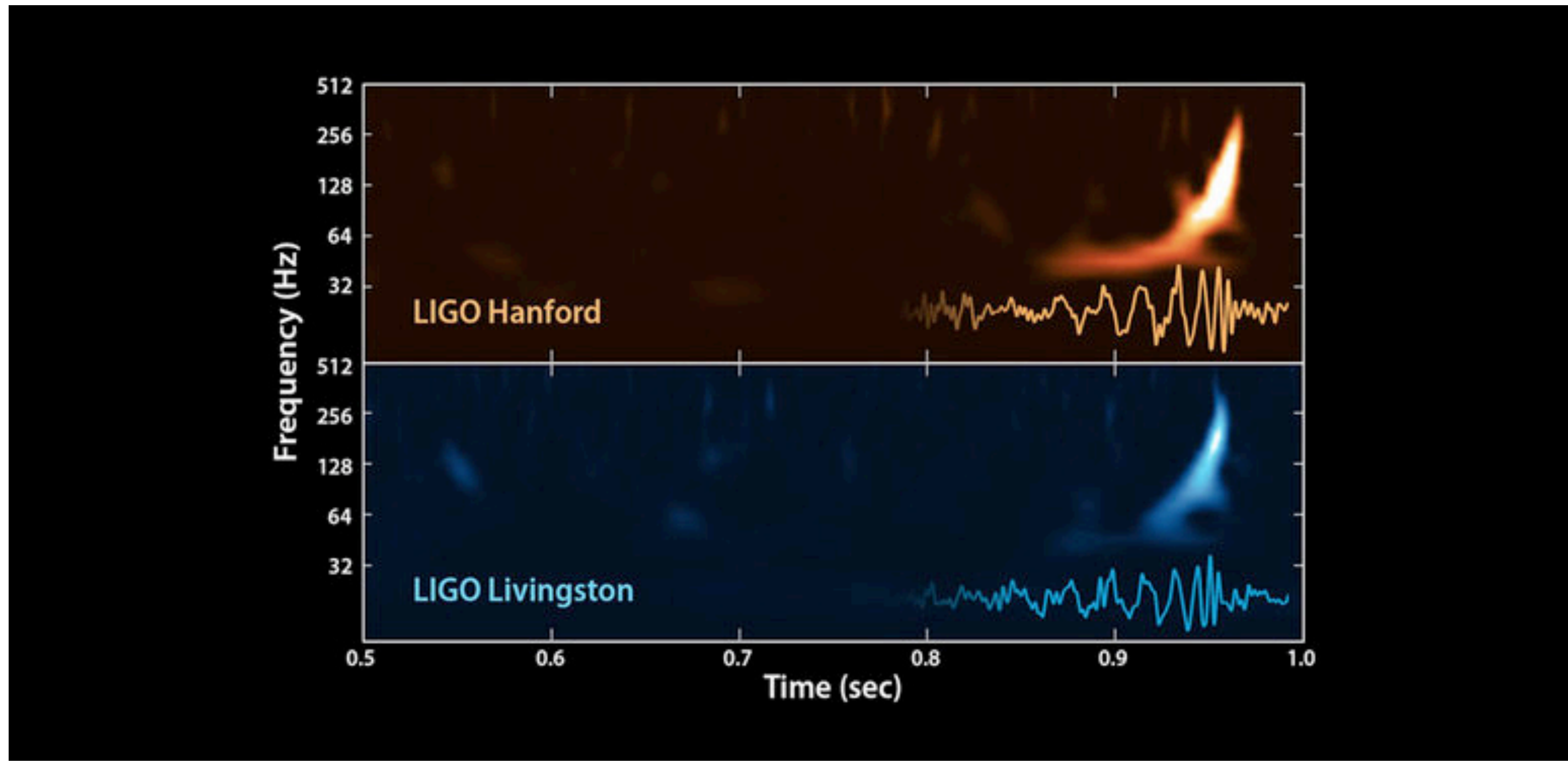

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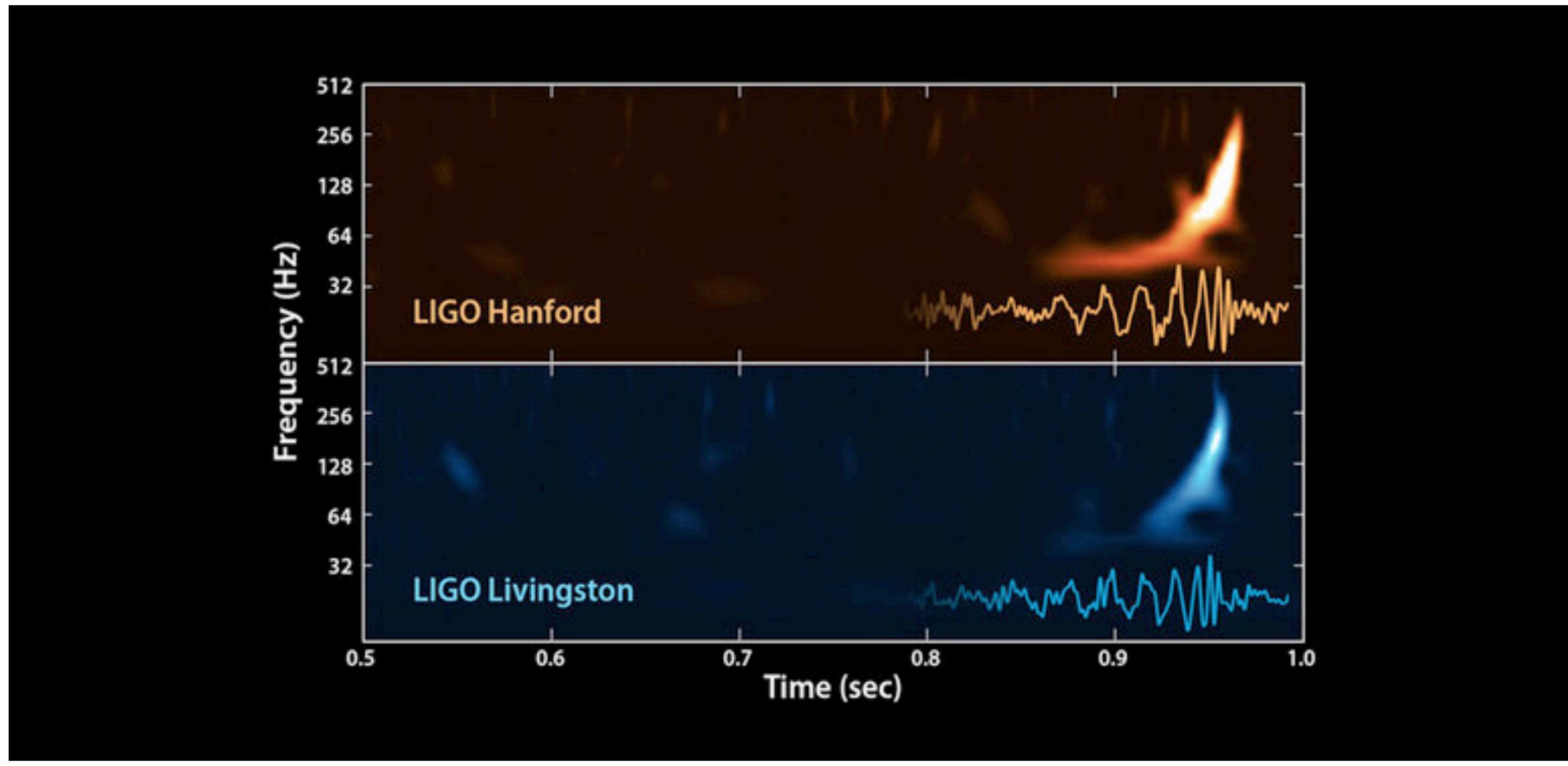

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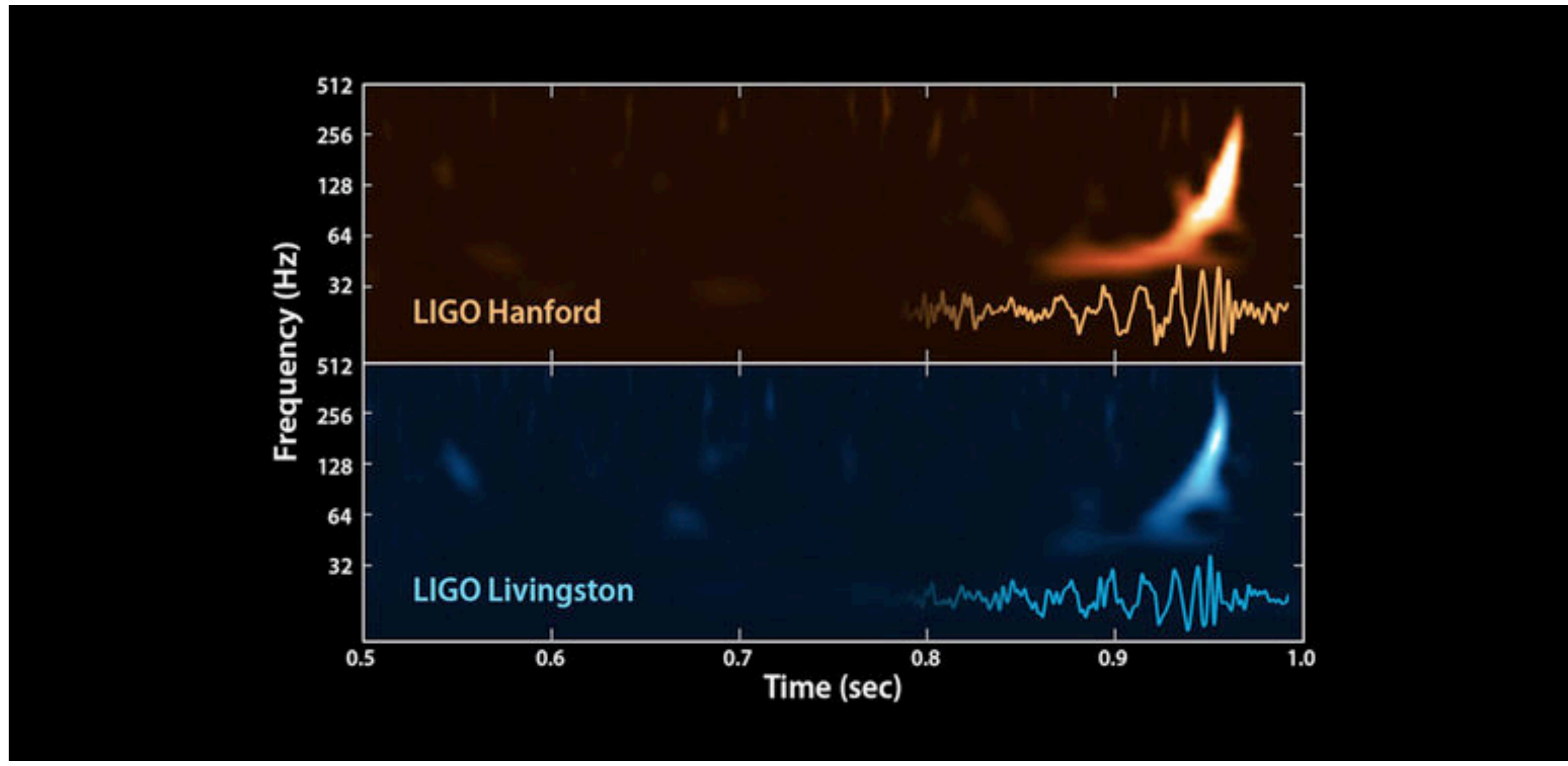

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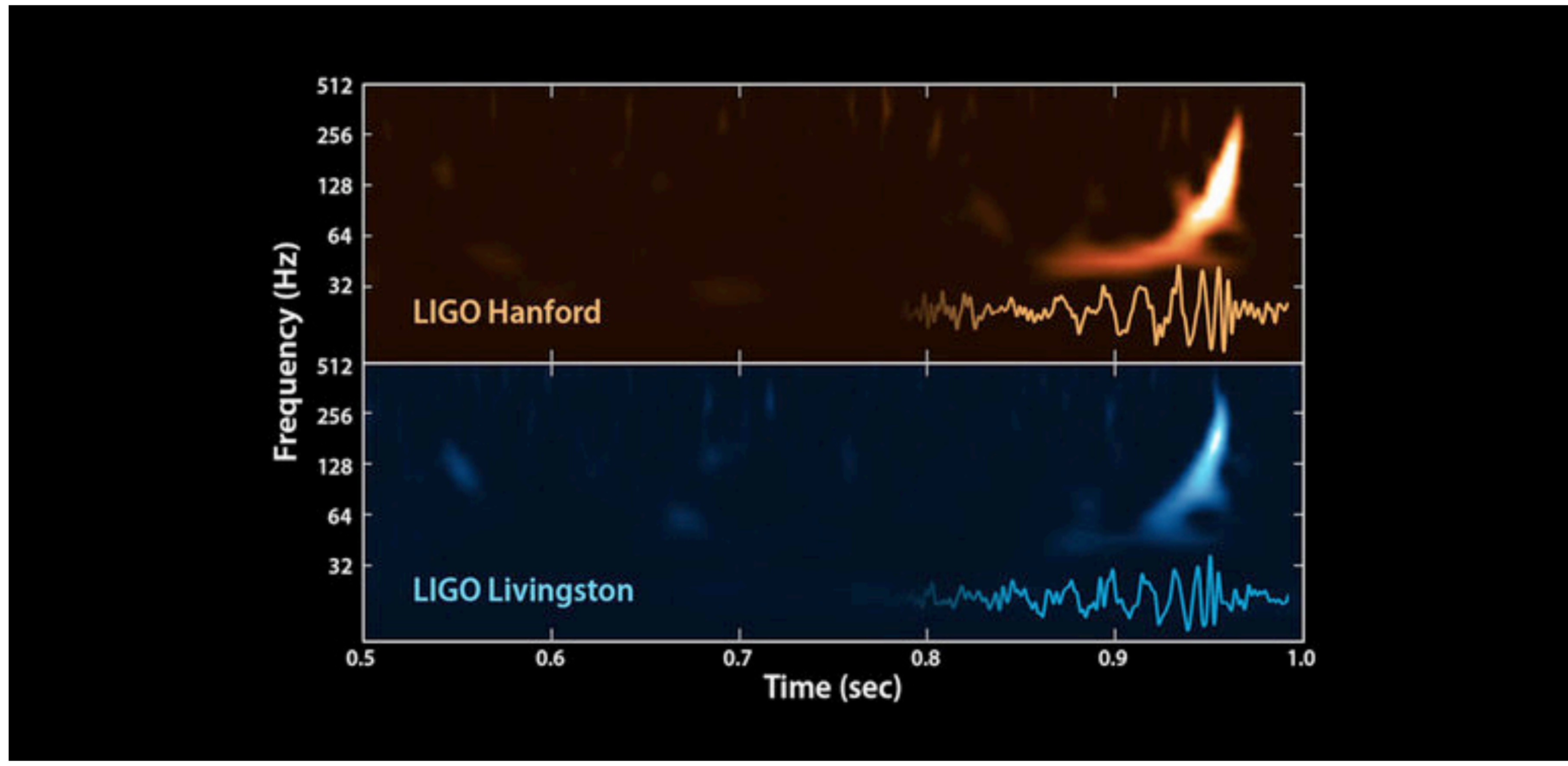

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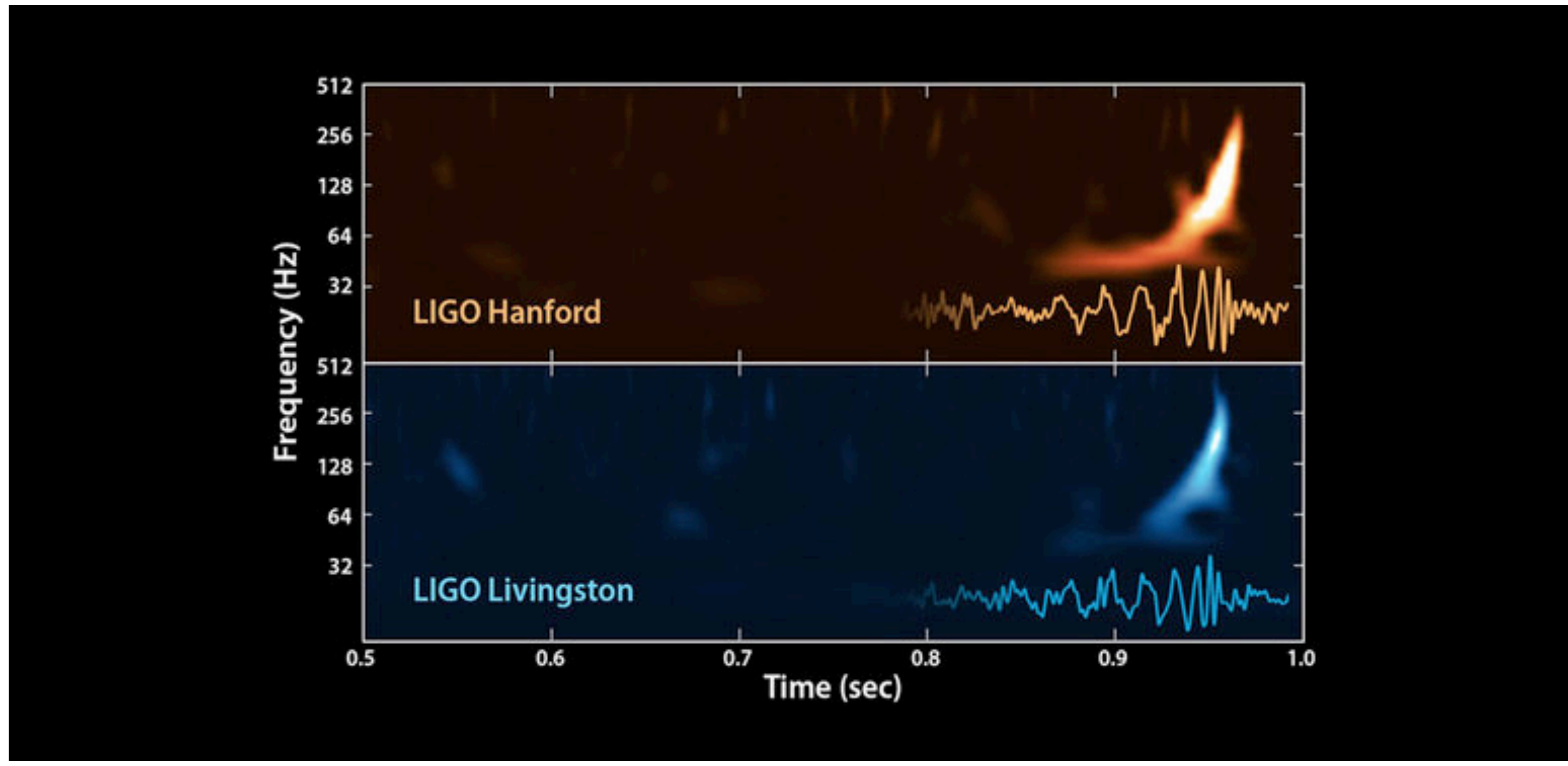

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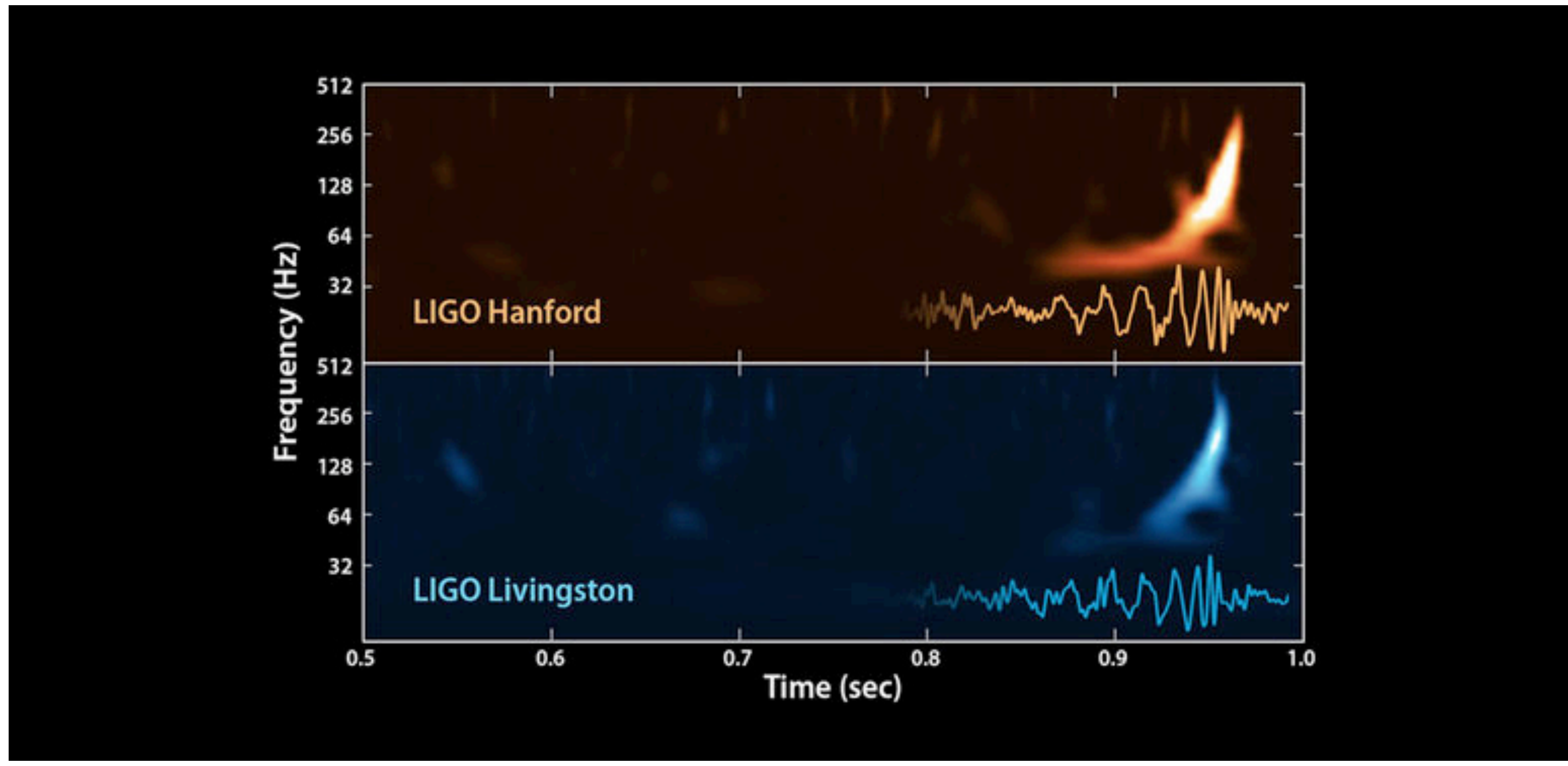
  - Better understanding of what the **minimal** computation is could lead to much simplified analysis.



Outlook

Amplitude toolbox for computations already provided many new efficient methods for computation

- Amplitude tools very useful for computations
  - Double-copy/KLT
  - Unitarity
  - Spinor-helicity
  - CHY formalism
  - Low energy limits of string theory

- Identifying IBP-relations solving DE equations/ integral
- Recycling tools from QCD computations
- Numerical programs for amplitude computation

Conclusion

Endless tasks ahead

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THANKS!