Graviton scattering on curved self-dual backgrounds via twistors/integrability

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Work with: Tim Adamo & Atul Sharma 2003.13501, 2103.16984, 2110.06066, 2203.02238 . . .

Compact gravity scattering formulae on SD backgrounds using integrability from twistors at $\mathscr{I}$.

Amplitudes on nontrivial backgrounds

Promise: Encode fully nonlinear effects, exact to all-orders!

Challenges:

- Construct momentum eigenstate analogues.
- Construct exact propagators on background.
- Perform space-time perturbation theory.

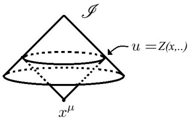
What has been done?

- 3-4 points on generic plane waves general: [2018 Adamo, Casali, M.,Nekovar], BMN: [Constable-et. al., Spradlin-Volovich].
- AdS/dS correlators: 5 pt [Goncalves, Perreira, Zhou], n-pt MUV [Green-Wen].
- That's it?

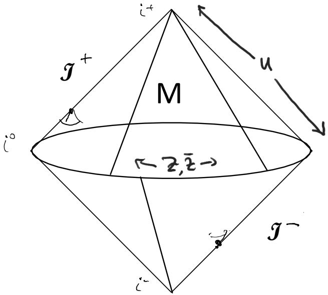
Here: give all multiplicity formulae on generic SD backgrounds.

Twistors at null infinity, and integrability

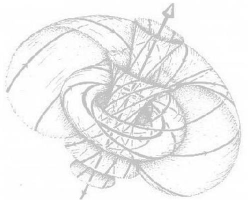
- Newman's good cuts attempt to rebuild space-time from $\mathscr{I}$ data.



- Yields instead $\mathcal{H}$-*space*: a complex *self-dual* space-time.



- We use Penrose's asymptotic twistor space at $\mathscr{I}$, reformulating Newman's good cuts.
- Penrose's construction embodies integrability of self-dual sector.
- Use amplitudes to give perturbations of $\mathcal{H}$-space approximating real space-time.



Gravity amplitudes at MHV: $- - + \ldots +$ helicity.

Scatter n gravitons with momenta k_i, $i = 1, \ldots n$.

- In 2-component spinors, null momenta $k_{i\alpha\dot{\alpha}} = \kappa_{i\alpha}\kappa_{i\dot{\alpha}}$.
- Spinor helicity: $\langle 1\,2\rangle := \kappa_{1\alpha}\kappa_2^{\alpha}\,,\ [1\,2] := \kappa_{1\dot{\alpha}}\kappa_2^{\dot{\alpha}}\,,$
- Hodges 2012 MHV formula, defines $n \times n$ matrix:

$$\mathbb{H}_{ij} = \begin{cases} \frac{[ij]}{\langle ij\rangle} & i \neq j \\ -\sum_k \frac{[ik]}{\langle ik\rangle} & i = j\,. \end{cases}$$

- Then: $\mathcal{M}(1, \ldots, n) = \langle 12\rangle^6 \det{}'\mathbb{H}\ \delta^4(\sum_i k_i)$.
- $\mathbb{H}$ is Laplace matrix & matrix-tree theorem $\rightsquigarrow_{\text{[Feng,He 2012]}}$
- Sum of tree diagrams with propagators $\frac{[ij]}{\langle ij\rangle}$ [Bern, et. al. '98]

For what theory??? $\mathcal{M} =$ Tree correlator $\langle V_1 \ldots V_{n-2}\rangle$.

MHV formula on self-dual background (schematic)

Can we exploit integrability of SD background?

- The $\kappa_{i\alpha}$ survive as contants.
- We can 'dress' the $\kappa_{i\dot\alpha}$ and send $[ij] \to [\![ij]\!]$ in formulæ.
- Can define (x-dependent) $\mathbb{H}$

But:

- there are, say, t interactions with background, $t < n - 2$,
- and fields generate tails after hitting background...

Nevertheless, define $(n+t) \times (n+t)$ generating matrix for t interactions with background

$$\mathcal{H} := \begin{pmatrix} \mathbb{H} & \mathfrak{h} \\ \mathfrak{h}^T & \mathbb{T} \end{pmatrix}$$

Gives contribution $\int_M d^4x \ \prod_{m=1}^{t} \partial_{\epsilon_m}^{p_m} \det{}' \mathcal{H}|_{\epsilon_m=0} \times \ldots$

Outline

1. Generating functional for the gravity MHV amplitude from the Plebanski scalar for SD background.
2. Lift to asymptotic twistor space at $\mathscr{I}$.
3. Twistor sigma model for Plebanski scalar and tree formulae.
4. Computation on background.
5. (Extension to full gravity tree S-matrix & on background.)
6. ($Lw_{1+\infty}$-symmetry.)

Plebanski scalar as MHV generating function

MHV generating function: SD metric g^+

$$\mathcal{M}(1^-, 2^-, g^+) := \frac{\delta^2 S_{EH}[g]}{\delta g \delta g}(h_1^-, h_2^-)\Big|_{g=g^+}$$

where h_1^-, h_2^- are ASD linear gravitons on SD background g^+.

- Take $h_i^-, i = 1, 2$ be plane waves at $\mathscr{I}$, momenta $\kappa_i^\alpha \kappa_i^{\dot\alpha}$.
- κ_i^α defines coordinates $(x^{\dot\alpha}, \tilde{x}^{\dot\alpha}) := (x^{\alpha\dot\alpha}\kappa_{1\alpha}, x^{\alpha\dot\alpha}\kappa_{2\alpha})$,
- $g^+ \leftrightarrow$ *Plebanski scalar* (Kahler scalar)

$$g^+ = \frac{\partial\partial\Omega(x^{\dot\alpha}, \tilde{x}^{\dot\alpha})}{\partial x^{\dot\alpha} \partial \tilde{x}^{\dot\beta}} dx^{\dot\alpha} d\tilde{x}^{\dot\beta}, \qquad \det \partial\tilde{\partial}\Omega = 1.$$

Proposition (Adamo, M., Sharma, 2103.16984)

Generating function reduces to

$$\mathcal{M}(1^-, 2^-, g^+) = \langle 12 \rangle^6 \int_M d^4x \, \Omega \, \mathrm{e}^{[\tilde\kappa_1 x] + [\tilde\kappa_2 \tilde{x}]}$$

Asymptotic Twistor space

Penrose's nonlinear graviton at $\mathscr{I}$ with $\bar\partial$-operator deformed by $\mathscr{I}$ data.

Twistor space $\mathcal{T} = \mathbb{C}^4$ or projective $\mathbb{P}\mathcal{T}^3$, homogeneous coords:

$$W = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}, \qquad W \sim aW\,, a \neq 0\,.$$

Poisson bracket: $\{f, g\} = \varepsilon^{\dot\alpha\dot\beta} \frac{\partial f}{\partial \mu^{\dot\alpha}} \frac{\partial g}{\partial \mu^{\dot\beta}} = \left[\frac{\partial f}{\partial \mu} \frac{\partial g}{\partial \mu}\right]$.

- Gravity data on $\mathcal{T}$ is

$$\mathbf{h} = h(\mu^{\dot\alpha}\bar\lambda_{\dot\alpha}, \lambda, \bar\lambda)[\bar\lambda d\bar\lambda] \in \Omega^{0,1}(2)\,.$$

- From $\mathscr{I}$ data: $\qquad h = \int^u \boldsymbol{\sigma}^0 du\,, \quad \boldsymbol{\sigma}^0 =$ asymptotic shear.
- For SD black hole: $h = h([\mu|T|\lambda\rangle, \lambda, \bar\lambda)$ works.

$\bar\partial$-operator on $\mathcal{T}$ is deformed by 'Hamiltonian' $\mathbf{h}$ to

$$\bar\partial_h f := \bar\partial_0 f + \{\mathbf{h}, f\}\,.$$

The self-dual space-time from holomorphic curves

To reconstruct self-dual space-time

$$(M^4, g) = \{ \text{ Holomorphic degree-1 } \mathbb{CP}^1\text{s in } P\mathcal{T}\}$$

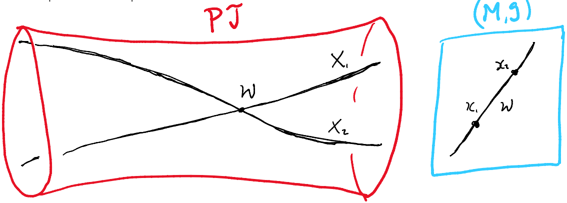
- Parametrize the $\mathbb{CP}^1_{x,\sigma} \subset \mathbb{P}\mathcal{T}$, with hgs coords (σ_0, σ_1) by

$$\lambda_\alpha = \left(\frac{1}{\sigma_0}, \frac{1}{\sigma_1}\right) = \frac{(1,z)}{\sigma_0}, \qquad \mu^{\dot\alpha} = \frac{x^{0\dot\alpha}}{\sigma_0} + \frac{x^{1\dot\alpha}}{\sigma_1} + M^{\dot\alpha},$$

- d-bar eq for $\mathbb{C}$-curves in deformed $\mathbb{P}\mathcal{T}$:

$$\bar\partial_\sigma \mu^{\dot\alpha} = \{\mu^{\dot\alpha}, \mathbf{h}\} = \varepsilon^{\dot\alpha\dot\beta}\frac{\partial \mathbf{h}}{\partial \mu^{\dot\beta}},$$

Correspondence with space-time



Sigma model action and MHV generating function

- For curve in $\mathcal{PT}$: $\mu^{\dot\alpha}(x,\sigma) := x^{0\dot\alpha}/\sigma_0 + x^{1\dot\alpha}/\sigma_1 + M^{\dot\alpha}(x,\sigma)$,
- Holomorphy follows from action

$$S[\mu^{\dot\alpha}, x] = \int D\sigma \left([M\bar\partial_\sigma M] + 2h(\lambda,\mu) \right) .$$

Key proposition: [Adamo, M., Sharma, 2103.16984]
Given small data h, for all $x \in M$, $\exists!\, \mu^{\dot\alpha}(x,\lambda)$ and then the on-shell action $S^{os}[x,h]$ yields:

1. The Kahler (Plebanski) scalar $\Omega(x^{0\dot\alpha}, x^{1\dot\alpha})$ for SD metric

$$g_+ = \frac{\partial^2\Omega}{\partial x^{0\dot\alpha}\partial x^{1\dot\beta}} dx^{0\dot\alpha} dx^{1\dot\beta}, \qquad \Omega(x) = S^{os}[x,h]$$

2. the generating function for MHV amplitudes

$$\mathcal{M}(1^-, 2^-, h^+) = \langle 1\, 2\rangle^6 \int_M d^4x\, \mathrm{e}^{(k_1+k_2)\cdot x}\, S^{os}[x,h^+] .$$

MHV generating function, trees and Hodge formula

Starting from the MHV generating function

$$\mathcal{M}(1^-,2^-,h) = \langle 1\,2\rangle^6 \int_M d^4x\, \mathrm{e}^{(k_1+k_2)\cdot x}\, S^{os}[x,h]$$

- perturbatively expand in h in momentum eigenstates

$$h = \sum_{i=3}^{n} h_i\,, \qquad h_i = \int \frac{ds}{s^3}\bar{\delta}^2(s\lambda_\alpha - \kappa_{i\alpha})\,\mathrm{e}^{is[\mu\,\tilde{\kappa}_i]}\,.$$

- On-shell action has tree expansion (ignoring $O(h_i^2)$)

$$S_{\mathbb{PT}}[M,h] = \langle V_{h_3}\ldots V_{h_n}\rangle_{\mathrm{tree}}$$

with vertex operators $V_h = \int_{\mathbb{CP}^1} hD\sigma$ and propagators

$$\mathbb{H}_{ij} := \frac{[\partial_\mu h_i\,\partial_\mu h_j]}{\langle i\,j\rangle} = \frac{[i\,j]}{\langle i\,j\rangle}h_i h_j\,, \qquad i \neq j$$

- Yields tree-diagram formalism of Bern et. al. 1998.

Matrix-tree theorem gives $\langle V_{h_3}\ldots V_{h_n}\rangle_{\mathrm{tree}} = \det{}'\mathbb{H}$

$\rightsquigarrow$ Hodges reduced determinant formula, [cf Feng-He'12].

Sigma model at higher MHV degree

For N^{k-2}MHV need k ASD particles:

- ASD wave functions as momentum eigenstates

$$\tilde{h}_r(W_r) = \int s^5 ds\, \bar{\delta}^2(s\lambda - \kappa) e^{is[\mu\kappa]} \in H^1(\mathcal{O}(-6))\,.$$

- Insert ASD particles at $W_r \in \mathbb{T}$ and $\sigma_r \in \mathbb{CP}^1$, $r = 1, \ldots, k$:

$$W(\sigma) = \sum_{r=1}^{k} \frac{W_r}{\sigma - \sigma_r} + (0, M^{\dot{\alpha}}) : \mathbb{CP}^1 \to \mathbb{PT}\,.$$

- There exists unique $W(\sigma)$ with M of weight $(-1, 0)$.
- Action is now simply

$$S[W(\sigma), W_r, \sigma_r, h] = \int_{\mathbb{CP}^1} d\sigma \left([M\, \bar{\partial} M] + 2h \right)$$

Propn: On-shell action $S^{os}[W_r, \sigma_r, h]$ generates N^{k-2}MHV tree-amplitudes

The full gravity S-matrix: N^{k-2}MHV amplitudes

The formula for k ASD particles on SD background h is:

$$\mathcal{M}(1^-,\ldots,k^-,h)=\int_{(\mathbb{CP}^1\times\mathbb{PT})^k} S^{os}[W_r,\sigma_r,h]\,\mathrm{det}'\tilde{\mathbb{H}}\prod_{r=1}^{k}\tilde{h}_r D^3W_r d\sigma_r\,.$$

here we have inserted $\mathrm{det}'\tilde{\mathbb{H}}$, for the 'conjugate' Hodge matrix

$$\tilde{\mathbb{H}}_{ij}=\begin{cases}\frac{\langle\lambda_r\lambda_s\rangle}{\sigma_r-\sigma_s} & r\neq s\\ -\sum_q\frac{\langle\lambda_r\lambda_q\rangle}{\sigma_r-\sigma_q}\,, & r=s\,.\end{cases}$$

Expanding $h=\sum_{i=k+1}^n h_i$ as before gives

$$\mathcal{M}=\int_{(\mathbb{CP}^1\times\mathbb{PT})^k}\langle h_{k+1}\ldots h_n\rangle_{\mathrm{tree}}\,\mathrm{det}'\tilde{\mathbb{H}}\prod_{r=1}^{k}\tilde{h}_r D^3W_r d\sigma_r\,,$$

$$=\int_{(\mathbb{CP}^1)^n\times\mathbb{PT}^k}\mathrm{det}'\mathbb{H}\,\mathrm{det}'\tilde{\mathbb{H}}\prod_{r=1}^{k}\tilde{h}_r D^3W_r d\sigma_r$$

proof: reduce to Cachazo-Skinner formula. [Adamo, M, Sharma 2103.16984]

Perturbations on SD background

Now expand $h = h_0 + \sum_i \epsilon_i h_i$ around background h_0 in

- $$\mathcal{M}(1,2,h) = \langle 1\,2\rangle^4 \int_M d^4x\, \mathrm{e}^{(k_1+k_2)\cdot x}\, S^{os}[x,h]$$
- Now perturb $M^{\dot\alpha} = M_0^{\dot\alpha}(x,\sigma) + m^{\dot\alpha}(x,\sigma)$ with $\bar\partial_{\bar\sigma} M_0^{\dot\alpha} = \frac{\partial h_0}{\partial \mu^{\dot\alpha}}$

$$S[M,x] = S[M_0] + S_2[m] + \sum_{p\geq 3} U_p\,.$$

- where the quadratic perturbation is

$$S_2[m] := \int D\sigma \left([m\bar\partial_\sigma m] + \left.\frac{\partial^2 h_0}{\partial\mu^{\dot\alpha}\partial\mu^{\dot\beta}}\right|_{M=M_0} m^{\dot\alpha} m^{\dot\beta}\right)$$

- and the p-vertices U_p are

$$U_p := \int D\sigma \left.\frac{\partial^p h_0}{\partial\mu^{\dot\alpha_1}\dots\partial\mu^{\dot\alpha_p}}\right|_{M=M_0} m^{\dot\alpha_1}\dots m^{\dot\alpha_p} = \int D\sigma\,[\bar\lambda\, m]^p\, \partial_u^{p-1}\boldsymbol{\sigma}^0\,.$$

Feynman diagrams on the sphere

We want correlator of vertex operators $V_{h_i} = \int_{\mathbb{CP}^1} h_i D\sigma$

$$h_i = \int \frac{ds}{s^3} \bar{\delta}^2(s\lambda_\alpha - \kappa_{i\alpha})\, \mathrm{e}^{is[M\,\tilde{\kappa}_i]}$$

- The quadratic term dresses the $\mathbb{CP}^1$ propagator

$$\langle m_1^{\dot\alpha}(x,\sigma_1) m_2^{\dot\beta}(x,\sigma_2)\rangle = \frac{H(x,\sigma_1)^{\dot\alpha}_{\dot\gamma} H(x,\sigma_2)^{\dot\beta\dot\gamma}}{\sigma_1 - \sigma_2}\,, \qquad H^{\dot\alpha}_{\dot\beta} := \frac{\partial M_0^{\dot\alpha}}{\partial x^{\dot\beta}}\,.$$

- This leads to *dressed* square brackets, i.e.:

$$\langle V_{h_i} V_{h_j}\rangle = \frac{[\![ij]\!]}{\langle ij\rangle} V_{h_i} V_{h_j}\,, \qquad [\![ij]\!] := \kappa_{i\dot\alpha} H^{\dot\alpha}_{i\dot\gamma} H_j^{\dot\beta\dot\gamma} \kappa_{j\dot\beta}\,.$$

- Diagrams can include t vertices $U_{p_1}, \ldots, U_{p_t}$.
- For connected trees $t \leq n-4$, in fact $\sum(p_r - 2) \leq n-4$.

So must compute $\langle V_{h_1} \ldots V_{h_{n-2}} U_{p_1} \ldots U_{p_t}\rangle_{S[m]}$ as sum of trees.

$\langle V_{h_1} \dots V_{h_{n-2}} U_{p_1} \dots U_{p_t} \rangle_{S[m]}$ needs trees on $t+n-2$ vertices.

- Define $(n-2+t) \times (n-2+t)$ matrix of propagators

$$\mathcal{H} := \begin{pmatrix} \mathbb{H} & \mathfrak{h} \\ \mathfrak{h}^T & \mathbb{T} \end{pmatrix}$$

- Propagators between V_{h_i} and V_{h_j}

$$\mathbb{H}_{ij} = \begin{cases} \frac{[\![ij]\!]}{\langle ij \rangle} & i \neq j \\ -\sum_l \mathbb{H}_{il} & i = j \end{cases}$$

- Propagators resp. between V_{h_i} and U_{p_r} and U_{p_r} and U_{p_s}

$$\mathfrak{h}_{ir} := -\varepsilon_r \frac{[\![ir]\!]}{\langle ir \rangle}, \qquad \mathbb{T}_{rs} = -\varepsilon_r \varepsilon_s \frac{[\![\bar{\lambda}_r \bar{\lambda}_s]\!]}{\langle rs \rangle}, \quad \mathbb{T}_{rr} = \dots$$

here the parameters ε_r count the valency of U_{p_r}.

Propn: Sum of trees generated by reduced determinant $\det' \mathcal{H}$.

Final formulae

For MHV amplitude, sum over all contributions

$$\mathcal{M}_n^0 = \sum_{\substack{\#\text{ of tails, } t \\ \text{mult. of tails, } p}} \int \mathrm{d}^4 x \left(\prod_{r=1}^{t} D\sigma_r \, \partial_u^{p_r - 1} \boldsymbol{\sigma}^0(u_r, \sigma_r) \, \partial_{\varepsilon_r}^{p_r} \right) \det{}' \mathcal{H}_t|_{\varepsilon_r = 0} \,.$$

- For radiative Gibbons-Hawking, SD black holes etc., $M^{\dot{\alpha}}(x, \sigma)$ etc. can be made explicit.
- At n points expect $n - 2$ space-time integrals, here just one.
- On plane wave $\mathbb{CP}^1$ and three d^3x integrals localize.
- Higher MHV degree version can be made as explicit but requires more integrations.

Conclusions & discussion

- Gravity tree amplitudes generated by on-shell action of sigma model for curves in $\mathbb{PT}$ (from cuts of $\mathscr{I}$).
- Gives value of Einstein-Hilbert action at MHV.
- degree of map $= k - 1$ at N^{k-2}MHV corresponds to rational approximation of true light cone cut.
- Story extends to $\Lambda \neq 0$, YM.
- Perturbations around SD backgrounds includes SD black holes.
- Can we extend to general worldlines?

Celestial holography and $Lw_{1+\infty}$:

- Penrose's nonlinear graviton realizes SD graviton phase space as loop group $Lw_{1+\infty}$.
- Geometric action of $Lw_{1+\infty}$ on $\mathbb{PT}$ is by Čech vertex operators for SD gravitons.

Thank You

Quantization?

- Einstein gravity tree $=$ tree sigma model correlator (MHV).
- Does full quantum sigma model correlator $\leftrightarrow$ gravity loops?

$$\langle 1\,2\rangle^{2n}\prod_{i=3}^{n}\frac{1}{\langle 1\,i\rangle^2\,\langle 2\,i\rangle^2}\,\exp\left[-\frac{\mathrm{i}\,\alpha}{8\pi}\sum_{j\neq i}\frac{[i\,j]}{\langle i\,j\rangle}\,\frac{\langle 1\,i\rangle^2\,\langle 2\,j\rangle^2}{\langle 1\,2\rangle^2}\right]\,.$$

- Does quantum sigma model realize $W_{1+\infty}$ or W-gravity?
- Moyal quantization of $\mu^{\dot\alpha}$-plane and 'palatial twistors'?