The Integrable Hypereclectic Spin Chain

Matthias Staudacher

Institute for Physics, Institute for Mathematics, and IRIS Adlershof
Mathematical Physics of Space, Time and Matter
Humboldt University, Berlin

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Based on

- Earlier work with Asger Ipsen and Leo Zippelius, arXiv:1812.08794.
- Earlier work with with Changrim Ahn, arXiv:2010.14515.
- Main work with Changrim Ahn and Luke Corcoran, arXiv:2112.04506.
- Upcoming work with Changrim Ahn (this week).

Luke Corcoran's scientific work at SAGEX

- L. Corcoran and M. Staudacher, *The dual conformal box integral in Minkowski space*, Nucl. Phys. B **964** (2021), 115310, `arXiv:2006.11292`.

- L. Corcoran, F. Loebbert, J. Miczajka and M. Staudacher, *Minkowski Box from Yangian Bootstrap*, JHEP **04** (2021), 160, `2012.07852`.

- C. Ahn, L. Corcoran and M. Staudacher, *Combinatorial solution of the eclectic spin chain*, JHEP **03** (2022), 028, `arXiv:2112.04506`.

- L. Corcoran, F. Loebbert and J. Miczajka, *Yangian Ward identities for fishnet four-point integrals*, JHEP **04** (2022), 131, `arXiv:2112.06928`.

Motivations

- There has been some recent interest in strongly twisted planar $\mathcal{N}=4$ Super Yang-Mills Theory. This is a non-unitary yet still conformal and integrable quantum field theory. It was proposed that the model is simpler than the undeformed theory, and that its integrability can be more easily understood.

- We looked into this in the simplest possible setting: The one-loop dilatation operator. We found that curious novel challenges arise for the integrability program.

Strongly Twisted $\mathcal{N}$=4 Super Yang-Mills Theory, I

Start from planar, integrable, three-parameter γ-deformed $\mathcal{N}$=4 SYM.

Perform double-scaling limit: [O. Gürdoğan, V. Kazakov '15; Sieg, Wilhelm '16; Kazakov et.al. '18].

$$g = \frac{\sqrt{\lambda}}{4\pi} \longrightarrow 0 \quad \text{and} \quad q_j = e^{-i\gamma_j/2} \longrightarrow \infty \quad \text{or} \quad q_j = e^{-i\gamma_j/2} \longrightarrow 0$$

such that for each $j = 1, 2, 3$ either $g\, q_j$ or else $g\, q_j^{-1}$ is held fixed.

This yields $2^3 = 8$ different strong twisting limits: Write $q_j := \varepsilon^{\mp 1}\, \xi_j^{\pm}$, replace $g \to \varepsilon\, g$, and take ε to zero. For $(q_1, q_2, q_3) = (\infty, \infty, \infty)$:

$$\begin{aligned} \mathcal{L}_{\text{int}} \;=\; & -g^2\, N\, \mathrm{Tr}\left((\xi_3^+)^2\, \phi_1^\dagger \phi_2^\dagger \phi^1 \phi^2 + (\xi_2^+)^2\, \phi_3^\dagger \phi_1^\dagger \phi^3 \phi^1 + (\xi_1^+)^2\, \phi_2^\dagger \phi_3^\dagger \phi^2 \phi^3 \right) \\ & -g\, N\, \mathrm{Tr}\left(i\sqrt{\xi_2^+ \xi_3^+} (\psi^3 \phi^1 \psi^2 + \bar{\psi}_3 \phi_1^\dagger \bar{\psi}_2) + \text{cyclic} \right) \end{aligned}$$

Gauge fields "decouple".

Strongly Twisted $\mathcal{N}$=4 Super Yang-Mills Theory, II

Look at the other 7 cases. For $(q_1, q_2, q_3) = (0, 0, 0)$ one has the equivalent

$$\mathcal{L}_{\text{int}} = N \,\mathrm{Tr}\left((\xi_3^-)^{-2} \phi_2^\dagger \phi_1^\dagger \phi^2 \phi^1 + (\xi_2^-)^{-2} \phi_1^\dagger \phi_3^\dagger \phi^1 \phi^3 + (\xi_1^-)^{-2} \phi_3^\dagger \phi_2^\dagger \phi^3 \phi^2 \right)$$
$$+ \mathrm{Tr}\left(i(\xi_2^- \xi_3^-)^{-\frac{1}{2}} (\psi^2 \phi^1 \psi^3 + \bar{\psi}_2 \phi_1^\dagger \bar{\psi}_3) + \text{cyclic} \right)$$

The other six limits are different, but once again equivalent to each other. For example, for $(q_1, q_2, q_3) = (\infty, \infty, 0)$ we have

$$\mathcal{L}_{\text{int}} = N \,\mathrm{Tr}\Bigg((\xi_3^-)^{-2} \phi_2^\dagger \phi_1^\dagger \phi^2 \phi^1 + (\xi_2^+)^{2} \phi_3^\dagger \phi_1^\dagger \phi^3 \phi^1 + (\xi_1^+)^{2} \phi_2^\dagger \phi_3^\dagger \phi^2 \phi^3$$
$$+ \sqrt{\frac{\xi_2^+}{\xi_3^-}} \left(\bar{\psi}_1 \phi^1 \bar{\psi}_4 - \psi^1 \phi_1^\dagger \psi^4 \right) - \sqrt{\frac{\xi_1^+}{\xi_3^-}} \left(\bar{\psi}_4 \phi^2 \bar{\psi}_2 - \psi^4 \phi_2^\dagger \psi^2 \right)$$
$$- i \sqrt{\xi_1^+ \xi_2^+} \left(\bar{\psi}_2 \phi_3^\dagger \bar{\psi}_1 + \psi^2 \phi^3 \psi^1 \right) \Bigg)$$

Dilatation Operator and Non-Hermitian Spin Chains

As in ordinary $\mathcal{N}=4$ SYM, the one-loop dilatation operator yields a nearest neighbor spin chain Hamiltonian $\hat{\mathbf{H}}$:

$$\mathfrak{D} = \mathfrak{D}_0 + g^2\,\hat{\mathbf{H}} + \mathcal{O}(g^4)$$

Dropping all fermions, and regarding only chiral composite ops $\mathrm{Tr}\,\phi_{j_1}\phi_{j_2}\phi_{j_3}\ldots$, one gets for $(q_1, q_2, q_3) = (\infty, \infty, \infty)$

$$\hat{\mathbf{H}} = \sum_{\ell=1}^{L} \hat{\mathbb{P}}^{\ell,\ell+1} \quad \text{acting on} \quad \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3$$

where the strongly twisted permutation op $\hat{\mathbb{P}}$ acts on sites $\ell, \ell+1$ as

$$\hat{\mathbb{P}}\,|11\rangle = 0 \qquad \hat{\mathbb{P}}\,|22\rangle = 0 \qquad \hat{\mathbb{P}}\,|33\rangle = 0$$

$$\hat{\mathbb{P}}\,|12\rangle = 0 \qquad \hat{\mathbb{P}}\,|23\rangle = 0 \qquad \hat{\mathbb{P}}\,|31\rangle = 0$$

$$\hat{\mathbb{P}}\,|21\rangle = \xi_3^+\,|12\rangle \qquad \hat{\mathbb{P}}\,|32\rangle = \xi_1^+\,|23\rangle \qquad \hat{\mathbb{P}}\,|13\rangle = \xi_2^+\,|31\rangle$$

The Hypereclectic Spin Chain

Specializing to $\xi_1^+ = \xi_2^+ = 0$, $\xi_3^+ = 1$ one gets the hypereclectic model:

$$\mathfrak{H} = \sum_{\ell=1}^{L} \mathfrak{P}^{\ell,\ell+1} \quad \text{acting on} \quad \underbrace{\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3}_{L-\text{ times}}$$

with periodic boundary conditions, and where $\mathfrak{P}$ acts on sites $\ell, \ell+1$ as

$$\begin{array}{lll} \mathfrak{P}\,|11\rangle = 0 & \mathfrak{P}\,|22\rangle = 0 & \mathfrak{P}\,|33\rangle = 0 \\ \mathfrak{P}\,|12\rangle = 0 & \mathfrak{P}\,|23\rangle = 0 & \mathfrak{P}\,|31\rangle = 0 \\ \mathfrak{P}\,|21\rangle = |12\rangle & \mathfrak{P}\,|32\rangle = 0 & \mathfrak{P}\,|13\rangle = 0\,. \end{array}$$

Could there be a simpler spin chain Hamiltonian?

As we shall see, this model is integrable, but has not yet been exactly solved. We will also see that its "spectrum" is actually more complicated than the one of the eclectic model with "generic" parameters $\xi_1^+, \xi_2^+, \xi_3^+$.

Integrability of the Eclectic Spin Chain, I

The R-matrix of the eclectic model reads

$$
\hat{\mathbf{R}}(u) = \left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & 1 & & & & & \\
 & & \xi_2^+ u & & & & 1 & & \\
\hline
 & 1 & & \xi_3^+ u & & & & & \\
 & & & & 1 & & & & \\
 & & & & & & & 1 & \\
\hline
 & & 1 & & & & & & \\
 & & & & & 1 & & \xi_1^+ u & \\
 & & & & & & & & 1
\end{array}\right)
$$

It satisfies the Yang-Baxter equation:

$$
\hat{\mathbf{R}}^{12}(u-u')\hat{\mathbf{R}}^{13}(u)\hat{\mathbf{R}}^{23}(u') = \hat{\mathbf{R}}^{23}(u')\hat{\mathbf{R}}^{13}(u)\hat{\mathbf{R}}^{12}(u-u')
$$

Integrability of the Eclectic Spin Chain, II

In standard fashion, the quantum monodromy matrix is then built as

$$\hat{\mathbf{M}}^{a,L}(u) = \hat{\mathbf{R}}^{a,L}(u) \cdot \hat{\mathbf{R}}^{a,L-1}(u) \cdot \ldots \cdot \hat{\mathbf{R}}^{a,2}(u) \cdot \hat{\mathbf{R}}^{a,1}(u)$$

Also satisfies the YBE. The transfer matrix is $\hat{\mathbf{T}}(u) := \mathrm{Tr}_a \, \hat{\mathbf{M}}(u)$, while

$$\hat{\mathbf{H}} = \mathbf{U}^{-1} \frac{d}{du} \hat{\mathbf{T}}(u) \bigg|_{u=0} \quad \text{with the shift operator} \quad \mathbf{U} = \hat{\mathbf{T}}(0)$$

It thus encodes a tower of commuting charges, including the Hamiltonian:

$$[\hat{\mathbf{T}}(u), \hat{\mathbf{T}}(u')] = 0 \quad \text{and hence} \quad [\hat{\mathbf{H}}, \hat{\mathbf{T}}(u')] = 0$$

This renders the eclectic spin chain integrable by two of the possible definitions of quantum integrability: Quantum YBE and charges in involution.

Integrability of the Hypereclectic Spin Chain

The R-matrix of the hypereclectic model reads

$$\mathfrak{R}(u) = \left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & 1 & & & & & \\
 & & & & & & 1 & & \\
\hline
 & 1 & & u & & & & & \\
 & & & & 1 & & & & \\
 & & & & & & & 1 & \\
\hline
 & & 1 & & & & & & \\
 & & & & & 1 & & & \\
 & & & & & & & & 1
\end{array}\right)$$

Being just the special case $\xi_1^+ = \xi_2^+ = 0$, $\xi_3^+ = 1$ of the eclectic model, the above construction works in the very same way. This proves the model's quantum integrability. But is it also exactly solvable?

Non-Diagonalizability of the (Hyper)eclectic Model

For hermitian Hamiltonians $\mathbf{H}$, we know that there must be $j = 1, \ldots, 3^L$ linearly independent eigenstates $|\psi_j\rangle$ satisfying, with $\omega_L := e^{\frac{2\pi i}{L}}$,

$$\mathbf{H}\,|\psi_j\rangle = E^j\,|\psi_j\rangle \quad \text{where} \quad \mathbf{U}\,|\psi_j\rangle = \omega_L^{k_j}\,|\psi_j\rangle \quad \text{and} \quad k_j \in \{0, \ldots, L-1\}$$

For the eclectic model, the eigenvalue equation has to be replaced by

$$\left(\hat{\mathbf{H}} - E^j\right)^{m_j}\,|\psi_j^{m_j}\rangle = 0 \quad \text{with} \quad m_j = 1, \ldots, l_j$$

The $|\psi_j^{m_j}\rangle$ are generalized eigenstates with generalized eigenvalues E^j.

Note that the Hamiltonian $\hat{\mathbf{H}}$ is still block-diagonal w.r.t. sectors of fixed numbers $L-M$ of fields ϕ_1, $M-K$ fields ϕ_2, and K fields ϕ_3. And the $|\psi_j^{m_j}\rangle$ may still be chosen to be eigenstates of $\mathbf{U}$ with eigenvalues $\omega_L^{k_j}$.

Chiral XY-Model

For $K = 0$ (no fields ϕ_3) the non-hermitian Hamiltonian is actually diagonalizable, either by Bethe ansatz, or else a Jordan-Wigner transformation:

$$E = \sum_{m=1}^{M} \frac{1}{u_m^-} \quad \text{and} \quad \omega_L^k = \prod_{m=1}^{M} \frac{1}{\xi_3^+ \, u_m^-}$$

where, in the sector of M fields ϕ_2, one has

$$(\xi_3 \, u_m^-)^L = 1 \quad \text{for} \quad m = 1, \ldots M$$

One easily checks the completeness of all the $\binom{L}{M}$ states of this sector. This clearly leads to the completeness of all 2^L states with $K = 0$.

Jordan Normal Form

For $K \neq 0$ the Hamiltonian $\hat{\mathbf{H}}$ is not diagonalizable. It turns out that all generalized eigenvalues are $E = 0$. Define $l \times l$ Jordan blocks by

$$J_l := \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix}.$$

The best one can do is to bring $\hat{\mathbf{H}}$ into Jordan Normal Form (JNF) by a similarity transform S, composed of b blocks of sizes l_j:

$$S \cdot \hat{\mathbf{H}} \cdot S^{-1} = \begin{pmatrix} J_{l_1} & & 0 \\ & \ddots & \\ 0 & & J_{l_b} \end{pmatrix} := l_1 l_2 \ldots l_b \quad \text{with} \quad l_1 + \ldots + l_b = 3^L - 2^L$$

Bethe Ansatz: Intricate, but Failing, I

Integrable spin chains are usually solved by Bethe ansatz. Applying it directly to the eclectic spin chain, it algebraically fails. Before taking $\varepsilon \to 0$ in the twisted model with $q_j = \varepsilon^{-1}\xi_j^+$ it works perfectly:

$$E = \varepsilon L + \varepsilon \sum_{m=1}^{M} \left(\frac{1}{u_m} - \frac{1}{u_m + 1} \right)$$

with the Bethe equations ($\xi := \xi_1^+ \xi_2^+ \xi_3^+$)

$$\left(\frac{u_m + 1}{u_m} \right)^L = \varepsilon^{3K-L} \frac{\xi_3^{+L}}{\xi^K} \prod_{\substack{j=1 \\ j \neq m}}^{M} \frac{u_m - u_j + 1}{u_m - u_j - 1} \prod_{i=1}^{K} \frac{u_m - v_i - 1}{u_m - v_i}$$

$$1 = \varepsilon^{3M-2L} \frac{\xi^{L-M}}{\xi_1^{+L}} \prod_{j=1}^{M} \frac{v_l - u_j + 1}{v_l - u_j} \prod_{\substack{i=1 \\ i \neq l}}^{K} \frac{v_l - v_i - 1}{v_l - v_i + 1}$$

Clearly very singular. Still, their limit may in most cases be analyzed.

Bethe Ansatz: Intricate, but Failing, II

E.g. for a rather generic (L, M, K) sector with $L > 3(M-K)$ fractional scaling solutions maybe found explicitly:

$$\begin{array}{lll} \text{(I)} & u_j = \varepsilon^{\alpha}\, u_j^- \,, & j = 1, \cdots, M-K \\ \text{(II)} & u_{l+M'} = -1 + \varepsilon^{\beta}\, u_l^+ \,, & l = 1, \cdots, K \\ \text{(III)} & v_l = -2 + \varepsilon^{\beta}\, u_l^+ + \varepsilon^{\gamma}\, \hat{v}_l, & l = 1, \cdots, K \end{array}$$

One may explicitly find the scaled roots $u_j^-, u_l^+, \hat{v}_l$ and the exponents

$$\alpha = \frac{L-(M+K)}{L-(M-K)} \qquad \beta = \frac{L-3(M-K)}{L-(M-K)}$$

$$\gamma = 2L - 3M - \frac{L-3(M-K)}{L-(M-K)}\,(K-1)$$

Proves $E = 0$. But all Bethe states collapse to a trivial "locked" state: $|\phi_1 \ldots \phi_1\, \phi_2 \ldots \phi_2 \ldots \phi_3\, \ldots \phi_3\rangle := |1 \ldots 1\, 2 \ldots 2\, 3 \ldots 3\rangle$. JNF ???

However, see linear combinations approach of [Nieto García, Wyss '21, Nieto García '22].

Universality Hypothesis for the Eclectic Spin Chain

For the eclectic chain, the JNF is identical for almost all $\xi_1^+, \xi_2^+, \xi_3^+$. *For the hypereclectic chain, the JNF is identical to the one of the generic eclectic chain, as long as the following filling conditions are satisfied:*

$$L - M \geq M - K \geq K \quad \Leftrightarrow \quad L \geq 2M - K \quad \text{and} \quad M \geq 2K.$$

Example: $L = 7, M = 3, K = 1$. Generic eclectic chain: JNF = 1 5 9.
For the hypereclectic chain, we have in the cyclic 15×15 sector

$$\begin{aligned} \text{JNF} &= 1\,5\,9 && \text{for permutations of } |1111223\rangle \text{ and } |2222113\rangle \\ \text{JNF} &= 1\,2\,3\,4\,5 && \text{for permutations of } |1111332\rangle \text{ and } |2222331\rangle \\ \text{JNF} &= 1^6\,2^3\,3 && \text{for permutations of } |3333112\rangle \text{ and } |3333221\rangle \end{aligned}$$

How to prove this hypothesis?

Example: Hypereclectic JNF for $M = 5$, $K = 1$

L	Sizes of Jordan Blocks
8	1 5 7 9 13
9	$1\ 5^2\ 9^2$ 11 13 17
10	$1\ 5^2$ 7 9^2 11 13^2 15 17 21
11	$1^2\ 5^2$ 7 9^3 11 13^3 15 17^2 19 21 25
12	$1\ 5^3$ 7 $9^3\ 11^2\ 13^3\ 15^2\ 17^3$ 19 21^2 23 25 29
13	$1^2\ 5^3$ 7 $9^4\ 11^2\ 13^4\ 15^2\ 17^4\ 19^2\ 21^3$ 23 25^2 27 29 33
14	$1^2\ 5^3\ 7^2\ 9^4\ 11^2\ 13^5\ 15^3\ 17^4\ 19^3\ 21^4\ 23^2\ 25^3$ 27 29^2 31 33 37
15	$1^2\ 5^4$ 7 $9^5\ 11^3\ 13^5\ 15^3\ 17^6\ 19^3\ 21^5\ 23^3\ 25^4\ 27^2\ 29^3$ 31 33^2 35 37 41
16	$1^2\ 5^4\ 7^2\ 9^5\ 11^3\ 13^6\ 15^4\ 17^6\ 19^4\ 21^6\ 23^4\ 25^5\ 27^3\ 29^4\ 31^2\ 33^3$ 35 37^2 39 41 45
17	$1^3\ 5^4\ 7^2\ 9^6\ 11^3\ 13^7\ 15^4\ 17^7\ 19^5\ 21^7\ 23^4\ 25^7\ 27^4\ 29^5\ 31^3\ 33^4\ 35^2\ 37^3$ 39 41^2 43 45 49
18	$1^2\ 5^5\ 7^2\ 9^6\ 11^4\ 13^7\ 15^5\ 17^8\ 19^5\ 21^8\ 23^6\ 25^7\ 27^5\ 29^7\ 31^4\ 33^5\ 35^3\ 37^4\ 39^2\ 41^3$ 43 45^2 47 49 53

Quite involved, even though when staring at it, one sees some structure . . .

From Anti-Locked to Locked States: An Example

Example: $L = 7, M = 3, K = 1$. Hyperelectic chain: JNF = 1 5 9.

Anti-locked state: $|65\rangle := |2211113\rangle$ Locked state: $|21\rangle := |1111223\rangle$

Clearly $\mathfrak{H}|65\rangle = |64\rangle = |2121113\rangle$ and $\mathfrak{H}|21\rangle = 0$. Acting by $\mathfrak{H}$, we get $|65\rangle \mapsto |64\rangle \mapsto |63\rangle + |54\rangle \mapsto |62\rangle + 2|53\rangle \mapsto |61\rangle + 3|52\rangle + 2|43\rangle \mapsto 4|51\rangle + 5|42\rangle \mapsto 9|41\rangle + 5|32\rangle \mapsto 14|31\rangle \mapsto 14|21\rangle \mapsto 0$, the 9×9 block.

Ansatz: $a|63\rangle + b|54\rangle \mapsto a|62\rangle + (a+b)|53\rangle \mapsto a|61\rangle + (2a+b)|52\rangle + (a+b)|43\rangle \mapsto (3a + b)|51\rangle + (3a + 2b)|42\rangle \mapsto (6a + 3b)|41\rangle + (3a + 2b)|32\rangle \mapsto (9a + 5b)|31\rangle \mapsto 0$ for $a = 5,\ b = -9$. This is the 5×5 block!

2. Ansatz: $a'|61\rangle + b'|52\rangle + c'|43\rangle \mapsto (a' + b')|51\rangle + (b' + c')|42\rangle \mapsto 0$ for $c' = -b' = a'$. This is the, remaining, 1×1 block!

We can encode this structure into the following generating function:

$$Z^{\mathrm{cyc}}_{7,3,1} = q^{-4} + q^{-3} + 2q^{-2} + 2q^{-1} + 3q^{0} + 2q + 2q^{2} + q^{3} + q^{4}$$

Partition Function Approach, $K = 1$

We think this procedure works in generality. Tested extensively. Its validity is based on our non-shortening conjecture. If true, the JNF is encoded in

$$Z^{\mathrm{cyc}}_{L,M,1}(q) = \mathrm{Tr}^{\mathrm{cyc}}_{L,M,1}\, q^{\hat{S}'} \quad \text{with} \quad \hat{S}' = \hat{S} - \tfrac{1}{2}\,\hat{S}_{\mathsf{max}}$$

$\hat{S}$ counts the $\#$ of 1s (with multiple counts) to the right of the 2s. Then

$$Z^{\mathrm{cyc}}_{L,M,1}(q) = \sum_{j=1}^{\infty} N_j\, [j]_q$$

where N_j is the $\#$ of length-j Jordan blocks, and $[j]_q$ is a q-number

$$[j]_q = \frac{q^{j/2} - q^{-j/2}}{q^{1/2} - q^{-1/2}} = \sum_{k=-\frac{j-1}{2}}^{\frac{j-1}{2}} q^k$$

In our example above we have $Z^{\mathrm{cyc}}_{7,3,1}(q) = [1]_q + [5]_q + [9]_q$.

Gaussian binomial coefficients, and on to $K > 1$

We managed to compute the $K = 1$ partition functions exactly. This yields Gaussian binomial coefficients, a.k.a. q-binomials [C. Ahn, MS, L. Corcoran '21]:

$$Z^{\mathrm{cyc}}_{L,M,1}(q) = \begin{bmatrix} L-1 \\ M-1 \end{bmatrix}_q \quad \text{with} \quad \begin{bmatrix} \ell+m \\ m \end{bmatrix}_q := \prod_{k=1}^{m} \frac{q^{\frac{\ell+k}{2}} - q^{-\frac{\ell+k}{2}}}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}$$

This nicely encodes all of the JNFs I showed you earlier in the table!

To treat the case of general K, we propose that one still has

$$Z^{\mathrm{cyc}}_{L,M,K}(q) = \mathrm{Tr}^{\mathrm{cyc}}_{L,M,K}\, q^{\hat{S}'} = \sum_{j=1}^{\infty} N_j\, [j]_q$$

where now $\hat{S}' = \sum_{j=1}^{K} \hat{S}'_k$, and $\hat{S}'_k$ counts as above within each of K bins.

Partition Function Approach, $K > 1$

However, this does not just result in a sum of K products of q-binomials: We need to take into account all non-trivial symmetries under cyclic shifts. The method of choice it the Pólya enumeration theorem. To apply it, an

Idea: Replace the spin chain of length L, with spins in $\{1, 2, 3\}$, by a shorter chain of length K, with spins in $\mathcal{A} = \{3, 13, 23, 113, 123, 213, 223, \ldots\}$. Note that $\mathcal{A} \subset \oplus_{L=1}^{\infty} \left(\mathbb{C}^3\right)^{\otimes L}$. Now define the "one-site" (i.e. $K = 1$) "grand canonical" partition function

$$\mathrm{bin}(x, y, z, q) := \sum_{L,M=1}^{\infty} Z^{\mathrm{cyc}}_{L,M,1}(q)\, x^{L-M} y^{M-1} z$$

Its natural generalization involves in addition a sum over all K:

$$Z_{\mathrm{cyc}}(x, y, z, q) = \sum_{L,M,K=1}^{\infty} Z^{\mathrm{cyc}}_{L,M,K}(q)\, x^{L-M} y^{M-K} z^{K}$$

Pólya Enumeration Theorem and General K

Pólya's theorem then yields

$$Z_{\text{cyc}}(x,y,z,q) = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log\left(1 - \text{bin}(x^n, y^n, z^n, q^n)\right)$$

Here $\phi(n)$ is Euler's totient function, defined as the number of positive integers less than n that are coprime to n (i.e. the number of those elements of $\{1,\ldots,n-1\}$ whose only divisor common with n is 1).

Actually, there also exists an elegant way to rewrite

$$\text{bin}(x,y,z,q) = z \sum_{m=0}^{\infty} y^m \prod_{\ell=0}^{m} \frac{1}{1 - q^{\ell - \frac{m}{2}} x} = z \sum_{\ell=0}^{\infty} x^\ell \prod_{m=0}^{\ell} \frac{1}{1 - q^{m - \frac{\ell}{2}} y}$$

The above should be the complete solution for the spectrum of the (Hyper)eclectic chain in the cyclic sector relevant to quantum field theory.

One last example: L=9, M=6, K=3

Using Mathematica™, our solution yields within seconds

$$Z^{\text{cyc}}_{9,6,3}(q) = q^{-9/2}+q^{-7/2}+4q^{-3}+4q^{-5/2}+8q^{-2}$$
$$+18q^{-3/2}+18q^{-1}+26q^{-1/2}+28q^{0}+26q^{1/2}+18q$$
$$+18q^{3/2}+8q^{2}+4q^{5/2}+4q^{3}+q^{7/2}+q^{9/2}\,.$$

This is quickly expressed through q-numbers as

$$Z^{\text{cyc}}_{9,6,3}(q) = 10\,[1]_q + 8\,[2]_q + 10\,[3]_q + 14\,[4]_q + 4\,[5]_q + 3\,[6]_q + 4\,[7]_q + [10]_q$$

One then reads off immediately

$$\text{JNF} = 1^{10}\ 2^{8}\ 3^{10}\ 4^{14}\ 5^{4}\ 6^{3}\ 7^{4}\ 10$$

Conclusions

- Inspired by strongly twisted $\mathcal{N}=4$ SYM, we considered novel classes of non-diagonalizable spin chains: The Eclectic and Hypereclectic models.

- We proved their quantum integrability by deriving their R-matrices.

- We showed that the Bethe ansatz equations make sense, and can even be partially solved explicitly, exhibiting rather non-trivial scaling behavior. However, vexingly, they appear to be quite clumsy for determining the "spectrum" of Jordan Normal Forms.

- Still, with a combination of linear algebra methods and combinatorics, and under two conjectures, we derived exact solutions for this spectrum.

To Do

- Prove the two key assumptions: The universality hypothesis, and the non-shortening conjecture.
- How to use the integrability of the (Hyper)eclectic model? Appearance of q-numbers and q-binomials is very suggestive: quantum groups?
- Derive the consequences of the JNF on strongly twisted $\mathcal{N}=4$ SYM. Should be very non-trivial examples of four-dimensional non-unitary logarithmic quantum field theories.
- Non-perturbative solutions via the quantum spectrum curve (QSC) have been proposed, largely ignoring the JNF structure. Implications?
- Higher loops, strong coupling, and dual "Fish Chain"? [N. Gromov, A. Sever '19]