Amplitude Singularities from Cluster Algebras & Tropical Geometry

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CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

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JHEP 10 (2021) 007 and JHEP 08 (2020) 005 with Niklas Henke

Motivation: From $\mathcal{N} = 4$ SYM to the real world

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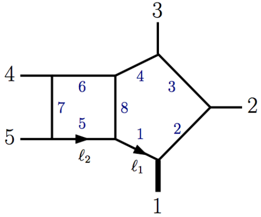
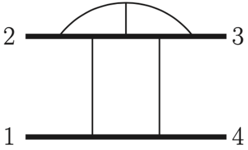
All of them crucial in recent state of the art calculations for collider and gravitational wave physics
[Abreu,Ita,Moriello,Page,Tschernow,Zeng'20] [Bern,Parra-Martinez,Roiban,Ruf,Shen,Solon,Zeng'21]





The Role of Cluster Algebras

Tremendously successful in describing singularities of n-particle amplitudes $\mathcal{A}_n$ in planar (color $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed) limit of $\mathcal{N} = 4$ SYM.
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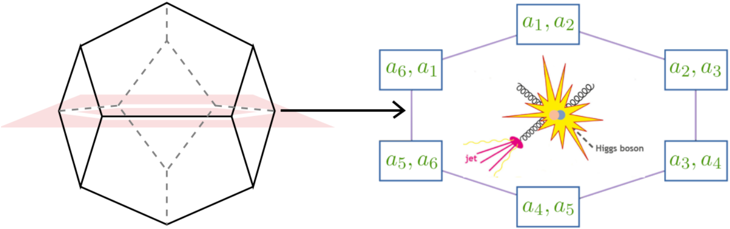
$\Rightarrow$ results for $n = 6, 7$ to unprecedented loop order. [Drummond,**GP**,Spradlin'14]
[Drummond,Foster,Gurdogan,**GP**'18] [Caron-Huot,Dixon,Dulat,Hippel,McLeod,**GP**'19A+B]



Recently observed to underlie analytic structure of a host of Feynman integrals and realistic processes such as Higgs+jet production in heavy-top limit of QCD! [Chicherin,Henn,**GP**;PRL 126 091603 (2021)]

Long-standing Burning Questions

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Explicit singularity predictions for

- $n = 8$ [Henke, Papathanasiou'19]
  see also [Arkani-Hamed,Lam,Spradlin'19] [Drummond,Foster,Gurdogan,Kalousios'19B]

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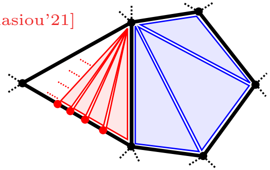
- $n = 8$ [Henke, Papathanasiou'19] see also [Arkani-Hamed,Lam,Spradlin'19] [Drummond,Foster,Gurdogan,Kalousios'19B]
- In principle any n, explicitly $n = 9$, [Henke, Papathanasiou'21] see also [Ren,Spradlin,Volovich'21]



In agreement with all known amplitude data.

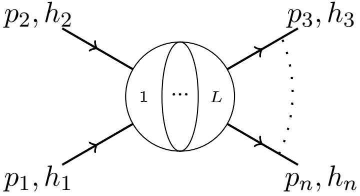
Outline

Introduction: Cluster Algebras and $\mathcal{N} = 4$ SYM

Relation to Tropical Grassmannians

Predictions for 8- and 9-particle Singularities

Planar $\mathcal{N} = 4$ Amplitudes: Symmetries and Kinematics



momenta $p_i^2 = 0$, helicities $h_i = \pm 1$,
degree $m = (\# h_i = -1) - 2$ (N^{m}MHV)
loop order L
amplitude $\mathcal{A}^{(L)}_{n,m}(\mathcal{X}_i(p_1, \ldots, p_n))$

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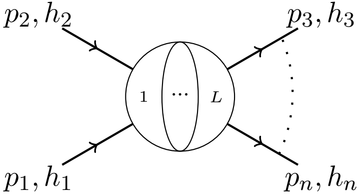

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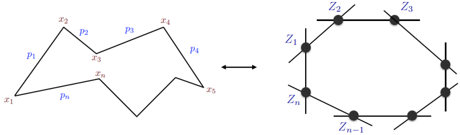
Dual conformal symmetry: $\mathcal{X}_i$ coordinates on $Gr(4,n)/(\mathbb{C}^*)^{n-1}$, i.e. $3n - 15$ independent components of n ordered *momentum twistors* $Z_i \in \mathbb{CP}^3$ [Drummond,Henn,Sokatchev,Smirnov'06] [Hodges'09]



$$p_i \equiv x_{i+1} - x_i\,, \quad x_i \sim Z_{i-1} \wedge Z_i$$

$$(x_i - x_j)^2 \sim \epsilon_{IJKL} Z^I_{i-1} Z^J_i Z^K_{j-1} Z^L_j = \det(Z_{i-1} Z_i Z_{j-1} Z_j) \equiv \langle i-1 i j-1 j \rangle$$

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Symbol $S(f_k)$ simultaneously takes account of all steps of recursion.

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The right variables

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- *variables* a_m *of a Grassmannian* $Gr(4, n)$ *cluster algebra* [Golden,Goncharov,Spradlin,Vergu,Volovich'13]

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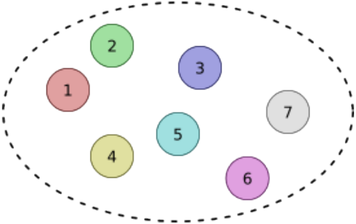

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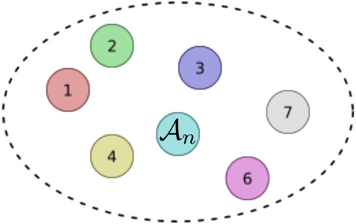

- *Finite number* of MPL with given alphabet & weight
- Identify $\mathcal{A}_n$ among them!

Application: The Steinmann Cluster Bootstrap for $\mathcal{N} = 4$ SYM Amplitudes

Evade Feynman diagrams by exploiting analytic structure

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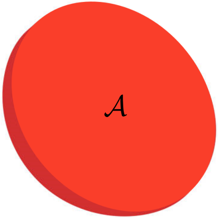

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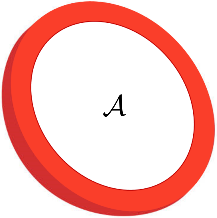

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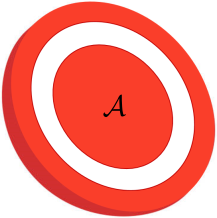

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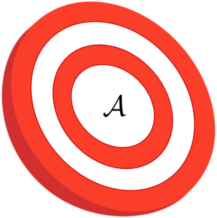

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Extended Steinmann	$\Leftrightarrow$ $\mathcal{A}_6^{(6)}, \mathcal{A}_{6,0}^{(7)}$
Coaction Principle	[Caron-Huot,Dixon,Dulat, McLeod,Hippel,**GP**]

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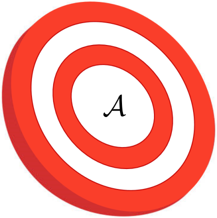

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Extended Steinmann $\Leftrightarrow$ Coaction Principle	$\mathcal{A}_6^{(6)}, \mathcal{A}_{6,0}^{(7)}$ [Caron-Huot,Dixon,Dulat, McLeod,Hippel,**GP**]

See also $S(A_n^{(2)}) \to A_n^{(2)}$, $S(\mathcal{A}_7) \to \mathcal{A}_7$ work [Golden(,Paulos),Spradlin(,Volovich)] [Dixon,Liu] [Golden,McLeod]

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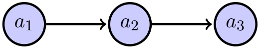

Initial Cluster

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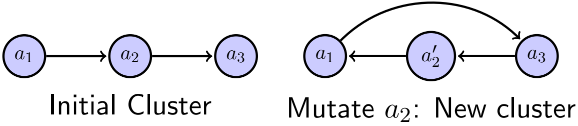

General rule for mutation at node k:

1. $\forall\ i \to k \to j$, add $i \to j$, reverse $i \leftarrow k \leftarrow j$, remove $\rightleftarrows$.

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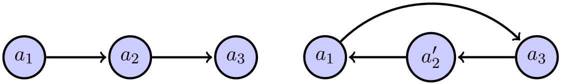


Initial Cluster — Mutate a_2: New cluster

$a_2' = (a_1 + a_3)/a_2$

and so on. . .

General rule for mutation at node k:

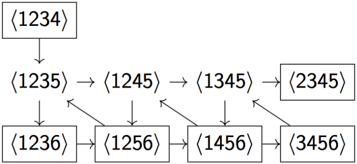
1. $\forall\ i \to k \to j$, add $i \to j$, reverse $i \leftarrow k \leftarrow j$, remove $\rightleftarrows$.
2. In new quiver/cluster, $a_k \to a_k' = \Big(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j \Big)/a_k$

Cluster Algebras [Fomin,Zelevinsky'01A]

They consist of

- A set of variables a_i, the *cluster ($\mathcal{A}$-)coordinates*
- Grouped into overlapping subsets $\{a_1, \ldots, a_d\}$ of *rank* d, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers. Example: $A_3 \simeq Gr(4,6)$ cluster algebra



- Further refinement: Include *frozen variables* $\boxed{a_{d+i}}$ that do not mutate
- Setting $\boxed{a_{d+i}} \to 1$ recovers previous definition

Geometric Realization of Cluster Algebras [Fomin,Zelevinsky'01B'02]

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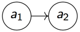

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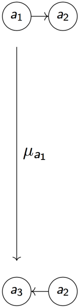

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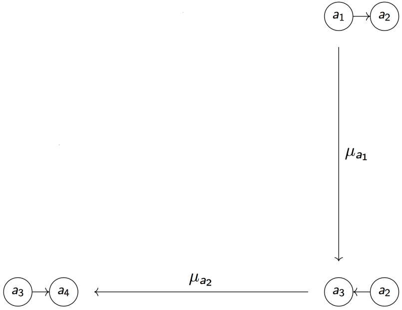

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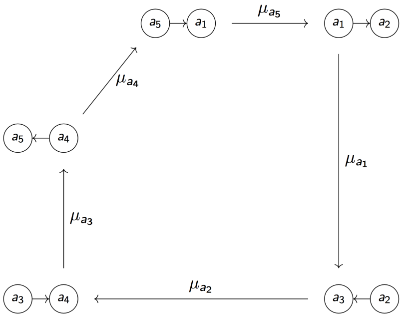

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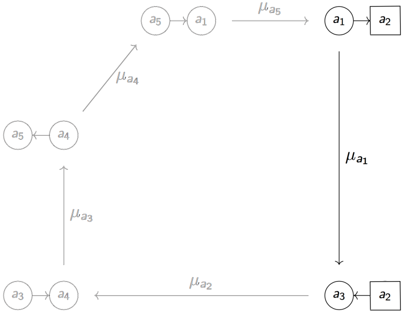


- Obtain subalgebras by *freezing* =forbidding mutation of certain nodes

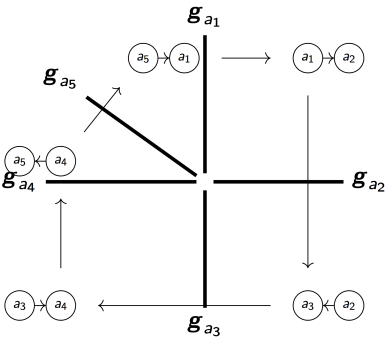
The Dual Cluster Fan

Equivalent description of cluster polytope

Take normal vectors (of undetermined length) to maximal dimension faces

- Give rise to *rays* (half-lines emanating from origin) ↔ cluster variables

A_2



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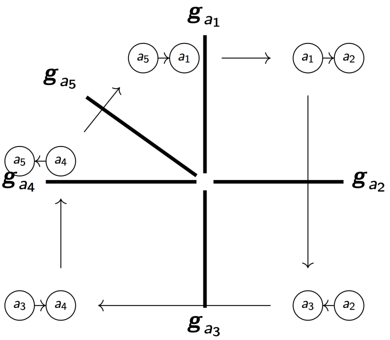
- Grouped in *cones* ↔ clusters

A_2



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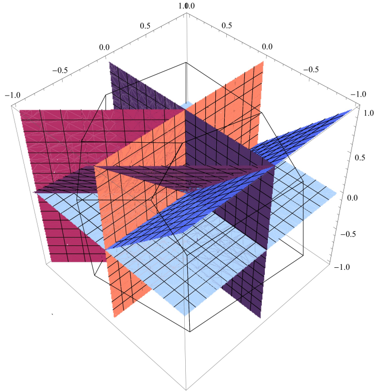

Collection of cones = *(polyhedral) fan* [Fomin,Zelevinsky'01B'02]

Back to First Burning Question

For $n \geq 8$, $Gr(4,8)$ cluster algebra associated to $\mathcal{A}_n$ becomes infinite!

$\Rightarrow$ infinite potential symbol letters render bootstrap inapplicable.

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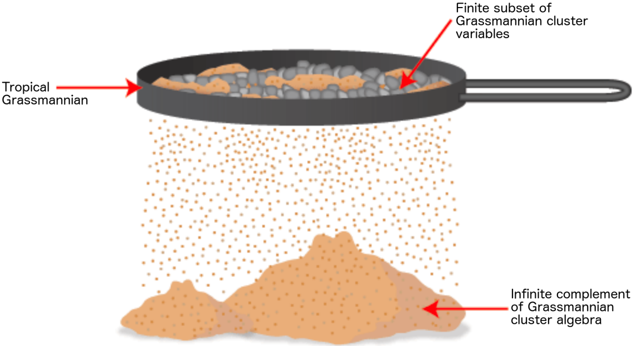

As we will see, *tropical Grassmannians* $Tr(4,n)$ provide a natural selection rule yielding a finite subset of cluster variables/rational letters of $\mathcal{A}_n$.

The (Positive) Tropical Grassmannian [Speyer,Sturmfels'03][Speyer,Williams'03]

- Parametrize kinematics with $Gr(4,n)$ initial cluster $\mathcal{X}$-*coordinates* x_i

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & * & * & * & \\ 0 & 1 & 0 & 0 & 1 & 1+x_1+x_1x_2 & * & * & \\ 0 & 0 & 1 & 0 & -1 & -1-x_1 & * & * & * \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \end{pmatrix}, \qquad x_i = \frac{\prod\limits_{\text{arrows } j\to i} a_j}{\prod\limits_{\text{arrows } j\leftarrow i} a_j}\,.$$

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  multiplication $\longrightarrow$ addition $\quad 0 \to \infty$

Example: $\langle 1346 \rangle = 1 + x_1 + x_1x_2 \longrightarrow \min(0, x_1, x_1 + x_2)$

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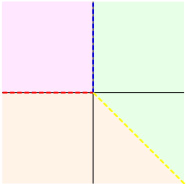
- *Tropical hypersurface* for $\langle 1346 \rangle$: $(d-1)$-dim. surface in $\mathbb{R}^d$ where minimum attained twice simultaneously



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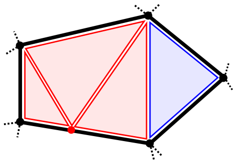
Solution of linear (in)equalities, inherently finite-dim. May similarly define

- *Rays* = 1-dim. intersections of tropical hypersurfaces, start at origin
- *Cones* = regions in $\mathbb{R}^d$ where all $min(\ldots)$ continuous
  = positive span of certain sets of $d=3n-15$ rays
- *Fan* = set of all cones

Tropical Grassmannians and Cluster Algebras

- Finite $Gr(k,n)$ cluster algebras *triangulate* $(p)Tr(k,n)$! [Speyer,Williams]

Illustration: Intersections of 3D cones with sphere ~ locally screen plane



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- • : $(p)Tr + Gr$ rays
- – : $(p)Tr + Gr$ boundaries
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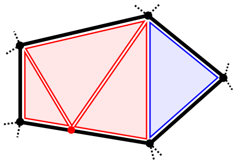

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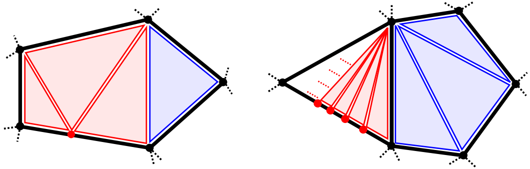

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> Idea: Cluster algebra ∞ due to infinitely redundant triangulations! Select *finite subset* of cluster variables corresponding to tropical rays

[Arkani-Hamed,Lam,Spradlin'19] [Henke,**GP**'19] [Drummond,Foster,Gurdogan,Kalousios'19B]

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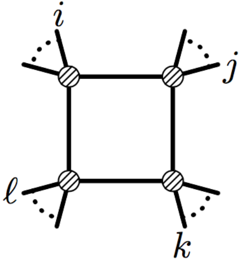

symbol letters contain $\sqrt{\Delta_{ijk\ell}}$,

$$\Delta_{ijk\ell} \equiv (f_{ij}f_{k\ell} - f_{ik}f_{j\ell} + f_{i\ell}f_{jk})^2 - 4 f_{ij}f_{jk}f_{k\ell}f_{i\ell}$$

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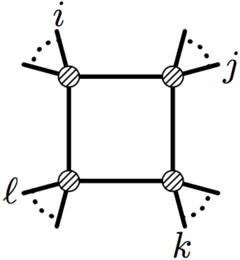

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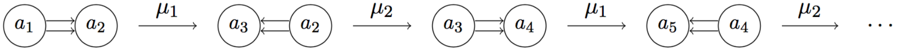
- Start at $n \geq 8$, and letters containing $\sqrt{\Delta_{ijk\ell}}$ indeed observed in explicit calculations of $\mathcal{A}^{(1)}_{8,1}$,$\mathcal{A}^{(2)}_{n,1}$ $\mathcal{A}^{(3)}_{8,0}$
[Henn, Herrmann,Parra-Martinez'18] [He,Li,Zhang'19'20] [Li,Zhang'21]

Square Root Letters from Infinite Mutation Sequences

For sequence of mutations mapping quiver back to itself, e.g. $A_1^{(1)}$



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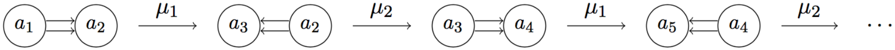

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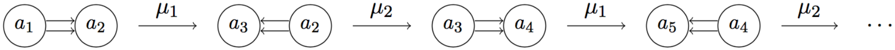

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Idea: Include infinite mutation sequences to obtain generalized cluster variables=square-root letters of amplitudes!

[Arkani-Hamed,Lam,Spradlin'19] [Henke, **GP**'19] [Drummond,Foster,Gurdogan,Kalousios'19B]

Application: $Gr(4,8)$ & Eight-particle Alphabet

Rational part:

- Start from initial cluster, mutate until redundant ray is reached
- Find 272 rational letters of degree up to 3 in $\langle ijkl\rangle$
- Includes the 196 $\mathcal{A}_{8,0}^{(3)}$ rational letters (which in turn contain the 172 $\mathcal{A}_{8,1}^{(2)}$ and 108 $\mathcal{A}_{8,0}^{(2)}$ rational letters resp.) [Li,Zhang'21]

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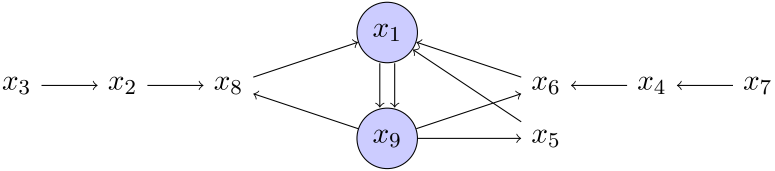

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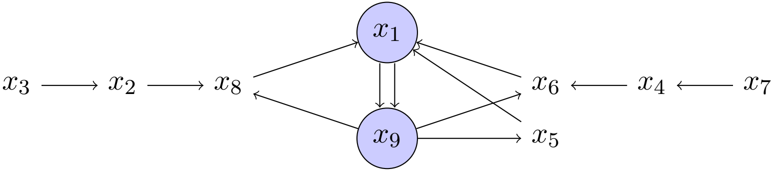


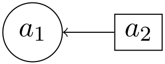
- Fine print: Limit value depends on cluster variables held frozen in infinite mutation sequence

The Role of Coefficients

There exists framework for simultaneously describing *any* choice of frozen variables: [Fomin,Zelevinsky'06]

$$\textit{Coefficients} \quad y_i = \frac{\prod_{\text{arrows } \boxed{j} \to i} a_j}{\prod_{\text{arrows } \boxed{j} \leftarrow i} a_j}.$$

- Can think of them as fundamental, define mutation rules they obey.
- Simplest case: *principal coefficients*, to each unfrozen node a_1,



$$y_1 = a_2$$

Square-root Letters from Infinite sequences II

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~~Ray $\leftrightarrow$ generalized cluster variable correspondence~~

- Implies 2 $pTr(4,8)$ limit rays $\rightarrow$ 18 square-root letters
- Indeed present in $\mathcal{A}_{8,1}^{(2)}, \mathcal{A}_{8,0}^{(3)}$ [He,Li,Zhang'19] [Li,Zhang'21]

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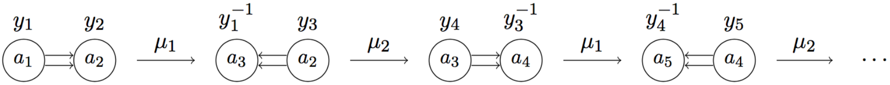
> Recently: $A_1^{(m)}$ infinite mutation sequence with general coefficients
> $\Rightarrow$ Proposal for $\mathcal{A}_n$ alphabet in principle $\forall n$, explicitly for $n = 9$!

[Henke,**GP**'21]

$A_1^{(1)}$ Sequences with General Coefficients



$$\lim_{i\to\infty} \frac{a_i}{a_{i-1}} = \frac{a_2}{a_1} \frac{K_1 \pm \sqrt{K_1^2 - 4K_2}}{2(1 + \hat{\oplus}\, y_1 \,\hat{\oplus}\, y_1 y_2)}$$

where

$$K_1 = 1 + x_1 + x_1 x_2\,, \quad K_2 = x_1 x_2\,, \quad x_i = y_i \frac{\prod_{\text{arrows } j\to i} a_j}{\prod_{\text{arrows } j\leftarrow i} a_j}\,,\ j \text{ unfrozen}\,,$$

and

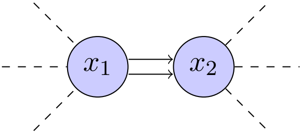
$$\prod_i f_i^{b_i} \,\hat{\oplus}\, \prod_i f_i^{c_i} = \prod_i f_i^{\min(b_i,c_i)}\,.$$

Also generalized to rank-$(m+1)$ $A_m^{(1)}$ sequences.

Generalized Cluster Variables for any $A_1^{(1)}$ Subalgebra of $Gr(4,n)$

Namely square-root letters for any amplitude multiplicity n

For any quiver containing $A_1^{(1)}$ with $\mathcal{X}$-coordinates x_1, x_2,



obtain limiting letters:

$$\phi_0 \equiv \frac{2 - K_1 + \sqrt{K_1^2 - 4K_2}}{-2 + K_1 + \sqrt{K_1^2 - 4K_2}}\,, \quad \tilde{\phi}_0 \equiv \frac{2K_2 - K_1 + \sqrt{K_1^2 - 4K_2}}{-2K_2 + K_1 + \sqrt{K_1^2 - 4K_2}}\,,$$
$$K_1 = 1 + x_1 + x_1 x_2\,, \quad K_2 = x_1 x_2$$

We showed that particular choice is motivated by closely related scattering diagrams approach. [Kontsevich,Soibelman'08] [Gross,Siebert'07] ... [Herderschee'21]

Application: $pTr(4,9)$ and Nine-particle Singularities

- 3078 cluster rays = rational letters of degree up to 6

Degree	1	2	3	4	5	6	7	8	9	10	Total
$pTr(4,9)$	117	576	1287	963	126	9	-	-	-	-	3078
$Tr(4,9)$	117	576	1854	3159	2943	1926	1296	531	180	63	12645

- 324 limit rays → 2349 square-root letters
- Contains alphabet of $\mathcal{A}_{9,1}^{(2)}$! [He,Li,Zhang'20]
- Also new types of square roots, e.g. $\Delta = A^2 - 4B$ with

$$A = 1 - \frac{\langle 6789\rangle\langle 13(278)\cap(246)\rangle^2}{\langle 1235\rangle\langle 1289\rangle\langle 3567\rangle\langle 1679\rangle^2} + \frac{\langle 1267\rangle\langle 23(146)\cap(178)\rangle\langle 46(278)\cap(129)\rangle}{\langle 1235\rangle\langle 1289\rangle\langle 3567\rangle\langle 1679\rangle^2}\,,$$

$$B = \frac{\langle 1267\rangle\langle 23\,(146)\cap(178)\rangle\langle 46\,(278)\cap(129)\rangle}{\langle 1235\rangle\langle 1289\rangle\langle 3567\rangle\langle 1679\rangle^2}\,.$$

Rational letters and radicands Δ also accounted for by tensor diagrams, but not complete square-root letters [Ren,Spradlin,Volovich'21]

Conclusions

Connection between cluster algebras and tropical Grassmannians provides candidate singularities/letters of $\mathcal{A}_n$ in principle $\forall n$!

- Selects finite subset of ∞ cluster algebras $\Rightarrow$ rational letters
- Limits of ∞ mutation sequences $\Rightarrow$ square-root letters
- Explicitly worked out for $n = 8, 9$
- Excellent agreement with fixed-order results & alternative approaches

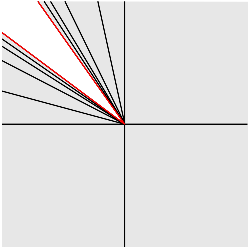
Moving Forward

- Efficient bootstrap of new results
- First-principle derivation of these remarkable mathematical structures
- Relevance for realistic gauge theories
  [Chicherin,Henn,**GP**;PRL 126 091603 (2021)] [He,Li,Ma,Wu,Yang,Zhang'22]
- Generalization beyond multiple polylogarithms

Open Questions

- 27 $pTr(4,9)$ rays unaccounted for by $A_1^{(1)}$ infinite mutation sequences
- Evidence that $Gr(4,n)$ cluster algebra does not entirely triangulate $pTr(4,n)$ with $n \geq 9$ for *any kind* of such mutation sequence
- Likely related to its mutation-infinite class. E.g. rank-2 example,

$$a_1 \Rrightarrow a_2$$



Indeed appears as $Gr(4,9)$ subalgebra

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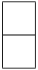
$$X^{IJ} = -X^{JI} =$$  of $SU(2,2)$

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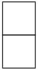
$$X^{IJ} = -X^{JI} =$$  of $SU(2,2)$

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- $(x_{i+i} - x_i)^2 = 0 \quad \Rightarrow X_i = Z_{i-1} \wedge Z_i$

The Kinematic Space of $\mathcal{N} = 4$ Amplitudes and Grassmannians

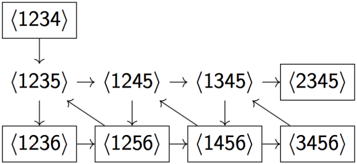
- Can realize kinematic space as $4 \times n$ matrix

$$(Z_1|Z_2|\ldots|Z_n) \in Gr(4,n)/(C^*)^{n-1}$$

modulo rescalings of the n columns and $SL(4)$ transformations $\Rightarrow$

$$\text{dimension} = 3n - 15.$$

- Closely related to *Grassmannian* $Gr(4,n)$: The space of 4-dimensional planes passing through origin in n-dimensional space.



- $Gr(4,n)$ cluster algebras provide compactification of *positive region* of kinematics with $\langle ijkl \rangle > 0$ for $i < j < k < l$.
[Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka'12]