Applications of Scattering Amplitudes to Gravitational Waves

SAGEX Final Workshop

June 21, 2022

Zvi Bern

ZB, C. Cheung, R. Roiban, C. H. Shen, M. Solon, M. Zeng,
arXiv:1901.04424 and arXiv:1908.01493.

ZB, A. Luna, R. Roiban, C. H. Shen, M. Zeng,
arXiv:2005.03071

ZB, J. Parra-Martinez, R. Roiban, M. Ruf. C.-H. Shen,
M. Solon, M. Zeng, arXiv:2101.07254; arXiv:2112.10750

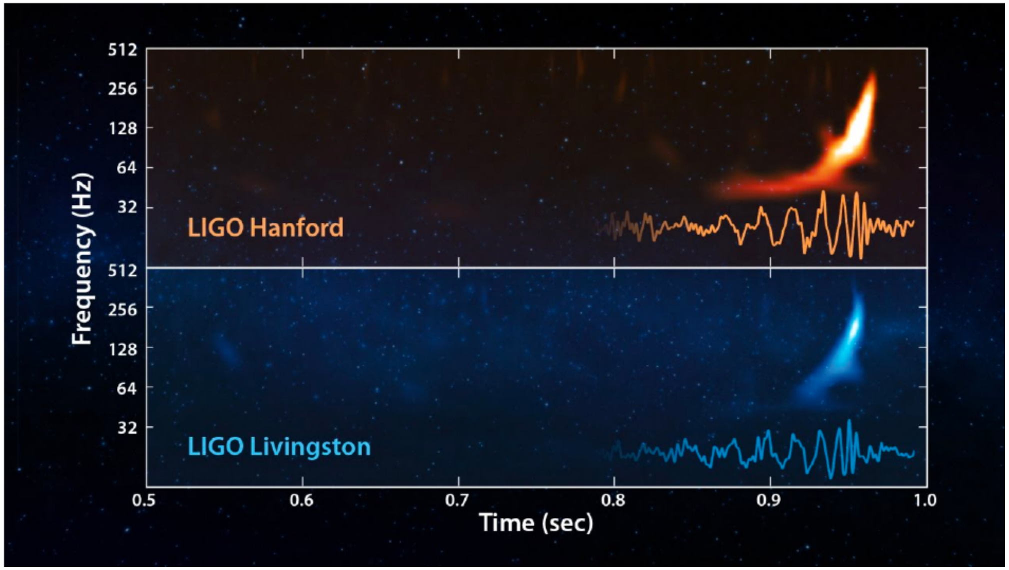
ZB, Kosmopoulos, Luna, Roiban, Teng, arXiv: 2203.06202

SAGEX
Scattering Amplitudes:
from Geometry to Experiment

UCLA Mani L. Bhaumik Institute for Theoretical Physics

Outline

Era of gravitational-wave astronomy has begun.



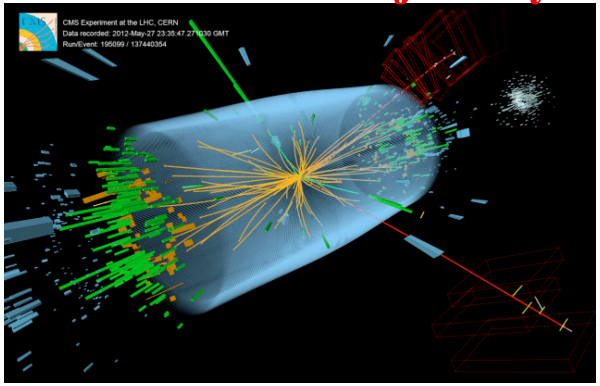
For an instant brighter in gravitational radiation than all the stars in the visible universe are in EM radiation!

How can QCD community, help out with core mission of LIGO/Virgo and future detectors?

Can Particle Theory Help with Gravitational Waves?

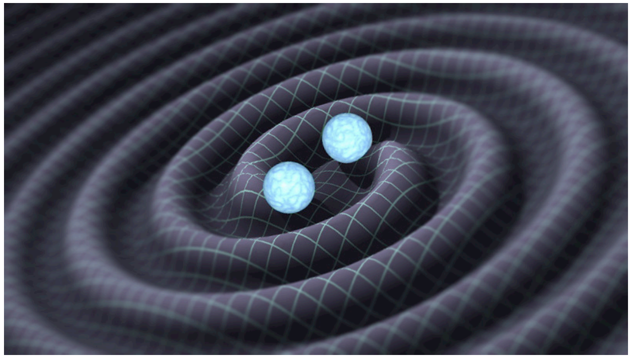
What does particle physics have to do with classical dynamics of astrophysical objects?

unbounded trajectory



gauge theories, QCD, electroweak
quantum field theory

bounded orbit



General Relativity
classical physics

Black holes and neutron stars are point particles as far as long-wavelength radiation is concerned.

Iwasaki (1971); **Goldberger, Rothstein** (2006), Porto; Vaydia, Foffa , Porto, Rothstein, Sturani; Kol; Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Neill and Rothstein; Levi, Steinhoff; Vines etc

Will explain that particle theory is well suited to push state-of-the-art perturbative calculations for gravitational-wave physics.

Importing QCD Methods

In particle physics we are very good at perturbation theory.
Vast experience with gauge theory.

Gravity ~ (gauge theory)^2

From this perspective gravity is similar to gauge theory

Highlight of imported field theory methods:

1. Unitarity method for building integrands. ZB, Dunbar, Dixon, Kosower
2. Method of regions. Beneke and Smirnov
3. NRQCD and EFT methods. Caswell and Lepage; Luke, Manohar, Rothstein
4. IBP reductions for Feynman integrals. Chetyrkin, Tkachov; Laporta
5. Method of differential equations for Feynman integrals. Kotikov, ZB, Dixon and Kosower; Gehrmann, Remiddi; Henn, Smirnov

Leverage advances in perturbative QCD to help with gravitational waves

Can Scattering Amplitudes Help with Gravitational Waves?

Two serious issues for applying this to gravitational waves:

1. We do quantum *not* classical perturbation theory.
2. Scattering process unbounded orbit. Want bound one for binary black hole gravitational-wave emission.

Two key topics for this talk:

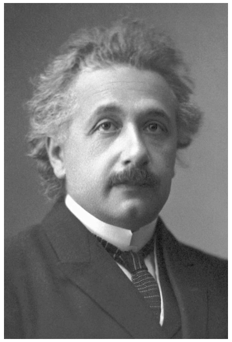
- Modern approach to perturbative gravity.
- How do we effectively deal with the above annoying issues?

Approach to General Relativity

Our appoach does *not* start from usual Einstein Field equations.

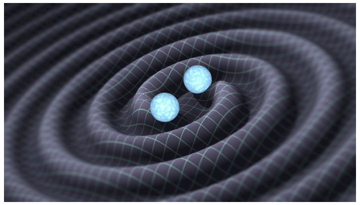
$$\xcancel{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

~~geometry~~



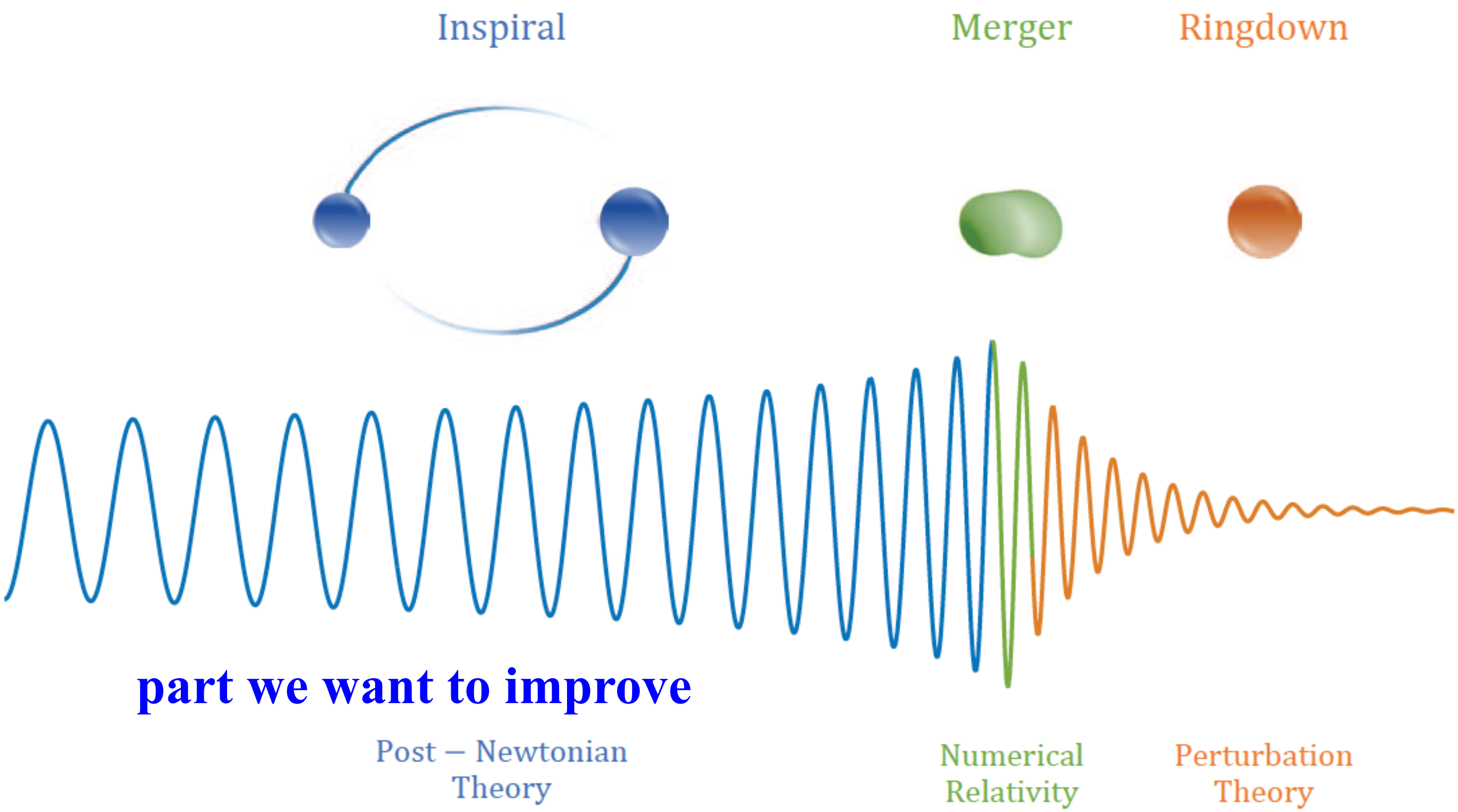
Gravitons are spin 2 particles

- **Not suited for all problems, but works well for asymptotically flat space-times in context of perturbation theory.**
- **Well suited for gravitational-wave physics from compact astrophysical objects**



Two Body Problem

From Antelis and Moreno, arXiv:1610.03567



part we want to improve

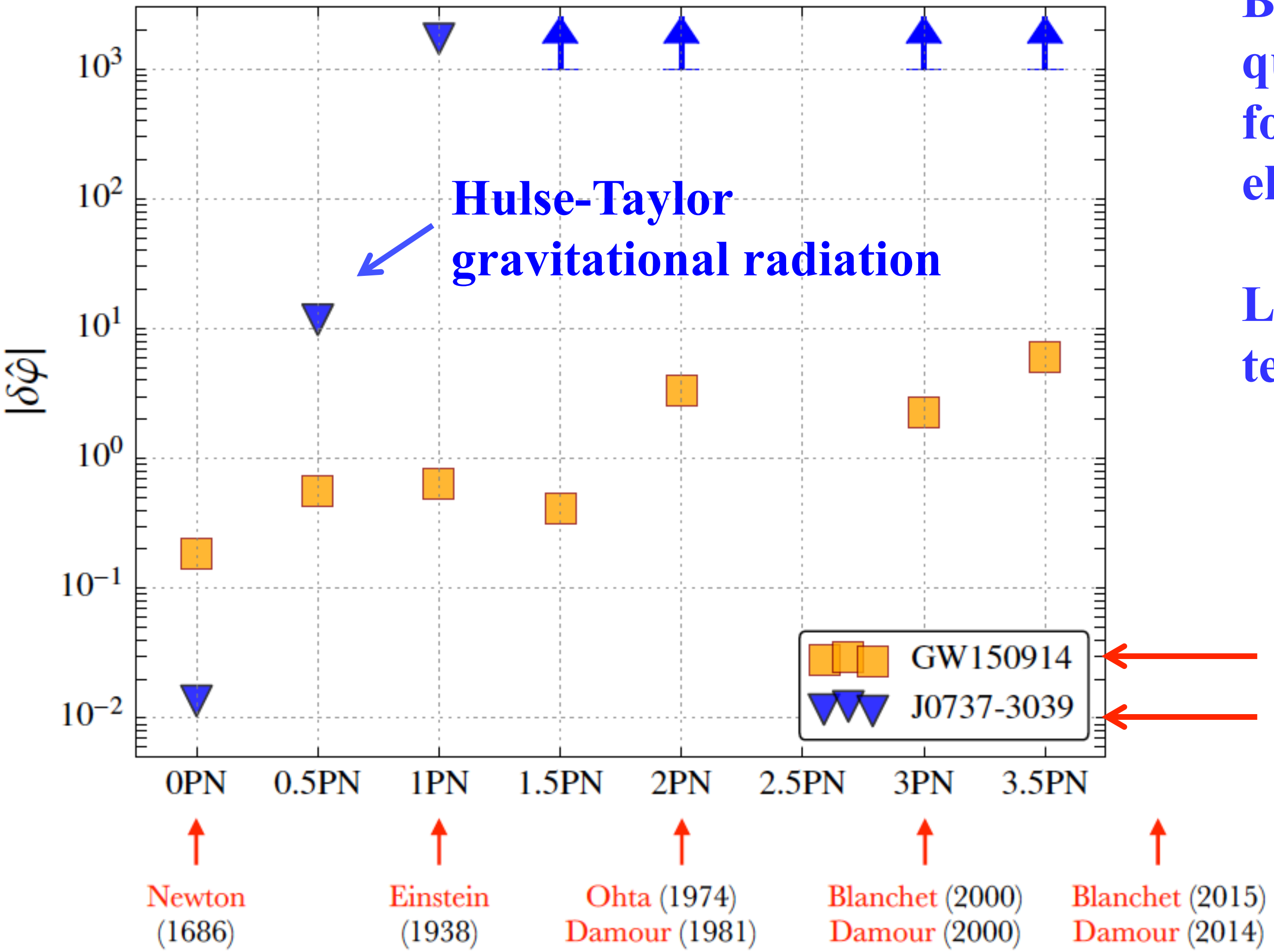
- **Small errors accumulate. Need for high precision.**
- **Input to EOB or other modeling to reliably approach merger.**

Buonanno and Damour

- **Two primary inputs: binding energy and frequency shift.**

Importance of higher orders for LIGO/Virgo

LIGO/Virgo Collaboration arXiv:1602.03841



Hulse-Taylor gravitational radiation

Binary pulsar confirms quadrupole radiation formula and not much else.

LIGO/Virgo tests PN terms from GR

LIGO

Binary pulsar

LIGO/Virgo sensitive to high PN orders.

Post Newtonian Approximation

For orbital mechanics:

$$v^2 \sim \frac{GM}{R} \ll 1$$

Expand in *G* and v^2

virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B/m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R}$$

Newton

$$+ \frac{1}{c^2}\left\{ -\frac{P^4}{8} + \frac{3\,\nu\,P^4}{8} + \frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2} - \frac{3\,P^2}{2} - \frac{\nu\,P^2}{2}\right) + \frac{G^2 m^2}{2R^2}\right\}$$

$$+ \ldots$$

1PN: Einstein, Infeld, Hoffmann; Droste, Lorentz

Hamiltonian known to 4PN order.

2PN: Ohta, Okamura, Kimura and Hiida.

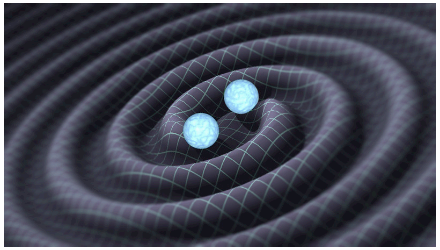
3PN: Damour, Jaranowski and Schaefer; L. Blanchet and G. Faye.

4PN: Damour, Jaranowski and Schaefer; Foffa (2017), Porto, Rothstein, Sturani (2019).

Which problem to solve?

ZB, Cheung, Roiban, Shen, Solon, Zeng

Some problems for (analytic) theorists:

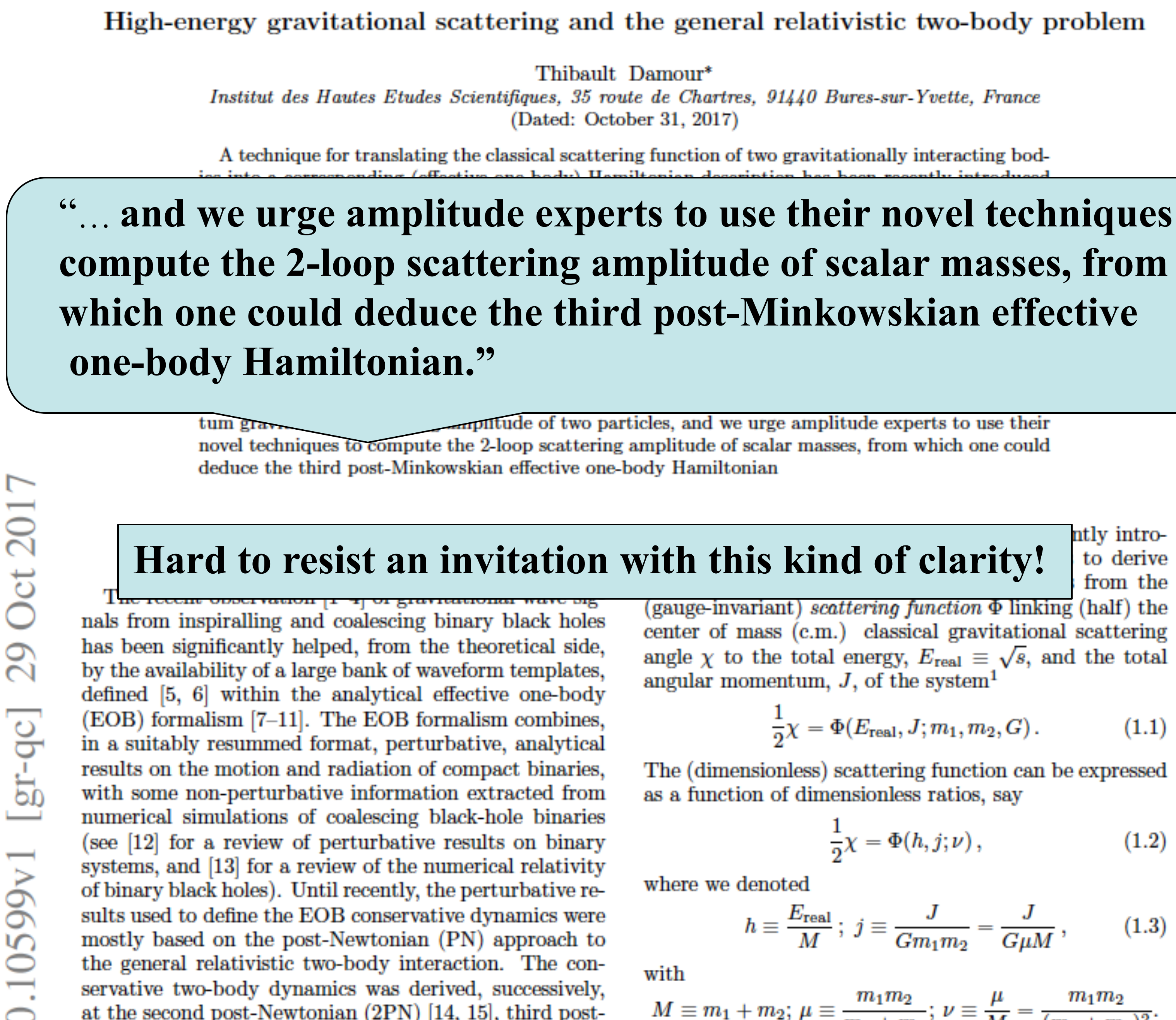


1. **Spin.**
2. **Finite size effects.**
3. **New physics effects.**
4. **Radiation.**
5. **High orders in perturbation theory.**

Which problem should we solve?

- **Needs to be difficult using standard methods.**
- **Needs to be of direct importance to LIGO theorists.**
- **Needs to be in a form that can in principle enter LIGO analysis pipeline.**

2-body Hamiltonian at 3rd and 4th post-Minkowskian order



High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour*
Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

(Dated: October 31, 2017)

A technique for translating the classical scattering function of two gravitationally interacting bod-

tum gra mplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian

"… and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian."

Hard to resist an invitation with this kind of clarity!

nals from inspiralling and coalescing binary black holes has been significantly helped, from the theoretical side, by the availability of a large bank of waveform templates, defined [5, 6] within the analytical effective one-body (EOB) formalism [7–11]. The EOB formalism combines, in a suitably resummed format, perturbative, analytical results on the motion and radiation of compact binaries, with some non-perturbative information extracted from numerical simulations of coalescing black-hole binaries (see [12] for a review of perturbative results on binary systems, and [13] for a review of the numerical relativity of binary black holes). Until recently, the perturbative results used to define the EOB conservative dynamics were mostly based on the post-Newtonian (PN) approach to the general relativistic two-body interaction. The conservative two-body dynamics was derived, successively, at the second post-Newtonian (2PN) [14, 15], third post-

ntly intro-
to derive
from the
(gauge-invariant) *scattering function* Φ linking (half) the center of mass (c.m.) classical gravitational scattering angle χ to the total energy, $E_{\rm real} \equiv \sqrt{s}$, and the total angular momentum, J, of the system[1]

$$\frac{1}{2}\chi = \Phi(E_{\rm real}, J; m_1, m_2, G)\,. \tag{1.1}$$

The (dimensionless) scattering function can be expressed as a function of dimensionless ratios, say

$$\frac{1}{2}\chi = \Phi(h, j; \nu)\,, \tag{1.2}$$

where we denoted

$$h \equiv \frac{E_{\rm real}}{M}\,;\; j \equiv \frac{J}{G m_1 m_2} = \frac{J}{G\mu M}\,, \tag{1.3}$$

with

$M \equiv m_1 + m_2;\ \mu \equiv m_1 m_2 \;;\ \nu \equiv \mu = m_1 m_2$

PN versus PM expansion for conservative two-body dynamics

From Buonanno Amplitudes 2018

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}}_{\downarrow} + \frac{1}{c^2}\Big[\ldots\Big] + \frac{1}{c^4}\Big[\ldots\Big] + \ldots$$

$$E(v) = -\frac{\mu}{2}\,v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	...
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	...
...						...	...	

(credit: Justin Vines)

$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

current known PN results

current known PM results

overlap between PN & PM results

unknown

- **PM results** (Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep)

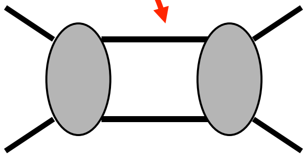
Generalized Unitarity Method

Use simpler tree amplitudes to build higher-order (loop) amplitudes.

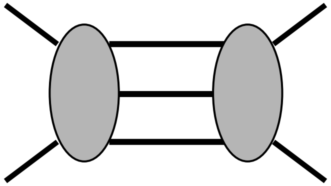
$E^2 = \vec{p}^{\,2} + m^2 \leftarrow$ **on-shell**

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:

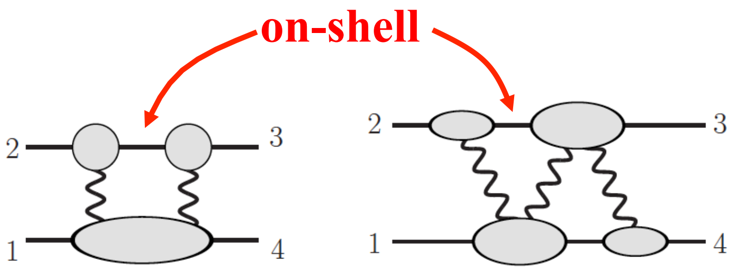


Three-particle cut:



- **Systematic assembly of complete loop amplitudes from tree amplitudes.**
- **Works for any number of particles or loops.**

Generalized unitarity as a practical tool for loops.



ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

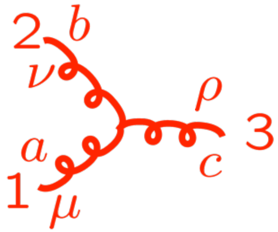
Idea used in the "NLO revolution" in QCD collider physics.
No gauge fixing in the formalism.

Simplicity of Gravity Scattering Amplitudes

People were looking at perturbative gravity amplitudes the wrong way.

On-shell **three vertices contains all information:** $E_i^2 - \vec{p}_i^{\,2} = 0$

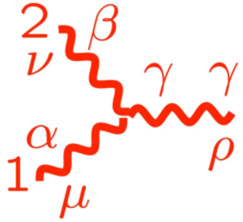
QCD:



color factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic})$$

Einstein gravity:



$$i\kappa(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1-k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Starting from this on-shell vertex any multi-loop amplitude can be constructed via modern methods

Gravitons are like two gluons!

Gravity ~ (gauge theory)^2

KLT Relation Between Gravity and Gauge Theory

KLT (1985)

Kawai-Lewellen-Tye string relations in low-energy limit:

gravity

gauge-theory color ordered

$$M_4^{\text{tree}}(1,2,3,4) = -i s_{12} A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)\,,$$

$$M_5^{\text{tree}}(1,2,3,4,5) = i s_{12} s_{34} A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5)$$

$$+ i s_{13} s_{24} A_5^{\text{tree}}(1,3,2,4,5)\, A_5^{\text{tree}}(3,1,4,2,5)$$

Inherently gauge invariant!

Generalizes to explicit all-leg form.

ZB, Dixon, Perelstein, Rozowsky

1. **Gravity amplitudes derivable from gauge theory.**
2. **Once gauge-theory amplitude is simplified, so is gravity.**
3. **Standard Lagrangian methods offer no hint why this is possible.**

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (BCJ)

Consider five-point tree amplitude:

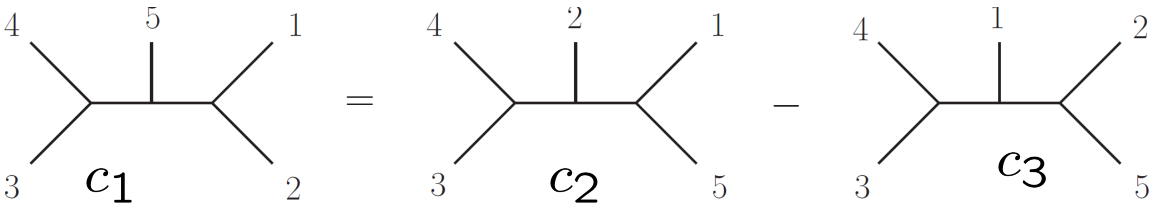
color factor

kinematic numerator factor

gauge theory

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

Feynman propagators



$$c_1 = f^{a_3a_4b}f^{ba_5c}f^{ca_1a_2} \qquad c_2 = f^{a_3a_4b}f^{ba_2c}f^{ca_1a_5} \qquad c_3 = f^{a_3a_4b}f^{ba_1c}f^{ca_2a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 + c_2 + c_3 = 0 \;\Leftrightarrow\; n_1 + n_2 + n_3 = 0}$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

Proven at tree level

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

Gravity from Gauge Theory

ZB, Carrasco, Johansson

gauge theory (QCD):

$$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i \, n_i}{D_i}$$

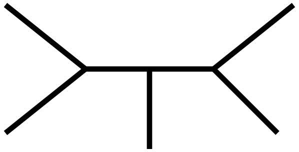
color factor

kinematic numerator factor

Feynman propagators

$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

$$\boxed{c_i \to n_i}$$



sum over diagrams with only 3 vertices

Einstein gravity:

$$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

Gravity and gauge theory kinematic numerators are the same!

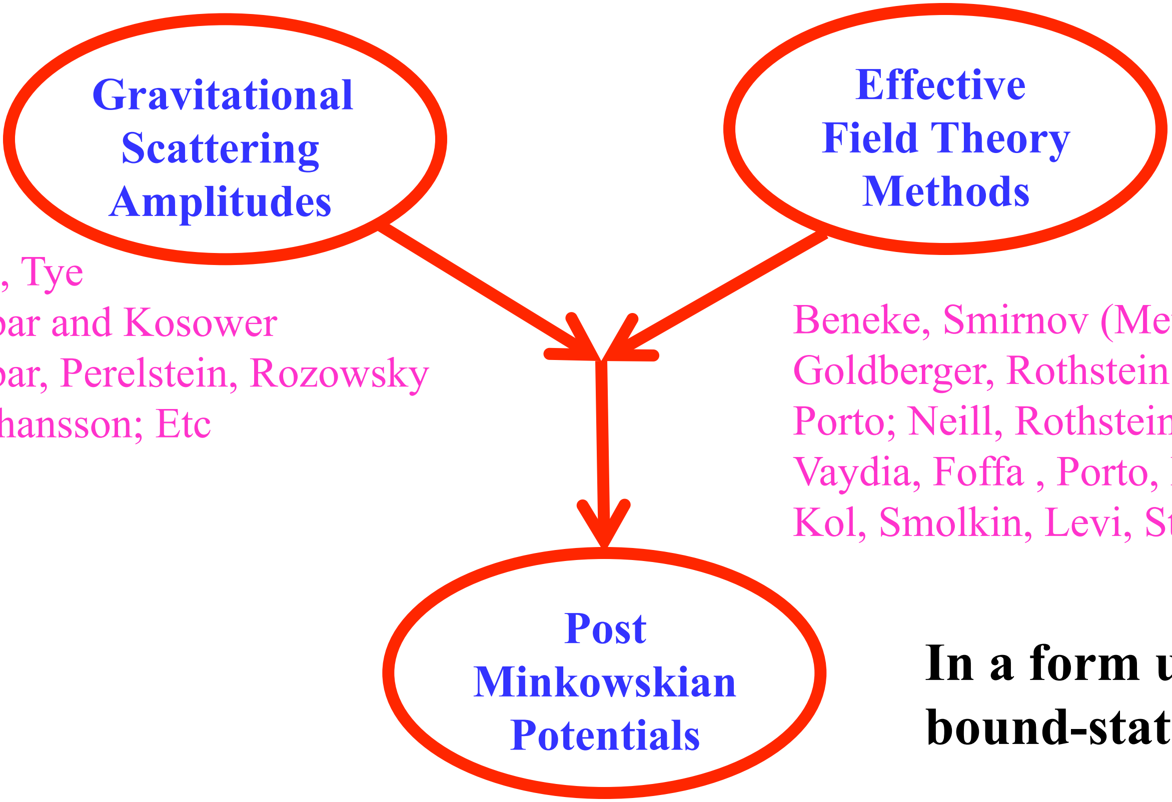
Gravity is just an ordinary gauge theory with color replaced with gauge-theory kinematic factors.

Directly feeds into our gravitational 2-body potential calculations

Effective Field Theory Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng

Cheung, Rothstein, Solon (2018)



Amplitudes community

Gravitational Scattering Amplitudes

Effective Field Theory Methods

EFT community

Kawai, Lewellen, Tye
ZB, Dixon, Dunbar and Kosower
ZB, Dixon, Dunbar, Perelstein, Rozowsky
ZB, Carrasco, Johansson; Etc

Beneke, Smirnov (Method of regions)
Goldberger, Rothstein;
Porto; Neill, Rothstein;
Vaydia, Foffa , Porto, Rothstein, Sturani;
Kol, Smolkin, Levi, Steinhoff, etc.

Post Minkowskian Potentials

In a form useful for bound-state problem

The EFT directly gives us a two-body Hamiltonian of a form appropriate to enter LIGO analysis pipeline (after importing into EOB or pheno models).

We prefer the EFT matching when pushing into new territory.

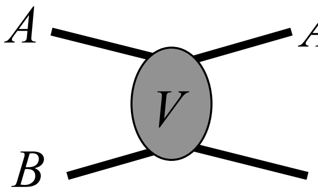
EFT is a Clean Approach

No need to re-invent the wheel.
Build EFT from which we can read off potential.

Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

$$L_{\rm kin} = \int_{\boldsymbol{k}} A^\dagger(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_A^2} \right) A(\boldsymbol{k}) + \int_{\boldsymbol{k}} B^\dagger(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_B^2} \right) B(\boldsymbol{k})$$

A, B scalars represents spinless black holes



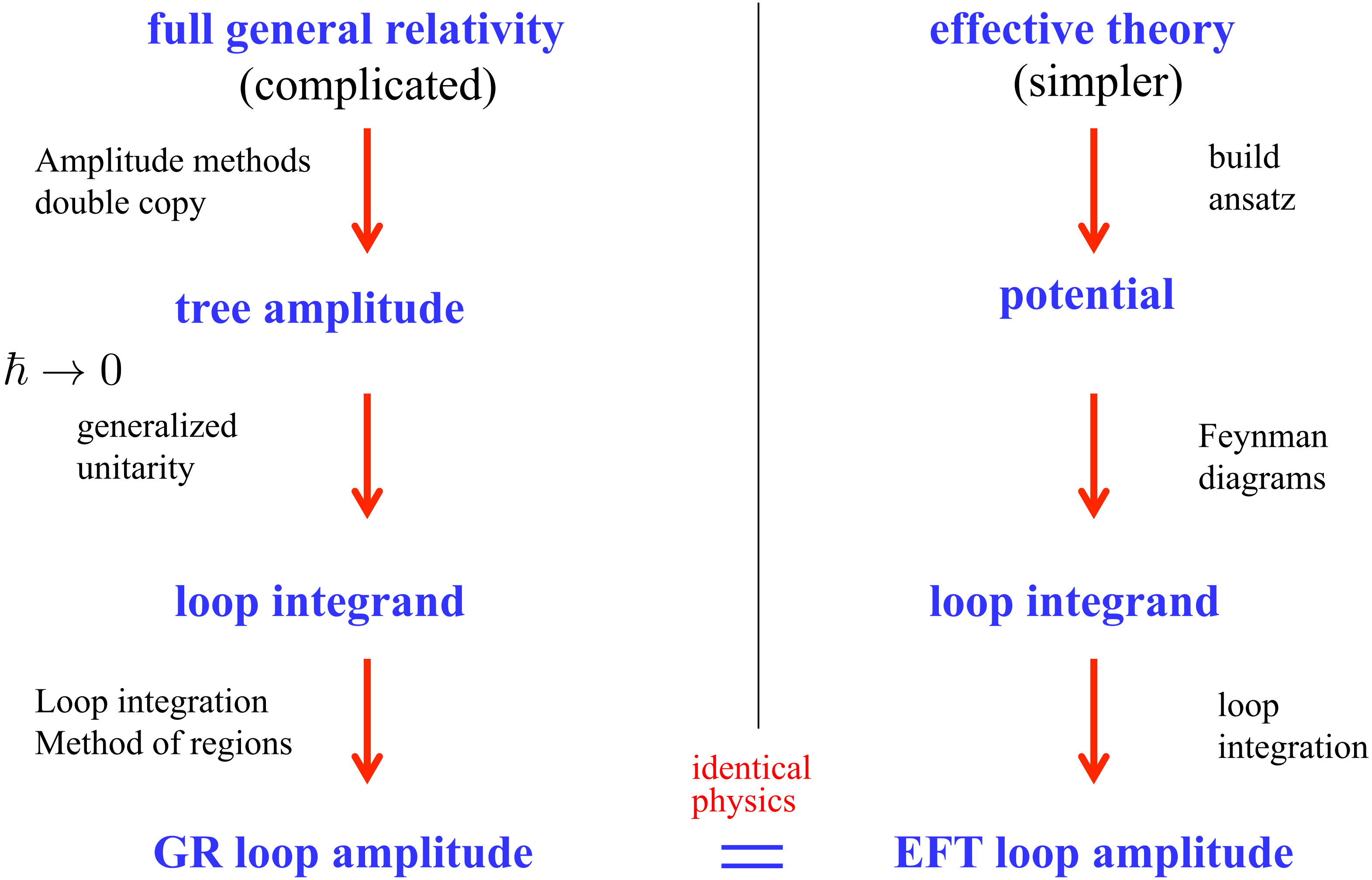
$$L_{\rm int} = -\int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\, A^\dagger(\boldsymbol{k}') A(\boldsymbol{k}) B^\dagger(-\boldsymbol{k}') B(-\boldsymbol{k})$$

two body potential

Match amplitudes of this theory to the full theory in classical limit to extract a potential which can then be directly used for bound state.

The EFT is used to define the potential and 2 body Hamiltonian. This gives us binding energy.

EFT Matching



full general relativity

(complicated)

Amplitude methods
double copy

tree amplitude

$\hbar \to 0$

generalized
unitarity

loop integrand

Loop integration
Method of regions

GR loop amplitude

effective theory

(simpler)

build
ansatz

potential

Feynman
diagrams

loop integrand

loop
integration

EFT loop amplitude

identical
physics

$=$

Roundabout but efficiently determines potential.

Alternative Methods

There are now multiple alternative ways that bypass EFT matching and subtraction of iterations.

- **Calculate physical observables** Kosower, Maybee, O'Connell
- **Eikonal Phase** Amati, Ciafaloni, Veneziano; Di Vecchia, Heissenberg, Russo, Veneziano
- **Amplitude Action Relation** ZB, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng
- **Exponential representation** Damgaard, Plante, Vanhove
- **Heavy mass field theory** Brandhuber, Chen, Travaglini, Wen; Damgaard, Haddad, Helset
- **World line formalisms** Goldberger, Rothstein; Levi, Steinhoff; Dlapa, Kälin, Liu, Porto; Jakobson, Mogul, Plefka, Steinhoff.

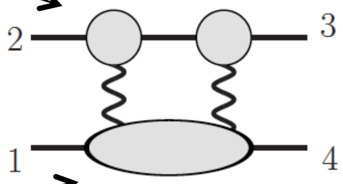
For pushing into new territory we still prefer EFT or KMOC.
All are fine. Key issue at high orders is efficient loop integration.

General Relativity: Unitarity + Double Copy

- **Long-range force: Two matter lines must be separated by on-shell propagators.**
- **Classical potential: 1 matter line per loop is cut (on-shell).**

Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for 2 PM 2 body Hamiltonian.



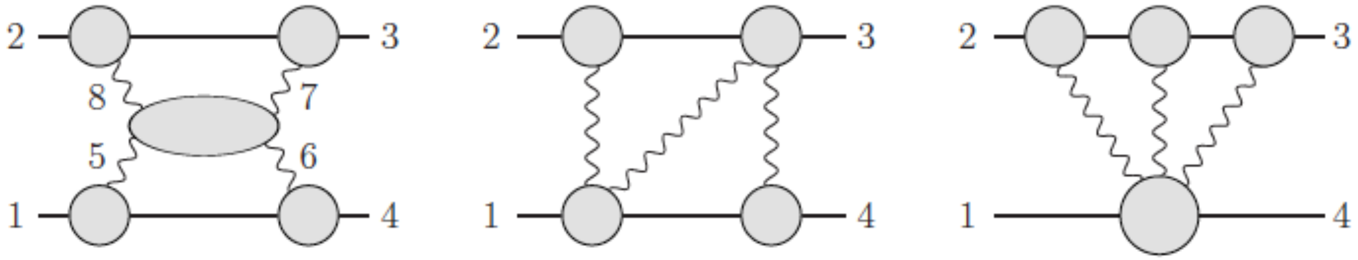
3 vertex, EM tensor

on-shell

Compton

Treat exposed lines on-shell (long range).
Pieces we want are simple!

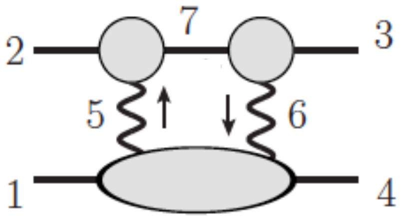
Independent generalized unitarity cuts for 3 PM.



Our amplitude tools fit perfectly with extracting pieces we want.

gravity **loops**

Generalized Unitarity Cuts



2^{nd} post-Minkowkian order

KLT relations

$$\begin{aligned}C_{\text{GR}} &= \sum_{h_5,h_6=\pm} M_3^{\text{tree}}(3^s,6^{h_6},-7^s)\,M_3^{\text{tree}}(7^s,-5^{h_5},2^s)\,M_4^{\text{tree}}(1^s,5^{-h_5},-6^{-h_6},4^s)\\ &= \sum_{h_5,h_6=\pm} it[A_3^{\text{tree}}(3^s,6^{h_6},-7^s)\,A_3^{\text{tree}}(7^s,-5^{h_5},2^s)\,A_4^{\text{tree}}(1^s,5^{-h_5},-6^{-h_6},4^s)]\\ &\qquad\qquad \times [A_3^{\text{tree}}(3^s,6^{h_6},-7^s)\,A_3^{\text{tree}}(7^s,-5^{h_5},2^s)\,A_4^{\text{tree}}(4^s,5^{-h_5},-6^{-h_6},1^s)]\end{aligned}$$

Problem of computing the generalized cuts in gravity is reduced to multiplying and summing gauge-theory tree amplitudes.

$$A_4^{\text{tree}}(1^s,2^+,3^+,4^s) = i\frac{m^2\,[2\,3]}{\langle 2\,3\rangle\,\tau_{12}} \qquad A_4^{\text{tree}}(1^s,2^+,3^-,4^s) = i\frac{\langle 3|\,1\,|2]^2}{s_{23}\tau_{12}}$$

$$\tau_{12} = 2p_1\cdot p_2$$

$$s_{23} = (p_1+p_2)^2$$

- **For spinless case, same logic works to all orders: KLT and BCJ work for massless *n*-point in *D*-dimension. Dimensional reduction gives massive case.**

Amplitude in Conservative Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (BCRSSZ)

To make story short. The O(G^3) or 3PM conservative terms are:

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2\xi}\Bigg[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3+12\sigma^2-4\sigma^4\right)\boxed{\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\Bigg] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}\Big[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32m^2\nu^2\left(1-2\sigma^2\right)^3 F_2\Big]$$

$$m = m_1 + m_2 \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

- **Amplitude remarkably compact.**
- **Arcsinh and the appearance of a mass singularity is new and robust feature. Cancels mass singularity of real radiation. No surprise.**

  Di Vecchia, Heissenberg, Russo, Veneziano; Damour
- **IR finite parts of amplitude directly connected to scattering angle.**

  Expanded on by Kälin, Porto; Bjerrum-Bohr, Cristofoli, Damgaard
- **Derived conservative scattering angle has simple mass dependence.**

  Observed by Antonelli, Buonanno, Steinhoff, van de Meent, Vines (1901.07102)
  Comprehensive understanding: Damour

Conservative O(G^3) 2-body Hamiltonian

BCRSSZ

The O(G^3) 3PM Hamiltonian:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})$$

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{3} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i,$$

Newton in here

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1 - 2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1 - 5\sigma^2\right) - \frac{4\nu\sigma\left(1 - 2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1 - 2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\left[\frac{1}{12}\left(3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3\right) - \frac{4\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}}\right.$$
$$- \frac{3\nu\gamma\left(1 - 2\sigma^2\right)\left(1 - 5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7 - 20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2\right)\left(1 - 2\sigma^2\right)}{4\gamma^3\xi^2}$$
$$\left. + \frac{2\nu^3(3 - 4\xi)\sigma\left(1 - 2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1 - 2\xi)\left(1 - 2\sigma^2\right)^3}{2\gamma^6\xi^4}\right],$$

$$m = m_1 + m_2 \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$
$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

- **Expanding in velocity gives infinite sequence of terms in PN expansion.**
- **Can be put into EOB form.** Antonelli, Buonannom Steinhoff,van de Meent, Vines

How do we know it is right?

Original check:

Compared to 4PN Hamiltonians after canonical transformation

Damour, Jaranowski, Schäfer; Bernard, Blanchet, Bohé, Faye, Marsat

Thibault Damour seriously questioned correctness.
Specific corrections proposed. Damour, arXiv:1912.02139v1

Subsequent calculations confirm our 3PM result:

1. **Papers confirming our result in 6PN overlap.**
   Blümlein, Maier, Marquard, Schäfer;
   Bini, Damour, Geralico
2. **Subsequent calculations reproducing our 3PM result.**
   Cheung and Solon; Kälin, Liu, Porto
3. **Scattering angle checks.** ZB, Ita, Parra-Martinez, Ruf
4. **Adding real radiation removes mass singularity.**
   Di Vecchia, Heissenberg, Russo, Veneziano; Damour

3PM results have passed highly nontrivial checks and careful scrutiny.

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$\widehat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$

$$c^2\,\widehat{H}_{1\mathrm{PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\Big\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r} + \frac{1}{2r^2},$$

$$c^4\,\widehat{H}_{2\mathrm{PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\Big\{\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r}$$
$$+\frac{1}{2}\Big\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},$$

$$c^6\,\widehat{H}_{3\mathrm{PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4 + \frac{1}{16}\Big\{\left(-7+42\nu-53\nu^2-5\nu^3\right)(\mathbf{p}^2)^3$$
$$+(2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\Big\}\frac{1}{r}$$
$$+\Big\{\frac{1}{16}\left(-27+136\nu+109\nu^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r^2}$$
$$+\Bigg\{\Bigg(-\frac{25}{8} + \left(\frac{\pi^2}{64} - \frac{335}{48}\right)\nu - \frac{23\nu^2}{8}\Bigg)\mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\right)\nu(\mathbf{n}\cdot\mathbf{p})^2\Bigg\}\frac{1}{r^3} + \Bigg\{\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\nu\Bigg\}\frac{1}{r^4},$$

G^4

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$
\begin{aligned}
c^8\, \widehat{H}^{\text{local}}_{\text{4PN}}(\mathbf{r},\mathbf{p}) = {} & \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right)(\mathbf{p}^2)^5 \\
& + \left\{ \frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\,\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2\right)\nu^2 \right. \\
& + \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^3 \\
& \left. + \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^4 \right\}\frac{1}{r} \\
& + \left\{ \frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu \right. \\
& + \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^2 \\
& \left. + \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^3 \right\}\frac{1}{r^2} \\
& + \left\{ \frac{105}{32}(\mathbf{p}^2)^2 + \left(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200}\right)(\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu \right. \\
& + \left(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800}\right)(\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^2 \\
& \left. + \left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^3 \right\}\frac{1}{r^3} \\
& + \left\{ \frac{105}{32}\mathbf{p}^2 + \left(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192}\right)\mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu \right. \\
& \left. + \left(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152}\right)\mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu^2 \right\}\frac{1}{r^4} \qquad \longleftarrow \; G^4 \\
& + \left\{ -\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400}\right)\nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45}\right)\nu^2 \right\}\frac{1}{r^5}. \qquad \longleftarrow \; G^5
\end{aligned}
$$

Mess is partly due to gauge choice and also expansion in velocity.

Ours is all orders in *p* at *G*^{3}

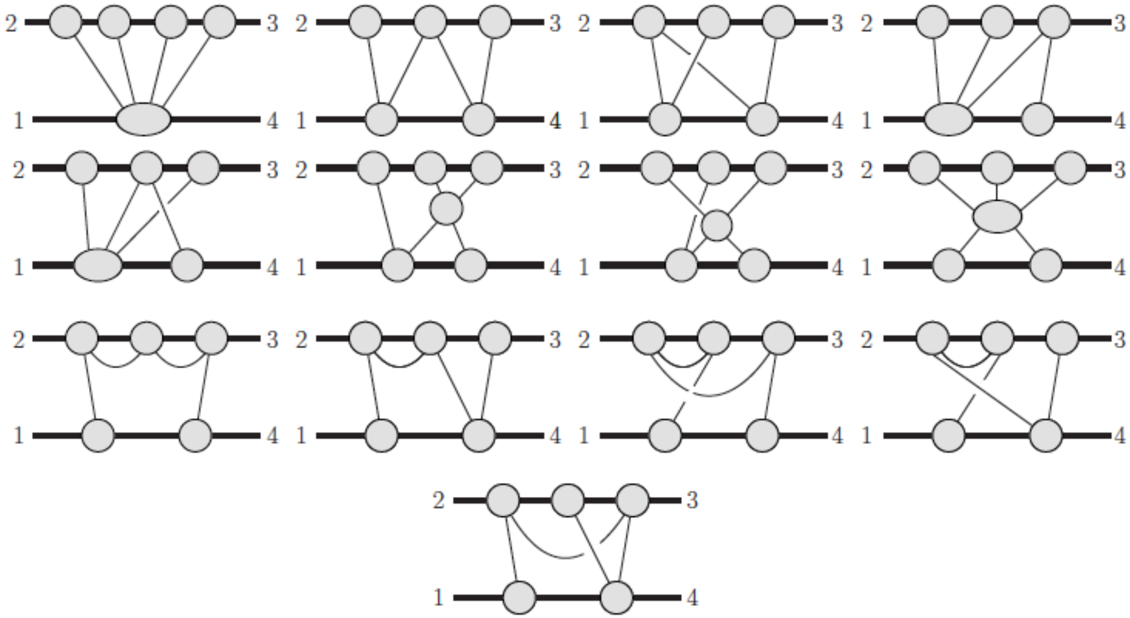
After a canonical transformation this matches our result in overlap

Higher Order Scalability: $O(G^4)$

Methods scale well to higher orders

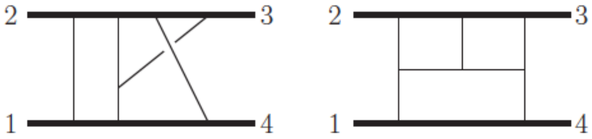
ZB, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng (2022)

At 4PM or $O(G^4)$ similar except cuts more complicated and integrals significantly harder.



High Loop Integration

ZB, Parra-Martinez, Roiban, Ruf. Shen, Solon, Zeng



Integration more challenging than at 2 loops.

Developed a new hybrid approach that combines ideas from various methods:

1. **Method of regions to separate potential and radiation.** Beneke and Smirnov
2. **Nonrelativistic integration. Velocity expand and then mechanically integrate. Get first few orders in velocity. Boundary conditions.** Cheung, Rothstein, Solon
3. **Integration by parts and differential equations. Imported from QCD.** Chetyrkin, Tkachov; Laporta; Kotikov, Bern, Dixon and Kosower; Gehrmann, Remiddi.

   Single scale integrals! **Parra-Martinez, Ruf, Zeng**

IBP: $$0 = \int \prod_i^L \frac{d^D \ell_i}{(2\pi)^D} \frac{\partial}{\partial \ell_i^\mu} \frac{N^\mu(\ell_k, p_M)}{Z_1 \ldots Z_n}$$

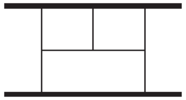
Solve linear relations between integrals in terms of master integrals.

DEs: $$\frac{\partial}{\partial s_i} I_j^{\text{master}} = \text{simplified via IBP}$$

Solve DEs either as series or basis of functions.

- **Many tools available: We use FIRE6, which is more than sufficient at 3 loops.** Smirnov, Chuharev
- **Elliptic integrals make an appearance. At end just a minor annoyance.**

Complete Conservative Contribution O(G^4)



ZB, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng (2022)

O(G^4) amplitude

test particle

$$\mathcal{M}_4^{\rm cons} = G^4 M^7 \nu^2 |\boldsymbol{q}| \pi^2 \left[\mathcal{M}_4^{\rm p} + \overbrace{\nu \left(4\mathcal{M}_4^{\rm t} \log\left(\tfrac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem}\right)}^{\text{1st self force}}\right] + \overbrace{\underbrace{\int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1} \tilde{I}_{r,3}}{Z_1}} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1}}^{\text{Iteration. No need to compute}}$$

tail effect

$$D = 4 - 2\epsilon$$

$$\mathcal{M}_4^{\rm t} = r_1 + r_2 \log\left(\tfrac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}, \qquad \mathcal{M}_4^{\rm p} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2-1\right)}$$

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5 \,\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_6 \,\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_7 \,\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right), \longleftarrow \text{elliptic}$$

$$\begin{aligned}\mathcal{M}_4^{\rm rem} &= r_8 + r_9 \log\left(\tfrac{\sigma+1}{2}\right) + r_{10} \tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11} \log(\sigma) + r_{12} \log^2\left(\tfrac{\sigma+1}{2}\right) + r_{13} \tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\tfrac{\sigma+1}{2}\right) + r_{14} \tfrac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} \\ &\quad + r_{15} \,\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + r_{16} \,\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) + r_{17} \tfrac{1}{\sqrt{\sigma^2-1}} \left[\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right].\end{aligned}$$

$$\nu = m_1 m_2/(m_1+m_2)^2 \qquad \sigma = p_1 \cdot p_2 / m_1 m_2,$$

r_i rational coefficients

This is *complete* conservative contribution.

$$\mathcal{M}_4^{\rm radgrav,f} = \frac{12044}{75} p_\infty^2 + \frac{212077}{3675} p_\infty^4 + \frac{115917979}{793800} p_\infty^6 - \frac{9823091209}{76839840} p_\infty^8 + \frac{115240251793703}{1038874636800} p_\infty^{10} + \cdots$$

First 3 terms match 6PN results of Bini, Damour, Geralico!

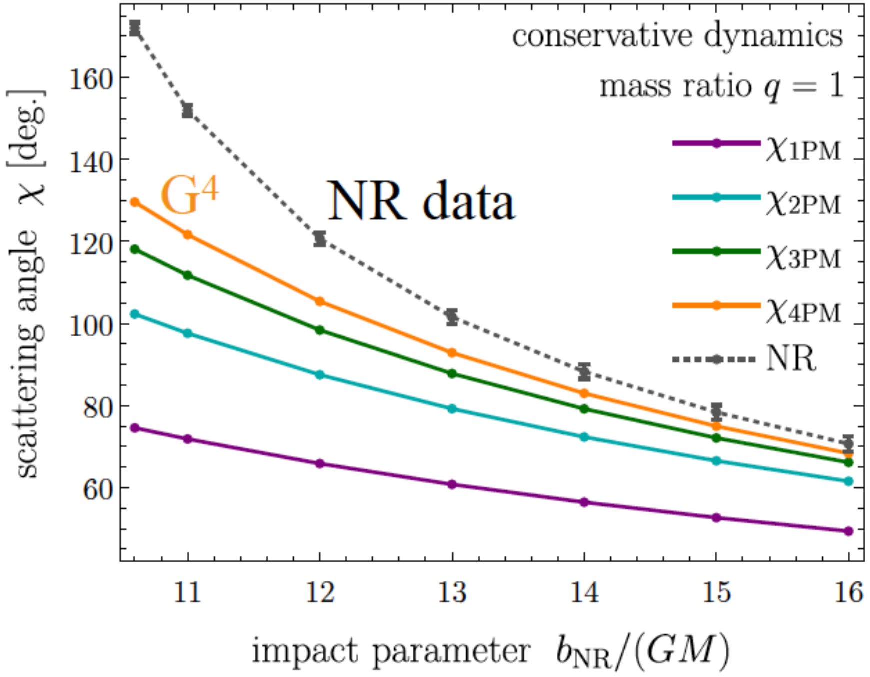
- **Some disagreement in one term with 6PN of Blumlein, Maier, Marquard, Schafer.**
- **Claim we and Bini et al dropped "memory" pieces by Diapa, Kalin, Liu, Porto.**

Comparison with Numerical Relativity

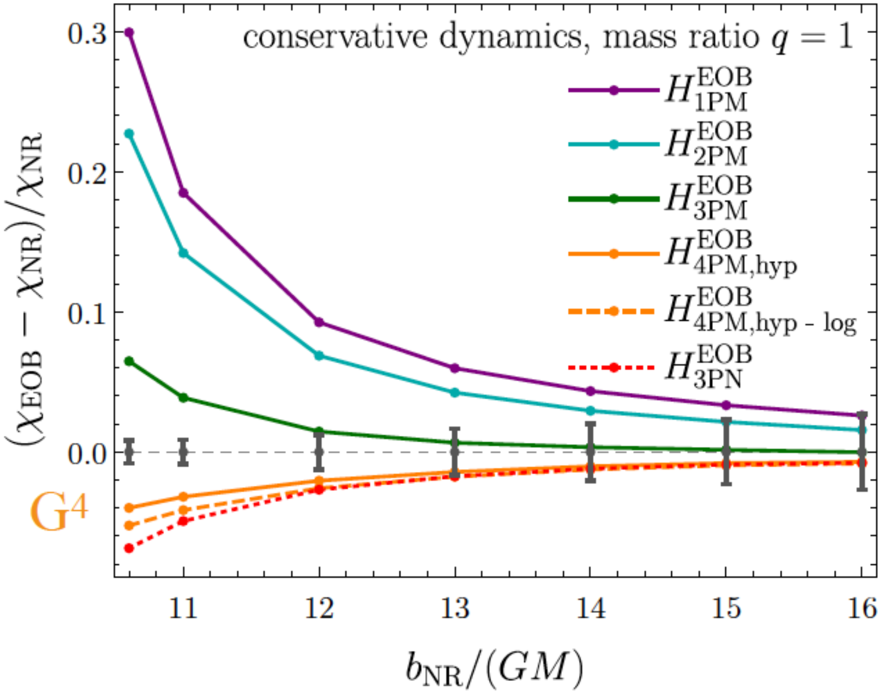
Khalil, Buonanno, Steinhoff, Vines

As an interesting check compare to numerical relativity:

Original angle in PM perturbation



EOB-improved angle



Numerical data from Damour, Guercilena, Hinder, Hopper, Nagar, Rezzolla

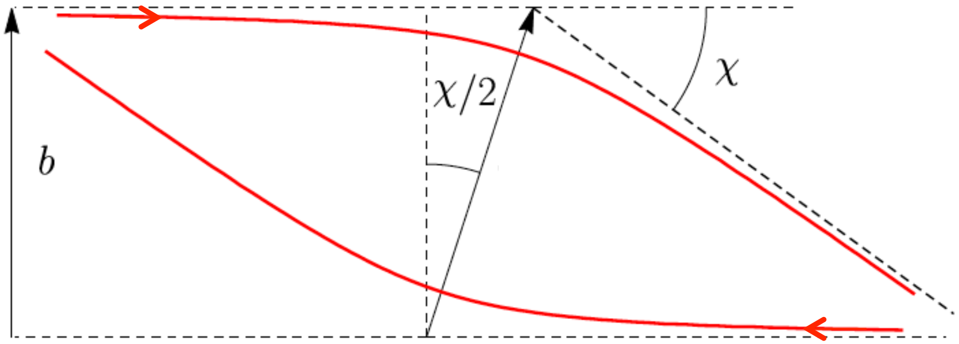
Surprisingly good agreement with numerical relativity.

Issues to be Resolved

1) **Disagreements with Bluemlein et al and Porto et al still needs to be fully resolved, though good progress.**

Progress: Damour's "Good Polynomiality": No 1/($m_1 + m_2$). Manifest in our scattering amplitude formalism. Not manifest in other approaches.

2) **We give a Hamiltonian valid for (unbound) hyperbolic motion. But analytic continuation (bound) elliptic motion at $O(G^4)$ is nontrivial. Needs to be resolved.**



Outlook

Amplitude methods have a lot of promise and their use has already been tested for a variety of nontrivial problems.

- **Pushing state of the art for high orders in G.** ZB, Cheung, Roiban, Parra-Martinez, Ruf, Shen, Solon, Zeng
- **Radiation.** Cristofoli, Gonzo, Kosower, O'Connell; Herrmann, Parra-Martinez, Ruf, Zeng; Di Vecchia, Heissenberg, Russo, Veneziano
- **Finite size effects.** Cheung and Solon; Haddad and Helset; Kälin, Liu, Porto; Cheung, Shah, Solon; ZB, Parra-Martinez, Roiban, Sawyer, Shen
- **Spin.** Vaidya; Geuvara, O'Connell, Vines; Chung, Huang, Kim, Lee; ZB, Luna, Roiban, Shen, Zeng; Kosmopoulos, Luna; Damgaard, Haddad, Helset; Aoude, Haddad, Helset;

We seem to have more Wilson coefficients than our GR friends. Is it physical or is it technical? Stay tuned… ZB, Kosmopoulos, Luna, Roiban, Teng

Amplitude methods are becoming a standard tool for gravitational wave problems.

Summary

- Particle physics give us new ways to think about problems of current interest in general relativity.
- Double-copy idea gives a unified framework for gravity and gauge theory.
- Combining with EFT methods gives a powerful tool for gravitational-wave physics in language LIGO/Virgo can use.
- Pushed state of the art: O(G^4) Newtonian-like conservative part.
- Methods nowhere close to exhausted.
- Higher orders in G, resummations in G, spin, finite-size effects, radiation and dissipation obvious paths to pursue.

Tools developed over the years for carrying out perturbative QCD computations are well suited for gravitational-wave problem and are being used to push the state of the art.