Graviton particle statistics from classical scattering amplitudes

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SAGEX
Scattering Amplitudes:
from Geometry to Experiment

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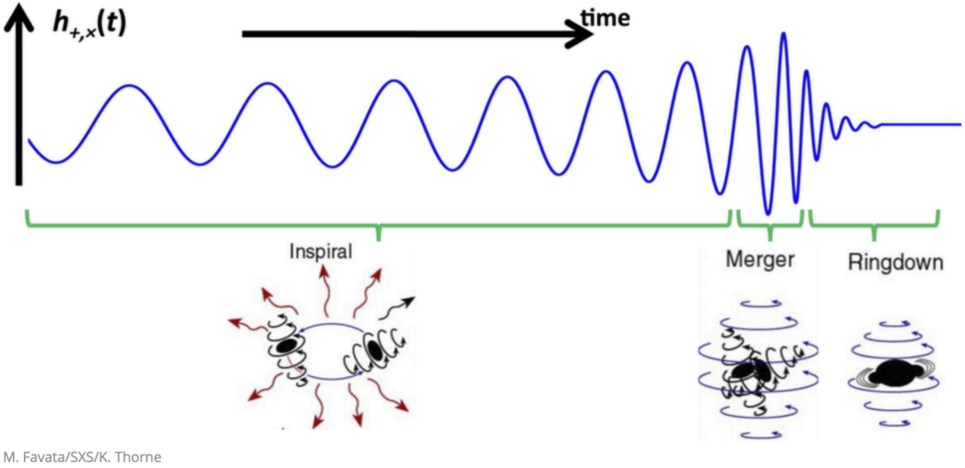
Mergers of binary black holes produce detectable gravitational waves



Detailed prediction of waveform needed for precision studies and tests of new physics. Techniques from amplitudes can help in the inspiral phase.

Classical gravitational observables from amplitudes

- Classical observables are analytically continued from bound orbits to hyperbolic scattering orbits. [Kälin, Porto]

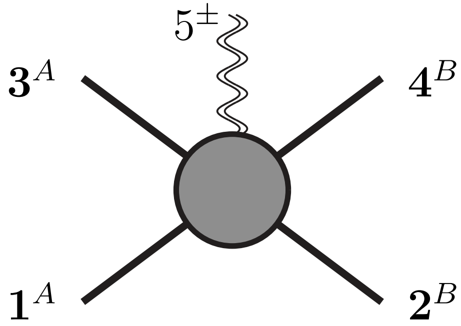
- Scatter two wavepackets with impact parameter b^μ and measure the change in an observable $\mathcal{O}$. [Kosower, Maybee, O'Connell]

$$\Delta\mathcal{O} = \langle \text{out} \,|\, \mathcal{O} \,|\, \text{out} \rangle - \langle \text{in} \,|\, \mathcal{O} \,|\, \text{in} \rangle$$

- S-matrix relations: $|\,\text{out}\rangle = S\,|\,\text{in}\rangle, \quad S = 1 + iT.$

Gravitational radiation

- Massless particles aren't localized
- Classical wave in a pure state ~ coherent state
- Coherence characterizes statistics of emitted particles: Poisson distribution



- For the two-body problem, we consider particle statistics of emitted gravitons and examine deviations from Poisson distribution.

- Aim: identify amplitudes that do/don't contribute to the classical limit.

- Related work motivates the close study of 2-graviton emission at tree level. [Luna, Nicholson, O'Connell, White; Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White]

- Amplitudes computed from a Lagrangian with an auxiliary field

- On-shell recursion gives compact expressions, better suited for analysis of the classical limit.

Classical observables: the KMOC formalism

[Kosower, Maybee, O'Connell]

$$|\psi_{\text{in}}\rangle = \int d\Phi(p_1, p_2)\, e^{ib\cdot p_1/\hbar}\, \psi_A(p_1)\psi_B(p_2)\,|p_1 p_2\rangle,$$

where $d\Phi(p) = d^4p\, \delta(p^2 - m^2)\, \theta(p^0)$,

and $\psi(p) \sim m^{-1} \exp\left[-\dfrac{p\cdot v}{\hbar \ell_c/\ell_w^2}\right]$.

Separation of scales for classical scattering: $\ell_c \ll \ell_w \ll b$.

Massive particles localized on classical trajectories as $\hbar \to 0$.

Coherent states for gravitons

[Cristofoli, Gonzo, Kosower, O'Connell]

- Massless particles (e.g. gravitons) cannot be localized; single gravitons are not classical.

- Every quantum state of radiation (or density matrix) is a superposition of coherent states. [Glauber]

- But our in state $|\psi_{\text{in}}\rangle$ is pure, and the S matrix is unitary, so the out state is a pure state. Thus it must be a single coherent state in the classical limit $\hbar \to 0$. [Hillery]

- Coherent state for a graviton of momentum k and helicity σ:

$$|\alpha_k^\sigma\rangle = \exp\left[\alpha_k a_\sigma^\dagger(k) - \alpha_k^* a_\sigma(k)\right]|0\rangle$$

[Glauber-Suradashan]

- Promote to infinite superposition of momenta:

$$|\alpha^\sigma\rangle = \exp\left[\int d\Phi(k)(\alpha(k)a_\sigma^\dagger(k) - \alpha^*(k)a_\sigma(k))\right]|0\rangle$$

Coherence and the Poisson distribution

Coherent state expanded in graviton-number states:

$$|\alpha^\sigma\rangle = \exp\left(-\frac{1}{2}\int d\Phi(k)\,|\alpha^\sigma(k)|^2\right)\sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{i=1}^{n}\left[d\Phi(k_i)\alpha^\sigma(k_i)\right]|k_1^\sigma\ldots k_n^\sigma\rangle$$

Probability of detecting n gravitons with helicity σ':

$$P_n^{\sigma'} = \delta_{\sigma\sigma'}\exp\left(-\int d\Phi(k)\,|\alpha^\sigma(k)|^2\right)\frac{1}{n!}\left(\int d\Phi(k)\,|\alpha^\sigma(k)|^2\right)^n$$

Poisson statistics are equivalent to coherence of the state.

Counting emitted gravitons

[cf. Gelis, Venugopalan in QCD]

Probability of emitting n gravitons:

$$\bar{P}_n = \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n=\pm} \int d\Phi(p_3) d\Phi(p_4) \int d\Phi(k) \,|\, \langle k_1^{\sigma_1} \ldots k_n^{\sigma_n} p_3 p_4 \,|\, S \,|\, p_1 p_2 \rangle \,|^2$$

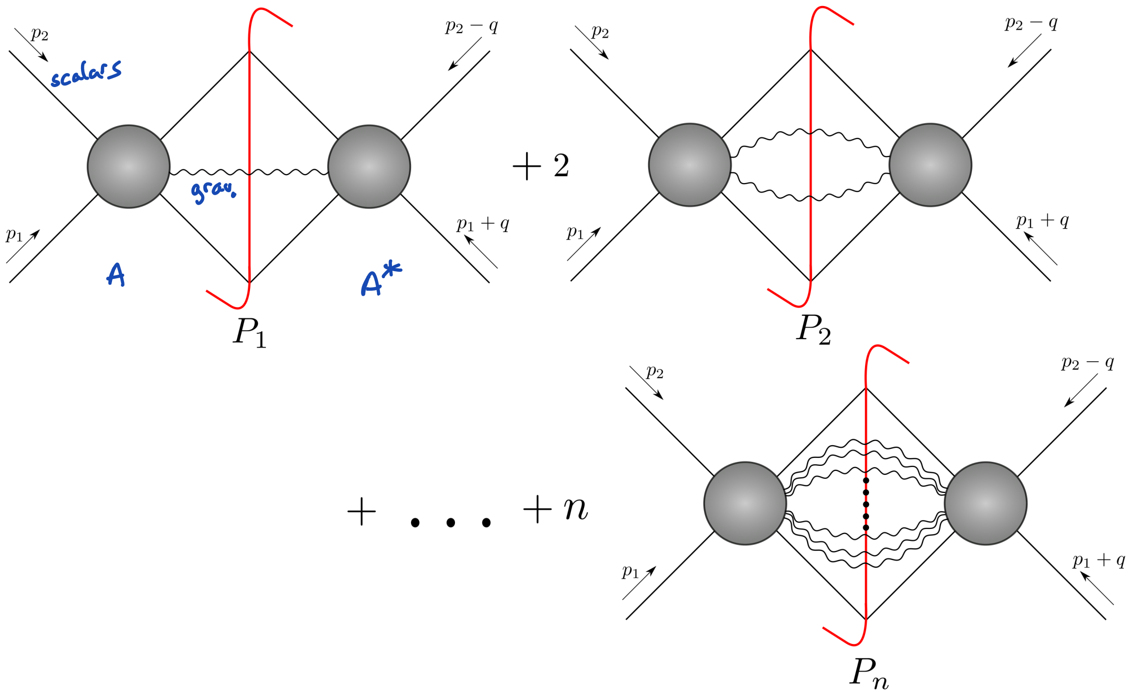
with an implicit IR cutoff.

Unitarity: $\sum_{n=0}^{\infty} \bar{P}_n = 1.$

Graviton number operator

$$\hat{N} = \sum_{\sigma=\pm} \int d\Phi(k)\, a_\sigma^\dagger(k) a_\sigma(k).$$

Mean: $\mu_{\text{out}} = \langle \psi_{\text{out}} | \hat{N} | \psi_{\text{out}} \rangle = \sum_{n=0}^{\infty} n\, P_n.$



Graviton particle statistics

- Mean: $\mu_{\text{out}} = \langle \psi_{\text{out}} | \hat{N} | \psi_{\text{out}} \rangle = \sum_{n=0}^{\infty} n\, P_n$

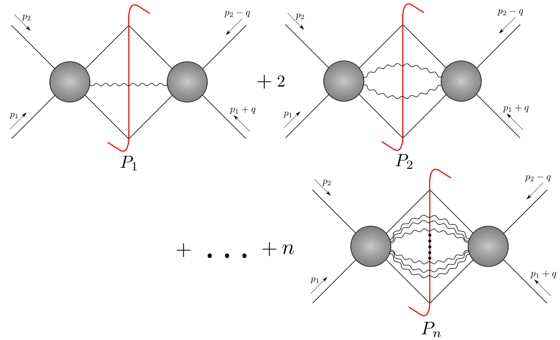
- Variance:

$$\Sigma_{\text{out}} = \langle \psi_{\text{out}} | (\hat{N})^2 | \psi_{\text{out}} \rangle - \left(\langle \psi_{\text{out}} | \hat{N} | \psi_{\text{out}} \rangle \right)^2 = \sum_{n=0}^{\infty} n^2 P_n - \left(\sum_{n=0}^{\infty} n\, P_n \right)^2$$

- In a Poisson distribution, Mean=Variance. Hence we define
$\Delta_{\text{out}} = \Sigma_{\text{out}} - \mu_{\text{out}}$
and check whether this deviation vanishes. Also check higher moments.

Graviton particle statistics

Expand in powers of the gravitational coupling G.



$$P_n = \sum_{L_1, L_2} G^{2+n+L_1+L_2} P_n^{(L_1,L_2)}$$

tree level

$$\text{Leading order: } \Delta_{\text{out}}\Big|_{\mathcal{O}(G^4)} = 2G^4 P_2^{(0,0)}.$$

Product of 6-point tree amplitudes. Do these amplitudes survive in the classical limit?

For the classical limit, we will examine $\hbar$ scaling and check whether $\hbar \Delta_{\text{out}} \to 0$.

Computing amplitudes

- The Einstein-Hilbert action suffers from a proliferation of vertices with gauge dependence.
- With an auxiliary field and explicit gauge fixing, a compact form of the tree-level Lagrangian is obtained with only cubic interactions. [Cheung, Remmen]
- We add minimally coupled scalars for the massive particles.

$$S_{\rm GR} = \frac{1}{16\pi G}\int d^4x\left[-\left(A^a_{bc}A^b_{ad} - \frac{1}{3}A^a_{ac}A^b_{bd}\right)\sigma^{cd} + A^a_{bc}\partial_a\sigma^{bc}\right]$$

where $\sigma^{ab} = \sqrt{-g}\,g^{ab}$;

$$S_{\rm matter} = -\sum_{j=A,B}\int d^4x\left[\frac{1}{2}\sigma^{ab}\partial_a\phi_j\partial_b\phi_j + \frac{1}{2}\sqrt{-\det(\sigma^{-1})}m_j^2\phi_j^2\right]$$

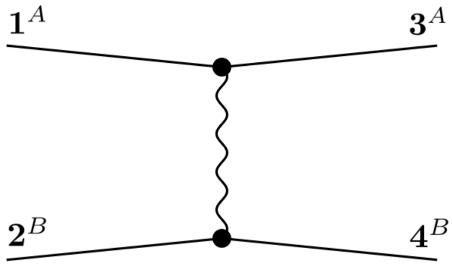
two scalars

$$\mathscr{L}_{\rm GF} = -\frac{1}{2}\eta_{cd}\,\partial_a\left(\sqrt{-g}\,g^{ac}\right)\partial_b\left(\sqrt{-g}\,g^{bd}\right)$$

With the massive scalars, interaction vertices do proliferate beyond cubic ones, but are under control at lower orders.

Interactions through order h^3: $hhh, hhA, hAA, h\phi\phi, hh\phi\phi, hhh\phi\phi$.

4-point amplitude

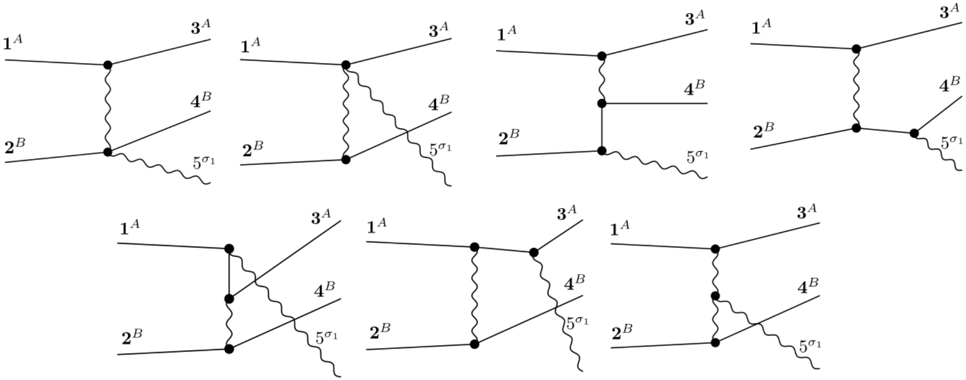


- $n = 0$. No graviton emission.
- Irrelevant for mean, variance, etc.; ingredient for recursive constructions.
- Single Feynman diagram

$\kappa = \sqrt{32\pi G}$

$$\mathscr{A}_4^{(0)}(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B) = -\frac{i\kappa^2}{2t}\left(\frac{1}{2}t\left(-m_A^2-m_B^2+s\right)+\frac{1}{2}\left(-m_A^2-m_B^2+s\right)^2-m_A^2m_B^2\right)$$

5-point amplitude



- 7 Feynman diagrams
- Prefer a compact formula, but with only one massless particle, there is no BCFW shift
- Introduce a new equal-mass shift

BCFW Recursion

[RB, Cachazo, Feng, Witten]

- Consider a tree-level amplitude as a function of momenta, $\mathscr{A}_n\left(\{p_i\}\right)$. Introduce a complex variable z through a momentum shift, $\hat{p}_i = p_i + zr_i$.

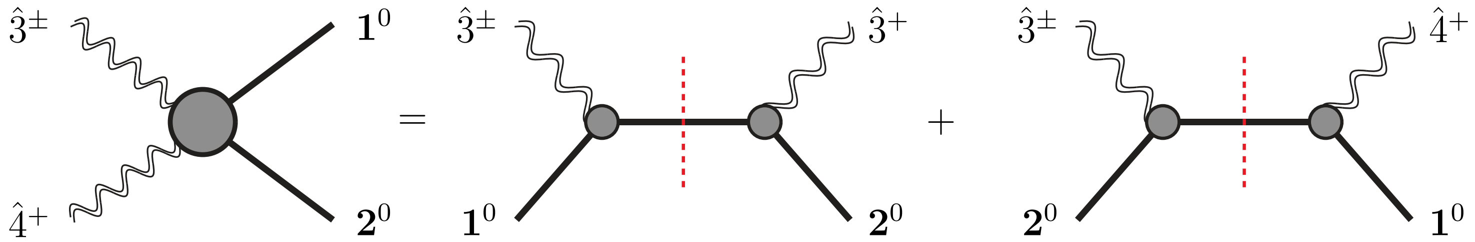
- If momentum is conserved, $\Sigma_i r_i = 0$, and momenta stay on shell, $\hat{p}_i^2 = m_i^2$, then the shifted function $\mathscr{A}_n(\{\hat{p}_i\})$ maintains properties of an amplitude.

- Further, if all $r_i \cdot r_j = 0$, then propagators depend linearly on z, so $\mathscr{A}_n(\{\hat{p}_i\})$ has only simple poles. If the residue at infinity (boundary term) vanishes, then we can apply Cauchy's residue theorem.

BCFW Recursion

- $\oint_{\gamma_\infty} dz \frac{\mathscr{A}_n(z)}{z} = \mathscr{A}_n(0) + \sum_I \operatorname*{Res}_{z=z_I} \left[\frac{\mathscr{A}_n(z)}{z} \right]$

- $\sum_I \operatorname*{Res}_{z=z_I} \left[\frac{\mathscr{A}_n(z)}{z} \right] = - \sum_I \sum_{\sigma=\pm} \mathscr{A}_L\big(\{\hat{p}_L\}, \hat{P}_I^{\sigma} \big) \frac{i}{P_I^2 - m_I^2} \mathscr{A}_R\big(-\hat{P}_I^{-\sigma}, \{\hat{p}_R\} \big)$



Convenient momentum shift with spinor variables:

$$\hat{p}_3^{a\dot{b}} = |3\rangle^a [\hat{3}|^{\dot{b}} = |3\rangle^a ([3| + z[4|)^{\dot{b}},$$

$$\hat{p}_4^{a\dot{b}} = |\hat{4}\rangle^a [4|^{\dot{b}} = (|4\rangle - z|3\rangle)^a [4|^{\dot{b}}.$$

Result:

$$\mathscr{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^+, 4^+) = -i\kappa^2 \frac{m_A^4 [34]^3}{\langle 34\rangle (s_{31} - m_A^2)(s_{32} - m_A^2)},$$

$$\mathscr{A}_4^{(0)}(\mathbf{1}^A, \mathbf{2}^A, 3^-, 4^+) = i\kappa^2 \frac{[4|1|3\rangle^4}{s_{34}(s_{31} - m_A^2)(s_{32} - m_A^2)}.$$

The equal-mass shift

- Since the masses are just parameters in the procedure, we can relax the on-shell condition: $\hat{p}_i^2 = \hat{m}_i^2$. Nonzero masses can vary, provided that any equal masses remain equal in the shift.

- For our 5-point amplitude, use

$$\hat{p}_5 = |\hat{5}\rangle[\hat{5}| = (|5\rangle + z\,(1-3)|5])[5|$$

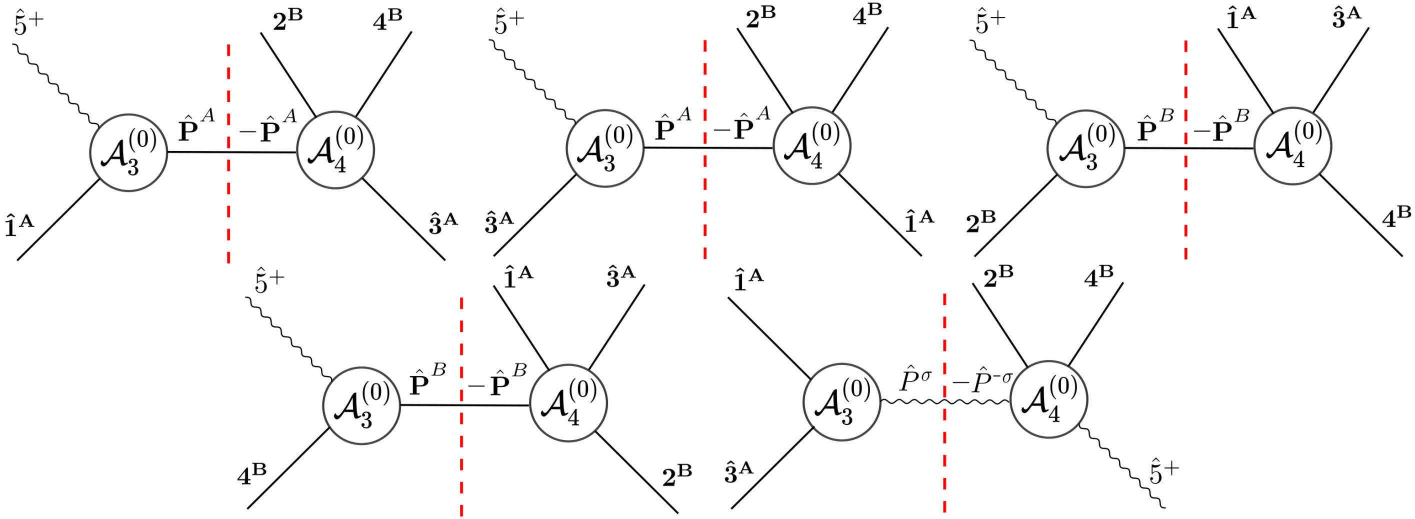
$$\hat{p}_1 = p_1 + z\,3|5][5|$$

$$\hat{p}_3 = p_3 - z\,1|5][5|$$

- $\hat{p}_1^2 = p_1^2 - z[5|13|5] = m_A^2 + z[5|31|5] = \hat{p}_3^2$.

- Vanishing of boundary term is hard to prove. We checked it with FD.

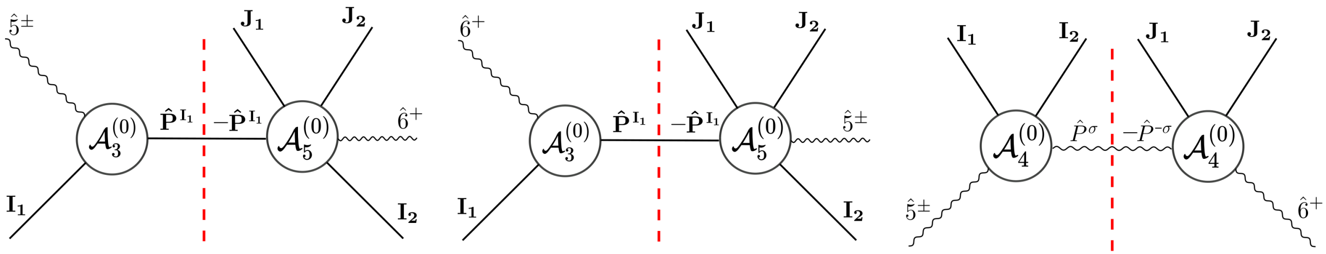
5-point factorization diagrams



5-point result

$$\mathcal{A}_5^{(0)}\left(\mathbf{1}^A,\mathbf{2}^B,\mathbf{3}^A,\mathbf{4}^B,5^+\right)=\frac{i\kappa^3}{8}\Bigg(\Bigg[\frac{-p_4\cdot p_2[5|13|5]^2}{s_{24}(s_{51}-m_A^2)(s_{53}-m_A^2)}+\frac{[5|K_AK_B|5]^2-8[5|13|5]^2}{16s_{13}s_{24}}$$
$$+\frac{(m_A^2+m_B^2)[5|13|5]\big(2(s_{13}-s_{24})[5|13|5]+[5|K_B|5\rangle[5|K_AK_B|5]\big)}{8(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}$$
$$-\frac{K_A\cdot K_B(s_{24}-s_{13})^2[5|13|5]\big(4s_{24}[5|42|5]-[5|K_A|5\rangle[5|K_AK_B|5]\big)}{32s_{13}s_{24}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}$$
$$-\frac{K_A\cdot K_B[5|42|5]\big([5|K_B|5\rangle[5|K_AK_B|5]-4(s_{13}+s_{24})[5|13|5]\big)}{8s_{13}s_{24}(s_{52}-m_B^2)(s_{54}-m_B^2)}$$
$$-\frac{K_A\cdot K_B[5|K_A|5\rangle[5|K_B|5\rangle\big([5|K_AK_B|5]^2-8[5|42|5]^2\big)}{64s_{13}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}$$
$$+\left(\mathrm{tr}(\not{K}_A\not{K}_B\not{K}_A\not{K}_B)+2[5|K_A|5\rangle^2+2[5|K_B|5\rangle^2-2s_{13}^2-2s_{24}^2\right)$$
$$\times\Bigg(\frac{[5|K_A|5\rangle[5|K_B|5\rangle[5|13|5][5|42|5]}{64s_{13}s_{24}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}$$
$$+\frac{[5|42|5](2(s_{13}-s_{24})[5|42|5]-[5|K_A|5\rangle[5|K_AK_B|5])}{64s_{13}(s_{51}-m_A^2)(s_{53}-m_A^2)(s_{52}-m_B^2)(s_{54}-m_B^2)}\Bigg)\Bigg]+\big[(1,3,K_A)\leftrightarrow(2,4,K_B)\big]\Bigg).$$

The 6-point amplitude



$\{I_1, I_2\} = \{1^A, 3^A\}$ or $\{2^B, 4^B\}$

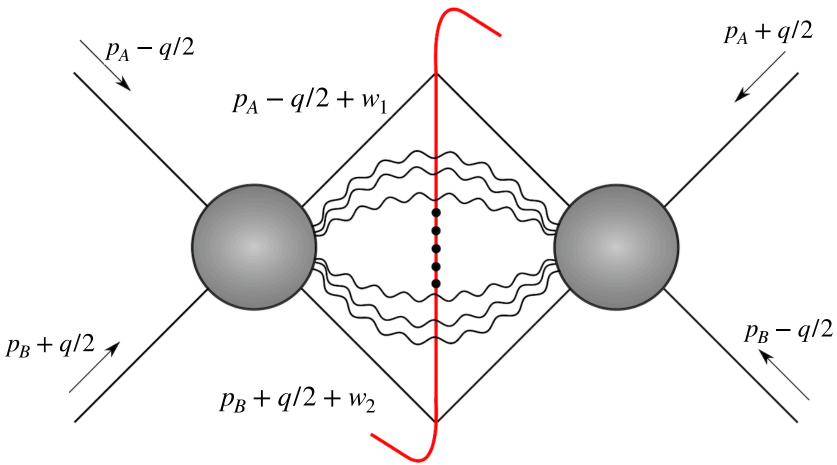
Usual BCFW shift on graviton pair.

Vanishing of boundary term verified from FD.

The classical limit

- KMOC told us how to set up the states. They also provide a detailed prescription for taking the classical limit. [Kosower, Maybee, O'Connell]

- Study how all quantities scale with $\hbar$, then take the limit $\hbar \to 0$.



- Momentum transfers q_j, w_j

- Wavenumbers $\bar{q} = q/\hbar,\ \bar{w}_j = w_j/\hbar,\ \bar{k}_i = k_i/\hbar$

- Coupling $\kappa \to \kappa/\sqrt{\hbar}$

Classical limit of the tree amplitudes

- Implement explicit scaling: $\bar{q} = q/\hbar,\ \bar{w}_j = w_j/\hbar,\ \bar{k}_i = k_i/\hbar, \kappa \to \kappa/\sqrt{\hbar}$
- Replace velocities on classical trajectories: $p_j = \tilde{m}_j v_j, \quad \tilde{m}_j^2 = m_j^2 - \hbar^2 \frac{\bar{q}^2}{4}$
- Naively expected behavior from Feynman diagrams:

$$\mathscr{A}_5 = C_1^{(5)} \hbar^{-\frac{9}{2}} + C_2^{(5)} \hbar^{-\frac{7}{2}} + \mathcal{O}\left(\hbar^{-\frac{5}{2}}\right),$$

$$\mathscr{A}_6 = C_1^{(6)} \hbar^{-6} + C_2^{(6)} \hbar^{-5} + C_3^{(6)} \hbar^{-4} + \mathcal{O}\left(\hbar^{-3}\right).$$

- Leading orders are suppressed:

$$C_1^{(5)} = 0, \qquad C_2^{(5)} \neq 0,$$
$$C_1^{(6)} = C_2^{(6)} = 0, \qquad C_3^{(6)} \neq 0.$$

Classical limit of the 5-point amplitude

- $n = 1$, and $\mathscr{A}_5 = C_2^{(5)} \hbar^{-\frac{7}{2}} + \mathcal{O}\left(\hbar^{-\frac{5}{2}}\right)$.
- Leading order matches a known result. [Luna, Nicholson, O'Connell, White]
- The scaling of the *energy* of emitted radiation must remain finite, so the leading-order scaling for a contribution to the classical limit is compensated by precisely $\hbar^{5/2+n}$ for each amplitude.
- Probabilities should be treated as $\hbar P_n$.
- The 5-point tree does give a classical contribution: $\lim_{\hbar \to 0} \hbar P_1^{(0,0)} \sim \mathcal{O}(1)$.

Classical limit of the 6-point amplitude

- $n = 2$, and $\mathscr{A}_6 = C_3^{(6)}\hbar^{-4} + \mathcal{O}\left(\hbar^{-3}\right)$.
- No classical contribution: $\lim_{\hbar\to 0} \hbar\, P_2^{(0,0)} = 0$.
- Leading order is new, and provides the first check of coherence.

$$\lim_{\hbar\to 0} \hbar\, \Delta_{\text{out}}\Big|_{\mathcal{O}(G^4)} = 0.$$

Coherence at higher orders

- At tree-level, we see the origin of $\hbar$ suppression from the BCFW shift, so we conjecture: $\lim_{\hbar \to 0} \mathscr{A}_{4+n} \sim \hbar^{-3-\frac{n}{2}}$. Hence only the 5-point amplitude provides a classical contribution.

- Conjecturally then, $\lim_{\hbar \to 0} \hbar P_n^{(0,0)} = 0$ for $n > 2$.

- Consistent with expectations of coherence. If coherence holds to higher orders in G and L, then there must be further relations among amplitudes, in the classical limit.

Coherence at higher orders

- Higher (factorial) moments: $\Gamma^{(m)} = \langle \psi | \hat{N}(\hat{N}-1)\ldots(\hat{N}-m+1) | \psi \rangle$.
- For a Poisson distribution, $\Gamma^{(m)} = \mu^m$.
- Thus we check the vanishing of $\Delta^{(m)} = \Gamma^{(m)} - \mu^m$. We have already done $m = 2$.

$$\Delta^{(m)} = \sum_n \sum_{L_1,L_2} G^{2+n+L_1+L_2} \frac{n!}{(n-m)!} P_n^{(L_1,L_2)}$$

$$- \sum_{n_1,\ldots,n_m} \sum_{L_1^{(1)},\ldots,L_1^{(m)}} \sum_{L_2^{(1)},\ldots,L_2^{(m)}} G^{2m+\sum_k \left[n_k + L_1^{(k)} + L_2^{(k)}\right]} \prod_j \left[n_j P_{n_j}^{\left(L_1^{(j)},L_2^{(j)}\right)} \right].$$

Relations among amplitudes

Up to $\mathcal{O}(G^7)$,

$$\lim_{\hbar\to 0} \hbar\Delta^{(2)} = \lim_{\hbar\to 0} \hbar \left(G^6(2P_2^{(2,0)} + 2P_2^{(0,2)}) \right)$$
$$+ \lim_{\hbar\to 0} \hbar \left(G^7(2P_2^{(3,0)} + 2P_2^{(0,3)} + 6P_3^{(2,0)} + 6P_3^{(0,2)} + 6P_3^{(1,1)} \right)$$
$$+ \lim_{\hbar\to 0} \hbar \left[G^6(2P_2^{(1,1)} - (P_1^{(0,0)})^2) + G^7(2P_2^{(1,2)} + 2P_2^{(2,1)} - 2P_1^{(0,1)}P_1^{(0,0)} - 2P_1^{(1,0)}P_1^{(0,0)}) \right],$$
$$\lim_{\hbar\to 0} \hbar\Delta^{(3)} = \lim_{\hbar\to 0} \hbar \left(G^7(6P_3^{(0,2)} + 6P_3^{(2,0)} + 6P_3^{(1,1)}) \right)$$

Thus, coherence implies $\lim_{\hbar\to 0} \hbar \left(G^7(6P_3^{(0,2)} + 6P_3^{(2,0)} + 6P_3^{(1,1)}) \right) = 0.$

We make one more assumption, $\lim_{\hbar\to 0} \hbar P_n^{(L,0)} = \lim_{\hbar\to 0} \hbar P_n^{(0,L)} = 0$ for $n \geq 2$.

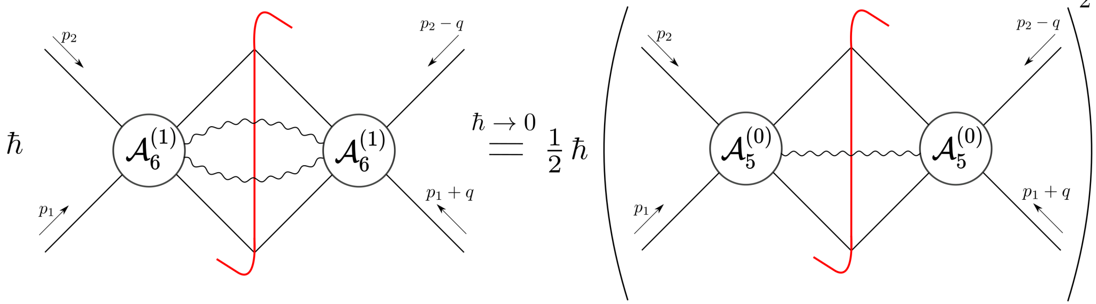
[cf. Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White]

Relations among amplitudes

- $\lim_{\hbar\to 0} \hbar P_3^{(1,1)} = 0$: the 7-point 1-loop amplitude is classically suppressed.

- $$\hbar P_2^{(1,1)} = \frac{1}{2}\hbar (P_1^{(0,0)})^2$$ as $\hbar \to 0$:
  $$\hbar (P_2^{(1,2)} + P_2^{(2,1)}) = \hbar (P_1^{(0,1)} P_1^{(0,0)} + P_1^{(1,0)} P_1^{(0,0)})$$

6- and higher-point amplitudes are related to 5-point at lower loop level.



Summary & Outlook

- “Amplitudes” techniques have been useful for precision calculations in the study of gravitational waves.

- Main result: classical suppression of a 6-pt tree amplitude as $\hbar \to 0$. Evidence for coherence of final semiclassical radiation state in binary scattering.

- Introduced an equal-mass shift for on-shell recursion.

- Conjectured higher-order relations in this framework, such that the 4- and 5-point amplitudes encode all information of the final state.

- Future directions:
  - explore higher-order relations, and connections to classical soft theorems
  - nonperturbative effects may spoil coherence
  - spin/tidal effects may spoil coherence
  - resummation of radiation reaction effects is desirable