SAGEX Closing Meeting, June 22, 2022

Talk 9: Computer Algebra and Special Function Algorithms for Feynman Integrals

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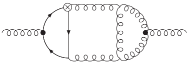
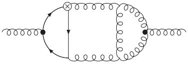
Evaluation of Feynman Integrals



behavior of particles

Evaluation of Feynman Integrals

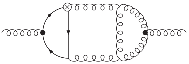


behavior of particles

$$\longrightarrow \quad \int \Phi(n, \epsilon, x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(n,\epsilon,x)dx$$

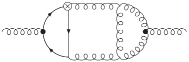
Feynman integrals

DESY

$$\sum f(n,\epsilon,k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(n, \epsilon, x)dx$$

Feynman integrals

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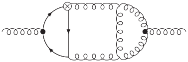
$$\sum f(n, \epsilon, k)$$

complicated
multi-sums

RISC
(Sigma-package)

expression in
special functions

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(n, \epsilon, x)dx$$

Feynman integrals

LHC at CERN

DESY

applicable

$$\sum f(n, \epsilon, k)$$

complicated multi-sums

RISC

(Sigma-package)

expression in special functions

$$F(\varepsilon,n)=\iiint\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||?$$

$$F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+F_{-1}(n)\varepsilon^{-1}+F_0(n)\varepsilon^0+\dots$$

$$F(\varepsilon,n)=\iiint\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times$$
$$\times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{n}{k}$$

where

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$F(\varepsilon,n)=\iiint\frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{n}{k}}_{=f_{-3}(n,k)\varepsilon^{-3}+f_{-2}(n,k)\varepsilon^{-2}+f_{-1}(n,k)\varepsilon^{-1}+\dots}$$

for general expansion methods see
J. Blümlein, CS, M. Saragnese, 2021. arXiv:2111.15501 [math-ph]

$$F(\varepsilon,n)=\iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}}\frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}}\frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\underbrace{\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Gamma\Big(-1-\frac{3\varepsilon}{2}\Big)\times \times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{n}{k}}_{=f_{-3}(n,k)\varepsilon^{-3}+f_{-2}(n,k)\varepsilon^{-2}+f_{-1}(n,k)\varepsilon^{-1}+\dots}$$

$$||$$

$$\Big(\sum_{k=1}^{n}f_{-3}(n,k)\Big)\varepsilon^{-3}+\Big(\sum_{k=1}^{n}f_{-2}(n,k)\Big)\varepsilon^{-2}+\Big(\sum_{k=1}^{n}f_{-1}(n,k)\Big)\varepsilon^{-1}+\dots$$

$$F(\varepsilon, n) = \iiint \frac{d^{4+\varepsilon}k_1}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_2}{(2\pi)^{4+\varepsilon}} \frac{d^{4+\varepsilon}k_3}{(2\pi)^{4+\varepsilon}} \frac{(\Delta.k_3)^n}{k_2^4((k_1-k_3)^2-m^2)(k_1-k_2)^2((k_3-p)^2-m^2)}$$

$$||$$

$$\sum_{k=1}^{n} \underbrace{(-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Gamma\Big(-1-\frac{3\varepsilon}{2}\Big) \times \times B\Big(2+k, \frac{\varepsilon}{2}\Big) B(-\varepsilon+k, -\varepsilon) B\Big(1-\frac{\varepsilon}{2}+k, 1+\frac{\varepsilon}{2}\Big) \binom{n}{k}}_{= f_{-3}(n,k)\varepsilon^{-3} + f_{-2}(n,k)\varepsilon^{-2} + f_{-1}(n,k)\varepsilon^{-1} + \dots}$$

$$||$$

$$\Big(\sum_{k=1}^{n} f_{-3}(n,k)\Big)\varepsilon^{-3} + \Big(\sum_{k=1}^{n} f_{-2}(n,k)\Big)\varepsilon^{-2} + \Big(\boxed{\sum_{k=1}^{n} f_{-1}(n,k)}\Big)\varepsilon^{-1} + \dots$$

Simplify

$$F_{-1}(n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

where

$$S_a(n) = \sum_{i=1}^{n} \frac{\mathsf{sign}(a)^i}{i^a} \text{ and } \zeta_a = \sum_{i=1}^{\infty} \frac{1}{i^a}$$

Simplify

$$F_{-1}(n)=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}\big(16n^3+144n^2+413n+384\big)(n+1)^2F_{-1}(n)&\\ -(n+2)(2n+5)\big(16n^3+112n^2+221n+113\big)F_{-1}(n+1)&\\ +(n+3)^2\big(16n^3+96n^2+173n+99\big)F_{-1}(n+2)&\\ =\frac{1}{2}\big(4n^2+21n+29\big)\zeta_2+\frac{-64n^5-500n^4-1133n^3+203n^2+3516n+3090}{3(n+2)(n+3)}&\end{aligned}$$

Simplify

$$F_{-1}(n)=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

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$$\begin{aligned}\big(16n^3+144n^2+413n+384\big)(n+1)^2F_{-1}(n)\\ -(n+2)(2n+5)\big(16n^3+112n^2+221n+113\big)F_{-1}(n+1)\\ +(n+3)^2\big(16n^3+96n^2+173n+99\big)F_{-1}(n+2)\\ =\frac{1}{2}\big(4n^2+21n+29\big)\zeta_2+\frac{-64n^5-500n^4-1133n^3+203n^2+3516n+3090}{3(n+2)(n+3)}\end{aligned}$$

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$$\begin{aligned}\{c_1\frac{1-4n}{n+1}+c_2\frac{-14n-13}{(n+1)^2}\\ +\frac{(4n-1)S_1(n)}{n+1}+\frac{(1-4n)S_1(n)^2}{6(n+1)}+\frac{(14n+13)S_1(n)}{3(n+1)^2}\\ +\frac{175n^2+334n+155}{12(n+1)^3}+\frac{(1-4n)S_2(n)}{6(n+1)}+\frac{\zeta_2}{8(n+1)}|c_1,c_2\in\mathbb{Q}\}\end{aligned}$$

Simplify

$$F_{-1}(n)=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$$\bigcap$$

$$\Big\{c_1\,\frac{1-4n}{n+1}+c_2\,\frac{-14n-13}{(n+1)^2}$$
$$+\frac{(4n-1)S_1(n)}{n+1}+\frac{(1-4n)S_1(n)^2}{6(n+1)}+\frac{(14n+13)S_1(n)}{3(n+1)^2}$$
$$+\frac{175n^2+334n+155}{12(n+1)^3}+\frac{(1-4n)S_2(n)}{6(n+1)}+\frac{\zeta_2}{8(n+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}$$

Simplify

$$F_{-1}(n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3} + \frac{2S_2(k)}{1+k} + \frac{\zeta_2}{2(1+k)} \Big)$$

|| (recurrence finding and solving)

$$\Big(\frac{1}{12} - \frac{1}{8}\zeta_2\Big) \frac{1-4n}{n+1} + 1 \, \frac{-14n-13}{(n+1)^2} + \frac{(4n-1)S_1(n)}{n+1} + \frac{(1-4n)S_1(n)^2}{6(n+1)} + \frac{(14n+13)S_1(n)}{3(n+1)^2} + \frac{175n^2+334n+155}{12(n+1)^3} + \frac{(1-4n)S_2(n)}{6(n+1)} + \frac{\zeta_2}{8(n+1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

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FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n)+\dots+a_d(n)F(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

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Special cases:

$$S_{2,1}(n)=\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}$$

(harmonic sums)

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases:

$$\sum_{k=1}^{n} \frac{2^k}{k} \sum_{i=1}^{k} \frac{2^{-i}}{i} \sum_{j=1}^{i} \frac{S_1(j)}{j}$$

(generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases:

$$\sum_{k=1}^{n} \frac{1}{(1+2k)^2} \sum_{j=1}^{k} \frac{1}{j^2} \sum_{i=1}^{j} \frac{1}{1+2i}$$ (cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases:

$$\sum_{j=1}^{n}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}$$

(binomial sums)

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{h=1}^{n} 2^{-2h}(1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2 \binom{2k}{k}}$$

(generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916].

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FIND all solutions expressible by indefinite nested products/sums
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A more general example:

$$\sum_{k=1}^{n}\left(\prod_{i=1}^{k}\frac{1+i+i^2}{i+1}\right)\sum_{j=1}^{k}\frac{1}{j\binom{4j}{3j}^2}$$

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FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a "closed form"

F(n)=combined solutions in terms of indefinite nested sums.

Sigma.m is based on difference ring/field theory

1. M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
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21. CS. Summation Theory II: Characterizations of $R\Pi\Sigma$-extensions and algorithmic aspects. J. Symb. Comput. 80(3), pp. 616-664. 2017.
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In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{n}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{n}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{n\},\{1\},ExpandIn\to\{\varepsilon,-3,-3\}]}$

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In[4]:= **mySum =**

$$\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left[2+k,\frac{\varepsilon}{2}\right]B[-\varepsilon+k,-\varepsilon]B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{n}{k};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}, ExpandIn** $\to$ **{**ε**, −3, −3}]**

Out[5]= $\left\{\frac{59n^2+120n+49}{9(n+1)^2}-\frac{2(n+3)S_1[n]}{3(n+1)}\right\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{n}(-1)^{k}e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!\mathrm{B}\Big[2+k,\frac{\varepsilon}{2}\Big]\mathrm{B}[-\varepsilon+k,-\varepsilon]\mathrm{B}\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{n}{k};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{n\},\{1\},ExpandIn} \to \{\varepsilon,-3,-2\}]$

Out[5]= $$\Big\{\frac{59n^2+120n+49}{9(n+1)^2}-\frac{2(n+3)S_1[n]}{3(n+1)},\\ -\frac{2(20n^3+58n^2+57n+22)}{3(n+1)^3}+\frac{2(n+2)(2n-1)S_1[n]}{3(n+1)^2}-\frac{S_1[n]^2}{n+1}-\frac{S_2[n]}{n+1}\Big\}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

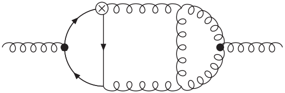
EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{mySum} =$

$$\sum_{k=1}^{n}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!B\Big[2+k,\frac{\varepsilon}{2}\Big]B[-\varepsilon+k,-\varepsilon]B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{n}{k};$$

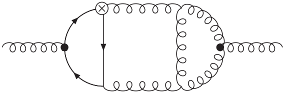
In[5]:= $\mathbf{EvaluateMultiSum[mySum,\{\},\{n\},\{1\},ExpandIn \to \{\varepsilon,-3,-1\}]}$

Out[5]=
$$\Big\{\frac{59n^2+120n+49}{9(n+1)^2}-\frac{2(n+3)S_1[n]}{3(n+1)},$$
$$-\frac{2(20n^3+58n^2+57n+22)}{3(n+1)^3}+\frac{2(n+2)(2n-1)S_1[n]}{3(n+1)^2}-\frac{S_1[n]^2}{n+1}-\frac{S_2[n]}{n+1},$$
$$\Big(\frac{1}{12}-\frac{1}{8}\zeta(2)\Big)\frac{1-4n}{n+1}+\frac{-14n-13}{(n+1)^2}+\frac{(4n-1)S_1(n)}{n+1}+\frac{(1-4n)S_1(n)^2}{6(n+1)}+$$
$$\frac{(14n+13)S_1(n)}{3(n+1)^2}+\frac{175n^2+334n+155}{12(n+1)^3}+\frac{(1-4n)S_2(n)}{6(n+1)}+\frac{\zeta(2)}{8(n+1)}\Big\}$$



[arXiv:1509.08324]

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$



[arXiv:1509.08324]

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

Simplify

$$||$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+ \Big(-\frac{4(13n+5)}{n^2(n+1)^2} + \Big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+ \big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
$$- 2(-1)^nS_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$- \frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

$\boxed{F_0(n)} =$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{\ldots} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

$$+ \Big(\ldots \frac{\ldots 1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$

$$+ \Big(2 + \ldots - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$

$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26 + 4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$

$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22 + 6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6 + 5(-1)^n\big)S_{-4}(n)$$

$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20 + 2(-1)^n\big)S_{2,-2}(n) + \big(-17 + 13(-1)^n\big)S_{3,1}(n)$$

$$- \frac{8(-1)^n(2n+1) + 4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24 + 4(-1)^n\big)S_{-3,1}(n) + \big(3 - 5(-1)^n\big)S_{2,1,1}(n)$$

$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{\ldots(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+ \Big(\ldots\frac{\ldots1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+ \Big(2 + \ldots_{2,1}(\ldots) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)\ldots\Big)S_2(n)^2$$
$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \Big(26 + 4(\ldots\Big)$$
$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22 + 6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
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$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

$$\boxed{F_0(n)} =$$

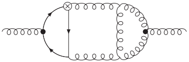
$$
\begin{aligned}
&\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{\ldots} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2\\
&+ \Big(\ldots \frac{\ldots 1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)\\
&+ \Big(2 + \ldots - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big) \ldots \Big)S_2(n)^2\\
&- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26 + 4(\ldots\Big)\\
&+ \Big(\frac{(-1)^n(5-3n)}{2n^2 \ldots} - \frac{5}{\ldots}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}\\
&+ \frac{4(3n - \ldots}{n(n+\ldots} \ldots -1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)\\
&+ \Big(\frac{(-1)\ldots}{n(\ldots} \ldots n) + \big(-6 + 5(-1)^n\big)S_{-4}(n)\\
&+ \Big(- \ldots 2(\ldots S_{2,-2}(n) + \big(-17 + 13(-1)^n\big)S_{3,1}(n)\\
&- \frac{8(-1)\ldots}{n(n+1)}S_{-2,1}(n) - \big(24 + 4(-1)^n\big)S_{-3,1}(n) + \big(3 - 5(-1)^n\big)S_{2,1,1}(n)\\
&+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)
\end{aligned}
$$

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$S_{-2,1,1}(n) = \sum_{i=1}^{n} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals

DESY

LHC at CERN

applicable

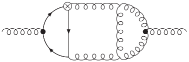
$$\sum f(n,\epsilon,k)$$

complicated
multi-sums

RISC
(Sigma-package)

expression in
special functions

Evaluation of Feynman Integrals



behavior of particles

$$\int \Phi(n, \epsilon, x) dx$$

Feynman integrals

DESY

$$\sum f(n, \epsilon, k)$$

complicated multi-sums

RISC

(Sigma-package)

expression in special functions

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_x f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_y f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

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The integrand is

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The integrand is

- hyperexponential in x,y,z:
- hypergeometric in n:

$$\frac{f(\varepsilon, n+1, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

$$
\begin{aligned}
a_0(n,\varepsilon) &= (n+1)(n+2)\big(8\varepsilon^{10} + 104\varepsilon^9(n+3) + 4\varepsilon^8\big(96n^2+601n+887\big) \\
&\quad + 4\varepsilon^7\big(12n^3+414n^2+1583n+1393\big) \\
&\quad - 8\varepsilon^6\big(264n^4+2436n^3+8643n^2+14518n+9947\big) \\
&\quad - 16\varepsilon^5\big(156n^5+1690n^4+6847n^3+12661n^2+9537n+717\big) \\
&\quad + 32\varepsilon^4\big(68n^6+1158n^5+8155n^4+30114n^3+61712n^2+67616n+31693\big) \\
&\quad + 64\varepsilon^3\big(40n^7+560n^6+2755n^5+3729n^4-14194n^3-61920n^2-89140n-46600\big) \\
&\quad - 128\varepsilon^2(n+2)\big(12n^7+254n^6+2249n^5+10758n^4+30173n^3+50610n^2 \\
&\quad + 49122n+22706\big) \\
&\quad + 256\varepsilon(n+2)^2(n+3)(n+4)\big(44n^4+501n^3+2044n^2+3455n+1976\big) \\
&\quad - 512(n+1)(n+2)^3(n+3)^2(n+4)\big(6n^2+47n+95\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_1(n,\varepsilon) = {} & (n+2)\big(-22\varepsilon^{11} - 2\varepsilon^{10}(157n+435) - \varepsilon^9\big(1500n^2+8611n+11745\big) \\
& - \varepsilon^8\big(2548n^3+22936n^2+63597n+54229\big) \\
& + 4\varepsilon^7\big(266n^4+1857n^3+6065n^2+14351n+15987\big) \\
& + 8\varepsilon^6\big(994n^5+12961n^4+67246n^3+174692n^2+226821n+116092\big) \\
& + 16\varepsilon^5\big(336n^6+5348n^5+33569n^4+104918n^3+165290n^2+108259n+6100\big) \\
& - 16\varepsilon^4\big(404n^7+7578n^6+61778n^5+284762n^4+802660n^3+1382074n^2 \\
& + 1340455n+560287\big) \\
& - 64\varepsilon^3\big(94n^8+1823n^7+14305n^6+55870n^5+96299n^4-37256n^3 \\
& - 447044n^2-704959n-379338\big) \\
& + 128\varepsilon^2(n+3)\big(30n^8+715n^7+7667n^6+48253n^5+194086n^4+507439n^3 \\
& + 835393n^2+785327n+320382\big) \\
& - 256\varepsilon(n+2)(n+3)^2\big(107n^6+2070n^5+16342n^4+67226n^3+151557n^2 \\
& + 176932n+83196\big) \\
& + 256(n+2)^3(n+3)^3(n+4)\big(30n^3+331n^2+1193n+1386\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_2(n,\varepsilon) = \big(&12\varepsilon^{12} + 12\varepsilon^{11}(17n+45) + 2\varepsilon^{10}\big(620n^2+3553n+4795\big) \\
&+ 2\varepsilon^9\big(1504n^3+14190n^2+41901n+38907\big) \\
&+ 4\varepsilon^8\big(172n^4+4983n^3+30942n^2+69119n+50850\big) \\
&- 4\varepsilon^7\big(1996n^5+24056n^4+113313n^3+269119n^2+337198n+185290\big) \\
&- 16\varepsilon^6\big(450n^6+8210n^5+59749n^4+227386n^3+486841n^2+563176n+275664\big) \\
&+ 16\varepsilon^5\big(340n^7+4314n^6+19137n^5+25532n^4-55105n^3-206516n^2-191528n \\
&- 23458\big) \\
&+ 32\varepsilon^4\big(140n^8+2940n^7+26550n^6+139926n^5+493839n^4+1240186n^3 \\
&+ 2161699n^2+2304248n+1100084\big) \\
&+ 64\varepsilon^3\big(4n^9+506n^8+8651n^7+63510n^6+236215n^5+395334n^4-105413n^3 \\
&- 1551017n^2-2362944n-1217770\big) \\
&- 128\varepsilon^2(n+3)\big(12n^9+314n^8+3782n^7+29105n^6+160727n^5+640273n^4 \\
&+ 1750874n^3+3052505n^2+3017094n+1276604\big) \\
&+ 256\varepsilon(n+2)(n+3)^2(n+4)\big(26n^6+825n^5+8967n^4+46529n^3+125411n^2 \\
&+ 168628n+88652\big) \\
&- 512(n+1)(n+2)^2(n+3)^3(n+4)^2\big(6n^3+98n^2+459n+655\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_3(n,\varepsilon) = \big(& -64\varepsilon^{12} - 8\varepsilon^{11}(113n+298) - 8\varepsilon^{10}\big(519n^2+2948n+3896\big) \\
& - 4\varepsilon^9\big(1444n^3+13839n^2+39746n+34305\big) \\
& + 4\varepsilon^8\big(1948n^4+17868n^3+63837n^2+112966n+84655\big) \\
& + 16\varepsilon^7\big(1456n^5+20460n^4+112365n^3+304963n^2+412258n+221769\big) \\
& - 8\varepsilon^6\big(320n^6+2050n^5+4192n^4+27408n^3+174901n^2+411759n+324872\big) \\
& - 16\varepsilon^5\big(1756n^7+33154n^6+265889n^5+1186719n^4+3218059n^3+5349388n^2 \\
& + 5071913n+2113696\big) \\
& + 32\varepsilon^4\big(188n^8+4802n^7+59527n^6+439922n^5+2025336n^4+5813984n^3 \\
& + 10076450n^2+9621283n+3878602\big) \\
& + 64\varepsilon^3\big(140n^9+2768n^8+22500n^7+99545n^6+287700n^5+723136n^4 \\
& + 1854572n^3+3714620n^2+4272517n+2031600\big) \\
& - 128\varepsilon^2\big(24n^{10}+830n^9+14362n^8+152630n^7+1053620n^6+4834279n^5 \\
& + 14824351n^4+29964399n^3+38244797n^2+27875896n+8824032\big) \\
& + 256\varepsilon(n+2)(n+3)(n+4)\big(118n^7+2639n^6+24247n^5+118311n^4+329565n^3 \\
& + 520306n^2+426076n+136854\big) \\
& - 512(n+1)(n+2)^2(n+3)^2(n+4)^2(n+5)\big(12n^3+97n^2+230n+144\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_4(n,\varepsilon) = \big(&64\varepsilon^{12} + 192\varepsilon^{11}(5n+14) + 16\varepsilon^{10}\big(297n^2 + 1769n + 2451\big) \\
&+ 16\varepsilon^9\big(453n^3 + 4462n^2 + 13094n + 11244\big) \\
&- 8\varepsilon^8\big(1084n^4 + 11117n^3 + 47258n^2 + 103981n + 94650\big) \\
&- 8\varepsilon^7\big(3304n^5 + 51138n^4 + 311957n^3 + 948722n^2 + 1440105n + 858544\big) \\
&+ 16\varepsilon^6\big(420n^6 + 5507n^5 + 36275n^4 + 169650n^3 + 536911n^2 + 952507n + 694370\big) \\
&+ 16\varepsilon^5\big(1828n^7 + 38868n^6 + 353301n^5 + 1801014n^4 + 5604391n^3 + 10664390n^2 \\
&+ 11433064n + 5260048\big) \\
&- 32\varepsilon^4\big(316n^8 + 8356n^7 + 105800n^6 + 802421n^5 + 3836854n^4 + 11588223n^3 \\
&+ 21401558n^2 + 22066744n + 9745752\big) \\
&- 64\varepsilon^3\big(116n^9 + 2424n^8 + 19923n^7 + 82966n^6 + 208191n^5 + 530980n^4 + 1847484n^3 \\
&+ 4687014n^2 + 6120858n + 3111104\big) \\
&+ 128\varepsilon^2\big(24n^{10} + 826n^9 + 14897n^8 + 172000n^7 + 1314686n^6 + 6710299n^5 \\
&+ 22873183n^4 + 51298261n^3 + 72551278n^2 + 58573022n + 20544948\big) \\
&- 256\varepsilon(n+2)(n+3)\big(106n^8 + 3278n^7 + 42903n^6 + 310942n^5 + 1366350n^4 \\
&+ 3729418n^3 + 6173159n^2 + 5657732n + 2191212\big) \\
&+ 512(n+1)(n+2)^2(n+3)^2(n+4)(n+5)(n+6)\big(12n^3 + 121n^2 + 396n + 431\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_5(n,\varepsilon) = (n+5)\big(&-128\varepsilon^{11} - 128\varepsilon^{10}(11n+26) - 32\varepsilon^9\big(115n^2+592n+647\big)\\
&+32\varepsilon^8\big(63n^3+430n^2+1665n+2384\big)\\
&+16\varepsilon^7\big(714n^4+7881n^3+33802n^2+66225n+47654\big)\\
&-16\varepsilon^6\big(234n^5+2444n^4+13989n^3+50862n^2+104083n+87848\big)\\
&-16\varepsilon^5\big(580n^6+10181n^5+76586n^4+319207n^3+772120n^2+1012046n+547832\big)\\
&+16\varepsilon^4\big(244n^7+5456n^6+61605n^5+401216n^4+1536277n^3+3408574n^2\\
&+4066436n+2026928\big)\\
&+64\varepsilon^3\big(26n^8+357n^7+583n^6-11139n^5-65193n^4-120264n^3+11864n^2\\
&+272830n+222624\big)\\
&-64\varepsilon^2(n+3)\big(12n^8+298n^7+4684n^6+49024n^5+306907n^4+1122441n^3\\
&+2350650n^2+2607576n+1185072\big)\\
&+256\varepsilon(n+2)(n+3)\big(25n^7+743n^6+8856n^5+55358n^4+197497n^3+404131N^2\\
&+439902N+196128\big)\\
&-256(N+1)(N+2)^2(N+3)^2(N+4)(N+6)(N+7)\big(6N^2+35N+54\big)\big).
\end{aligned}
$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

Example: A master integral from Ladder and V-topologies

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$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

`Sigma.m` $\downarrow$ (2 hours)

$$F(\varepsilon,n)=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\cdots+F_4(n)\varepsilon^4+O(\varepsilon^5)$$

We get
$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

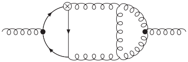
$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

$$F_{-1}(n) = (-1)^n\left(\frac{2\left(9n^5+81n^4+295n^3+533n^2+500n+204\right)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)}\right)$$
$$+\frac{2\left(18n^5+150n^4+490n^3+755n^2+536n+132\right)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)}$$
$$+\left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)}\right)S_2(n)$$
$$+\left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)}\right)S_{-2}(n)$$

Calculations based on Tactic 1:

- I. Bierenbaum, J. Blümlein, S. Klein, and CS. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl.Phys. B* 803(1-2):1–41, 2008.
- J. Ablinger, J. Blümlein, S. Klein, C. Schneider. Modern Summation Methods and the Computation of 2- and 3-loop Feynman Diagrams. Nucl. Phys. B (Proc. Suppl.) 205-206, pp. 110-115, 2010.
- J. Ablinger, I. Bierenbaum, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, F. Wissbrock. Heavy Flavor DIS Wilson coefficients in the asymptotic regime. Nucl. Phys. B (Proc. Suppl.) 205-206, pp. 242-249, 2010.
- J. Ablinger, J. Blümlein, S. Klein, CS, F. Wissbrock. The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity. Nucl. Phys. B, 844: 26-54, 2011.
- J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, CS, F. Wissbrock Massive 3-loop Ladder Diagrams for Quarkonic Local Operator Matrix Elements. Nuclear Physics B. 864: 52-84, 2012.
- J. Blümlein, A. Hasselhuhn, S. Klein, CS. The $O(\alpha_s^3 n_f T_F^2 C_{A,F})$ Contributions to the Gluonic Massive Operator Matrix Elements. Nuclear Physics B: 866: 196-211, 2013.
- J. Ablinger, J. Blümlein, C. Raab, CS, F. Wissbrock. Calculating Massive 3-loop Graphs for Operator Matrix Elements by the Method of Hyperlogarithms. Nuclear Physics B 885, pp. 409-447. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, CS, F. Wißbrock. Three Loop Massive Operator Matrix Elements and Asymptotic Wilson Coefficients with Two Different Masses. Nucl. Phys. B. 921, pp. 585-688. 2017.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. The Two-mass Contribution to the Three-Loop Gluonic Operator Matrix Element $A_{gg,Q}^{(3)}$. Nucl. Phys. B 932, pp. 129-240. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, M. Saragnese, CS, K. Schönwald. The three-loop polarized pure singlet operator matrix element with two different masses. Nuclear Physics B 952(114916), pp. 1-18. 2020.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. The Two-mass Contribution to the Three-Loop Polarized Operator Matrix Element $A_{gg,Q}^{(3)}$. Nuclear Physics B 955, pp. 1-70. 2020.

Evaluation of Feynman Integrals

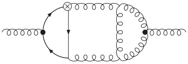


Behavior of particles

$$\longrightarrow \quad \int \Phi(n, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles → $\int \Phi(n, \epsilon, x) dx$

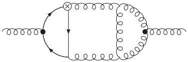
Feynman integrals

↓ **DESY**

$$Dy = A\,y$$

coupled systems of linear DEs

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals

DESY

$$Dy = A\,y$$

coupled systems of linear DEs

RISC

(new coupled system solver)

expression in special functions

Tactic 2: Solve coupled systems of differential equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda \times \lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

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$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinComb}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinComb}(..., D^i\hat{R}_i(x), \dots)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\dots,\hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x),\dots,\hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} = A(x) \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} + \begin{pmatrix}\hat{R}_1(x)\\ \dots \\ \hat{R}_\lambda(x)\end{pmatrix}$$

$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

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DE-solver

Tactic 2': Solve linear DEs and extract hypergeometric structures

(I) A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1} f(x)|_{x=0} \in \mathbb{K}$

(I) A differential equation solver (`HarmonicSums.m`)

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together with initial values $f(0), \ldots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

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Special cases of iterated integrals over hyperexponential functions:

$$H_{1,-1}(x) = \int_0^x \frac{1}{1-\tau_1} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

(harmonic polylogarithms)

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

(I) A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x),\dots,b_\lambda(x)\in\mathbb{K}[x]$

$$b_0(x)f(x)+\dots+b_\lambda(x)D^\lambda f(x)=0;$$

together with initial values $f(0),\dots,D^{\lambda-1}f(x)|_{x=0}\in\mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$H_{2,-2}(x)=\int_0^x\frac{1}{2-\tau_1}\int_0^{\tau_1}\frac{1}{2+\tau_2}d\tau_2 d\tau_1$$

(generalized polylogarithms)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

(I) A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x) f(x) + \dots + b_\lambda(x) D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1} f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{1+\tau_1+\tau_1^2} \int_0^{\tau_1} \frac{1}{1+\tau_2^2} d\tau_2 d\tau_1$$ (cyclotomic polylogarithms)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

(I) A differential equation solver (`HarmonicSums.m`)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

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together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{\sqrt{1+\tau_1}} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1 \qquad \text{(radical integrals)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

(I) A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

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together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{1-\tau_1+\eta\tau_1} \int_0^{\tau_1} \sqrt{1-\tau_2}\sqrt{1-\tau_2+\eta\tau_2}\, d\tau_2 d\tau_1$$

(generalized radical integrals)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

(I) A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1} f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

A more general example:

$$\int_0^x e^{\int_1^{\tau_1} \frac{1}{1+y+y^2} dy} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

HarmonicSums can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11+20x)f'(x)+(1+x)(35+134x)f''(x) + 3(1+x)^2(4+37x)f^{(3)}(x)+18x(1+x)^3f^{(4)}(x)=0$$

$$\downarrow$$

$$\Big\{c_1+c_2\int_0^x\frac{1}{1+\tau_1}\,d\tau_1+c_3\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1+\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2d\tau_1 + c_4\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1-\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2d\tau_1 \mid c_1,c_2,c_3,c_4\in\mathbb{K}\Big\}$$

(II) Connection: DE $\longleftrightarrow$ REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

be a (formal) power series. Then:

(II) Connection: DE $\longleftrightarrow$ REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

be a (formal) power series. Then:

There exist $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$ with

$$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0.$$

$$\Updownarrow$$

There exist $a_0(x), \ldots, a_\delta(x) \in \mathbb{K}[x]$ with

$$a_0(n) F(n) + \cdots + a_\delta(n) F(n+\delta) = 0.$$

(II) Connection: DE $\longleftrightarrow$ REC

Let

$$f(x)=\sum_{n=0}^{\infty}F(n)x^n$$

be a (formal) power series. Then:

There exist $b_0(x),\dots,b_\lambda(x)\in\mathbb{K}[x]$ with $$b_0(x)f(x)+\dots+b_\lambda(x)D^\lambda f(x)=0.$$

$\Updownarrow$ algorithmic

There exist $a_0(x),\dots,a_\delta(x)\in\mathbb{K}[x]$ with $$a_0(n)F(n)+\dots+a_\delta(n)F(n+\delta)=0.$$

Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x)$$
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for

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Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

for

$$\left(x^6 - 32x^5 + 256x^4\right) f^{(6)}(x) + \left(23x^5 - 528x^4 + 2560x^3\right) f^{(5)}(x)$$
$$+ \left(171x^4 - 2552x^3 + 6272x^2\right) f^{(4)}(x) + 2\left(245x^3 - 2002x^2 + 1728x\right) f^{(3)}(x)$$
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$$\Big\downarrow \texttt{Sigma.m}$$

$$F(n)=\frac{1}{\binom{2n}{n}^2}\Big(c_1+c_2S_1(n)\Big)=\frac{(1)_n(1)_n(1)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n n!}\frac{1}{16^n}\Big(c_1+c_2S_1(n)\Big)$$

Example 2: Find a power series solution

$$f(x) = c_1 \cdot {}_3F_2\left[\begin{matrix}1,1,1\\ \frac{1}{2},\frac{1}{2}\end{matrix};\frac{x}{16}\right] + c_2 \sum_{n=0}^{\infty} \frac{S_1(n)}{\binom{2n}{n}^2} x^n$$

for

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(III) A partial linear DE-solver

Find a power series solution

$$f(x_1,\dots,x_r)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}F(n_1,\dots,n_r)x_1^{n_1}\dots x_r^{n_r}$$

for

$$\sum_{(s_1,\dots,s_r)\in T}\underbrace{b_{(s_1,\dots,s_r)}(x_1,\dots,x_r)}_{\in\mathbb{K}[x_1,\dots,x_r]}D_{x_1}^{s_1}\dots D_{x_r}^{s_r}f(x_1,\dots,x_r)=0 \qquad \underset{\text{finite}}{T\subset\mathbb{N}^r}$$

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Note:

$\boxed{\exists \text{ a solution } f\in\mathbb{K}[x_1,\dots,x_r]}$ is algrithmically undecidable

[otherwise Hilbert's 10th problem would be alg. decidable; see Abramov/Petkovšek (2012)]

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$$\sum_{(s_1,\ldots,s_r)\in S}\underbrace{a_{(s_1,\ldots,s_r)}(n_1,\ldots,n_r)}_{\in\mathbb{K}[n_1,\ldots,n_r]}F(n_1+s_1,\ldots,n_r+s_r)=0 \qquad S\subset\mathbb{Z}^r \text{ finite}$$

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But: there are methods to hunt for solutions based on

M. Kauers, CS, *Partial denominator bounds for partial linear difference equations*, in: Proc. ISSAC'10 (2010)

M. Kauers, CS, *A refined denominator bounding algorithm for multivariate linear difference equations*, in: Proc. ISSAC'11 (2011)

J. Blümlein, M. Saragnese, CS, *Hypergeometric Structures in Feynman Integrals*, arXiv:2111.15501 [math-ph]

$$
\begin{aligned}
&(n+1)^2\left(k+n^2+2\right)\left(3kn^2-4k^2-5kn-12k+2n^3+2n^2-8n-8\right)F(n,k+1)\\
&+(n+1)^2\left(k+n^2+3\right)\left(2k^2-2kn^2+2kn+6k-n^3-n^2+4n+4\right)F(n,k+2)\\
&\qquad+(n+1)^2(k+n+1)\left(2k-n^2+n+4\right)\left(k+n^2+1\right)F(n,k)\\
&\qquad\qquad-(k+1)n^2(n+2)^2\left(k+n^2+2n+2\right)F(n+1,k)\\
&\qquad\qquad\qquad+kn^2(n+2)^2\left(k+n^2+2n+3\right)F(n+1,k+1)=0
\end{aligned}
$$

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&\quad+(n+1)^2(k+n+1)\left(2k-n^2+n+4\right)\left(k+n^2+1\right)F(n,k)\\
&\quad-(k+1)n^2(n+2)^2\left(k+n^2+2n+2\right)F(n+1,k)\\
&\quad+kn^2(n+2)^2\left(k+n^2+2n+3\right)F(n+1,k+1)=0
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$$

$$\Big\downarrow\; W=\{S_1(k),S_1(n+k),S_{2,1}(n+k)\}$$

degree bound 5

$$(n+1)^2\left(k+n^2+2\right)\left(3kn^2-4k^2-5kn-12k+2n^3+2n^2-8n-8\right)F(n,k+1)$$
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degree bound 5

37 solutions $\dfrac{p}{(1+n)^2(1+k+n^2)}$ with

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37 solutions $\dfrac{p}{(1+n)^2(1+k+n^2)}$ with

$$p\in\Big\{1+\frac{1}{2}nS_1(k+n),k,n,kn,kn^2,kn^3,kn^4,kS_1(n),knS_1(n),kn^2S_1(n),kn^3S_1(n),kS_1(n)^2,$$
$$knS_1(n)^2,kn^2S_1(n)^2,kS_1(n)^3,knS_1(n)^3,kS_1(n)^4,kS_{2,1}(n),knS_{2,1}(n),kn^2S_{2,1}(n),kn^3S_{2,}$$
$$kS_1(n)S_{2,1}(n),knS_1(n)S_{2,1}(n),kn^2S_1(n)S_{2,1}(n),kS_1(n)^2S_{2,1}(n),knS_1(n)^2S_{2,1}(n),$$
$$kS_1(n)^3S_{2,1}(n),kS_{2,1}(n)^2,knS_{2,1}(n)^2,kn^2S_{2,1}(n)^2,kS_1(n)S_{2,1}(n)^2,knS_1(n)S_{2,1}(n)^2,$$
$$kS_1(n)^2S_{2,1}(n)^2,kS_{2,1}(n)^3,knS_{2,1}(n)^3,kS_1(n)S_{2,1}(n)^3,kS_{2,1}(n)^4\Big\}$$

(IV) A solver for systems of partial DEs (arXiv:2111.15501)

Find a power series solution

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F(n,m)x^n y^m$$

for

$$(x-1)yD_{xy}f(x,y) + (x(2\varepsilon+\tfrac{7}{2}) - \varepsilon + 1)D_x f(x,y)$$
$$+(x-1)xD_x^2 f(x,y) + y(2\varepsilon+1)D_y f(x,y) + \tfrac{3}{2}(2\varepsilon+1)f(x,y) = 0,$$

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$$\begin{aligned}x(y-1)D_{xy}f(x,y) + x(4-\varepsilon)D_x f(x,y) + (y-1)yD_y^2 f(x,y)\\ +(y(\tfrac{13}{2}-\varepsilon) - \varepsilon + 1)D_y f(x,y) + \tfrac{3(4-\varepsilon)}{2} f(x,y) = 0.\end{aligned}$$

$$\downarrow$$

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$$-\tfrac{3}{2}(\varepsilon-4)F(n,m) - m(\varepsilon-1)F(n,m+1) = 0.$$

$\downarrow$ simplified Ore-Sato Theorem

$$F(n,m) = \Big(\prod_{i=1}^{n}\frac{(1+2i)(3+i-\varepsilon)}{2i(-2+i+\varepsilon)}\Big)\prod_{i=1}^{m}\frac{(1+2i+2n)(i+2\varepsilon)}{2i(-2+i+n+\varepsilon)}$$

(IV) A solver for systems of partial DEs (arXiv:2111.15501)

Find a power series solution

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F(n,m)x^ny^m=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}$$

for

$$(x-1)yD_{xy}f(x,y)+(x(2\varepsilon+\tfrac{7}{2})-\varepsilon+1)D_xf(x,y)$$
$$+(x-1)xD_x^2f(x,y)+y(2\varepsilon+1)D_yf(x,y)+\tfrac{3}{2}(2\varepsilon+1)f(x,y)=0,$$

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$\downarrow$

$$\tfrac{3}{2}(2\varepsilon+1)F(n,m)-n(\varepsilon-1)F(n+1,m)=0,$$
$$-\tfrac{3}{2}(\varepsilon-4)F(n,m)-m(\varepsilon-1)F(n,m+1)=0.$$

$\downarrow$ simplified Ore-Sato Theorem

$$F(n,m)=\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}$$

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \underbrace{\frac{\left(\frac{3}{2}\right)_{m+n} (4-\varepsilon)_n (1+2\varepsilon)_m}{m! n! (-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1} + F_0(n,m)\varepsilon^0 + \ldots}$$

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \underbrace{\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1}+F_0(n,m)\varepsilon^0+\ldots}$$

$$F_{-1}(n,m) = -\frac{1}{6}\frac{x^m y^n (3+n)!\left(\frac{3}{2}\right)_{m+n}}{n!(-2+m+n)!}$$

$$F_0(n,m) = \left[\ldots 6S_1(n) + 6S_1(m+n) - 12S_1(m)\right]\frac{x^m y^n (3+n)!\left(\frac{3}{2}\right)_{m+n}}{n!(-2+m+n)!}$$

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \underbrace{\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1}+F_0(n,m)\varepsilon^0+\ldots}$$

$$= \varepsilon^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{-1}(n,m) + \varepsilon^0 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_0(n,m) + \ldots$$

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$$= \varepsilon^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{-1}(n,m) + \varepsilon^0 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_0(n,m) + \ldots$$

$\Big\|$ `Sigma.m`

$$\varepsilon^{-1}\Big[-\frac{P_1(x,y)}{64(-1+x)^2(-1+y)^5(x-y)^3} - \frac{15x^6}{4(-1+x)^2(x-y)^4}\sum_{i=1}^{\infty}\frac{x^i\left(\frac{3}{2}\right)_i}{i!}$$

$$+\frac{P_2(x,y)}{64(-1+y)^5(x-y)^4}\sum_{i=1}^{\infty}\frac{y^i\left(\frac{3}{2}\right)_i}{i!}\Big] + \varepsilon^0\Big[\ldots\Big] + \ldots$$

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \underbrace{\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1}+F_0(n,m)\varepsilon^0+\dots}$$

$$= \varepsilon^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{-1}(n,m) + \varepsilon^0 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_0(n,m) + \dots$$

$||$ `Sigma.m`

$$\varepsilon^{-1}\Big[-\frac{P_1(x,y)}{64(-1+x)^2(-1+y)^5(x-y)^3} - \frac{15x^6}{4(-1+x)^2(x-y)^4}\sum_{i=1}^{\infty}\frac{x^i\left(\frac{3}{2}\right)_i}{i!}$$

$$+ \frac{P_2(x,y)}{64(-1+y)^5(x-y)^4}\sum_{i=1}^{\infty}\frac{y^i\left(\frac{3}{2}\right)_i}{i!}\Big] + \varepsilon^0\Big[\dots\Big] + \dots$$

$||$

$$\varepsilon^{-1}\Big[-\frac{15x^6}{4(x-y)^4(1-x)^{7/2}} - \frac{15y^3Q(x,y)}{64(x-y)^4(1-y)^{13/2}}\Big] + \varepsilon^0\Big[\dots\Big] + \dots$$

Back to Tactic 2: Solve coupled systems of differential equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\dots,\hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x),\dots,\hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} = A(x) \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} + \begin{pmatrix}\hat{R}_1(x)\\ \dots \\ \hat{R}_\lambda(x)\end{pmatrix}$$

$\downarrow$ uncoupling algorithms (Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

DE-solver

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\dots,\hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x),\dots,\hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} = A(x) \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} + \begin{pmatrix}\hat{R}_1(x)\\ \dots \\ \hat{R}_\lambda(x)\end{pmatrix}$$

$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

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DE-solver ← → REC-solver

Tactic 2: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

Tactic 2: the DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$ $\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$

Tactic 2: the DE-REC approach

DE system
$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

$\xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}}$

uncoupled DE system
$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)),\, k > 1$$

$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

Tactic 2: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$

$$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$$

$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

↓ holonomic closure prop. ↓

linear recurrence

$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

Tactic 2: the DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$

$$\hat{I}_k(x) = \text{expr}_k(\hat{I}_1(x)), k > 1$$

↓ holonomic closure prop. ↓

$$\hat{I}_1(x) = \sum_{n=0}^{\infty} I_1(n) x^n$$

linear recurrence

$$\sum_i a_i'(n) I_1(n+i) = r'(n)$$

← Sigma's REC-solver ←

closed form solutions of $I_1(n)$ in the class of nested sums and products – if this is possible

Tactic 2: the DE-REC approach (SolveCoupledSystem package)

DE system
$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)
uncoupling algorithm

uncoupled DE system
$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\hat{\mathbf{I}}_{\mathbf{k}}(\mathbf{x}) = \text{expr}_{\mathbf{k}}(\hat{\mathbf{I}}_{\mathbf{1}}(\mathbf{x})), \mathbf{k} > \mathbf{1}$$

extract coefficient
SumProduction package

holonomic closure prop.

linear recurrence
$$\sum_i a'_i(n) I_1(n+i) = r'(n)$$

`Sigma's` REC-solver

closed form solutions of
$I_1(n), \mathbf{I_2(n)}, \ldots, \mathbf{I_n(n)}$
in the class of nested
sums and products
– if this is possible

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \mathsf{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$$

$$I_i(n) = \varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \cdots + \varepsilon^{o_i} F_{o_i}(n) + \ldots$$

General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

$\downarrow$ solver for $\hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$

$$I_i(n) = \varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \cdots + \varepsilon^{o_i} F_{o_i}(n) + \ldots$$

Complications:

- different uncoupling methods and inputs lead to different orders o_i and recurrence orders
- different complexity in solving and providing boundary conditions
- extra complication: recursively defined systems

General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
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$\downarrow$ solver for $\hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$

$$I_i(n) = \varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \dots + \varepsilon^{o_i} F_{o_i}(n) + \dots$$

Complications:

- different uncoupling methods and inputs lead to different orders o_i and recurrence orders
- different complexity in solving and providing boundary conditions
- extra complication: recursively defined systems

Nikolai Fadeev: refined methods to find optimal uncoupling strategy

General strategy: physical problem $\hat{P}(x)$

$\big\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$\big\downarrow$ solver for $\hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$

$$I_i(n) = \varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \dots + \varepsilon^{o_i} F_{o_i}(n) + \dots$$

$\big\downarrow$ plug into $\hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$

$$P(n) = \varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n) + \varepsilon^{0} P_{0}(n) + \dots$$

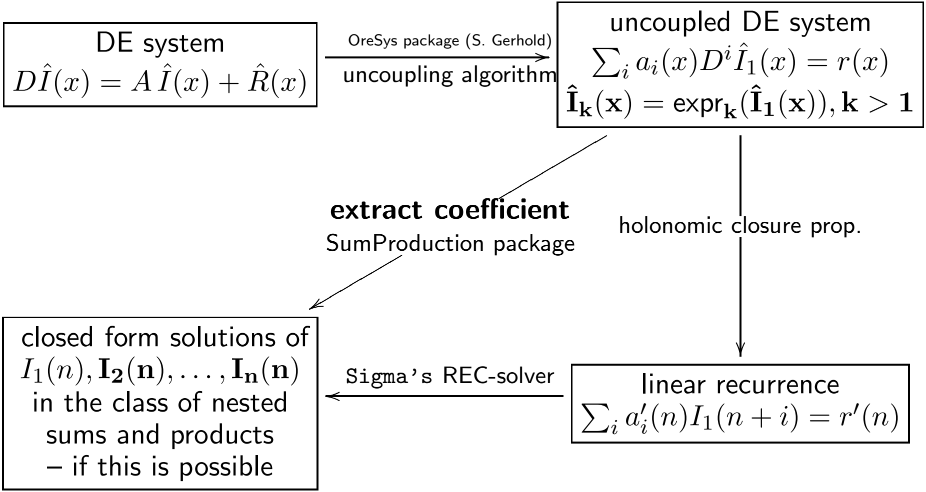
Calculations based on Tactic 2:

- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(n)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, pp. 263-288. 2014.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, pp. 280-317. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, pp. 733-823. 2014.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, pp. 48-151. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, pp. 612-644. 2015.
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), pp. 1-19. 2015.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), pp. 1-19. 2016.
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, pp. 33-112. 2016.
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D 97(094022), pp. 1-44. 2018.
- J. Ablinger, J. Blümlein, A. De Freitas, CS, K. Schönwald. The two-mass contribution to the three-loop pure singlet operator matrix element. Nucl. Phys. B(927), pp. 339-367. 2018. ISSN 0550-3213.
- J. Blümlein, A. De Freitas, CS, K. Schönwald. The Variable Flavor Number Scheme at Next-to-Leading Order. Physics Letters B 782, pp. 362-366. 2018.
- J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit. Physics Letters B 782, pp. 528-532. 2018.

Tactic 4: Compute large moments and guessing recurrences

[coming, e.g., from IBP methods]

Tactic 2: the DE-REC approach (SolveCoupledSystem package)



J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

Tactic 3: compute large moments (SolveCoupledSystem package)

DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$

OreSys package (S. Gerhold)
uncoupling algorithm

uncoupled DE system
$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$
$\hat{I}_k(x) = \mathrm{expr}_k(\hat{I}_1(x)), k > 1$

extract coefficient
SumProduction package

holonomic closure prop.

linear recurrence
$\sum_i a'_i(n) I_1(n+i) = r'(n)$

Sigma's REC-solver

closed form solutions of
$I_1(n)$
in the class of nested
sums and products
– if th

Given $r'(0), r'(1), \dots, r'(5114)$,
compute $I_1(0), I_1(1), \dots, I_1(5114)$

J. Blümlein, CS, 2017, arXiv:1701.04614 [hep-ph].

General strategy: physical problem $\hat{P}(x)$

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$$

$$I_i(n) = \underbrace{\varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_{0}(n) + \dots}_{\text{only numbers}}$$

$n = 0, 1, \dots, 8000$

$$\Big\downarrow \text{plug into } \hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(n)}_{\text{numbers}} + \dots$$

$n = 0, 1, \dots, 8000$

coupled systems for $P(n)$

`SolveCoupledSystem.m` ↓

large no. of moments

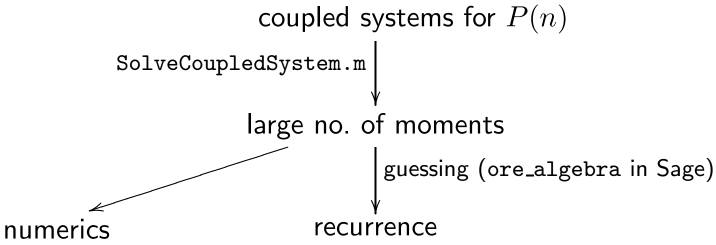
coupled systems for $P(n)$

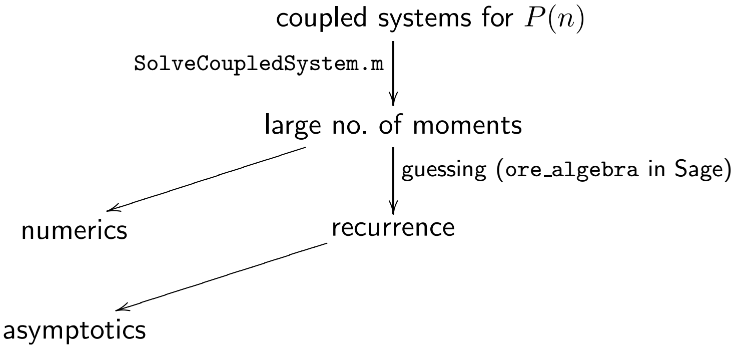
↓ `SolveCoupledSystem.m`

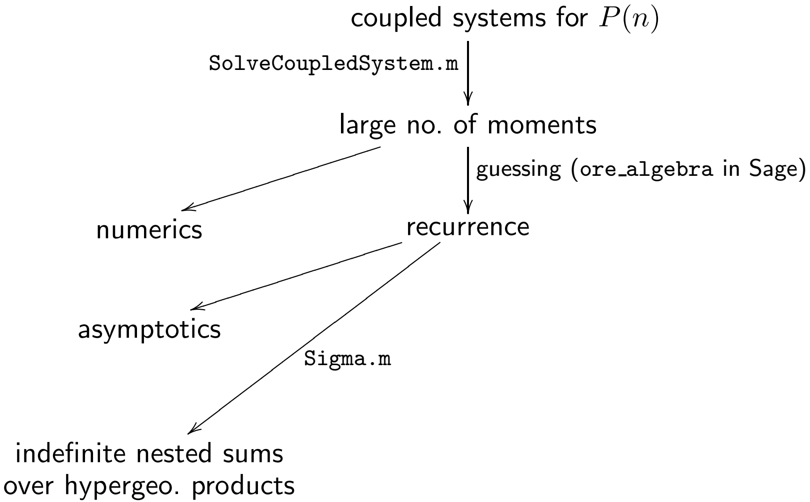
large no. of moments

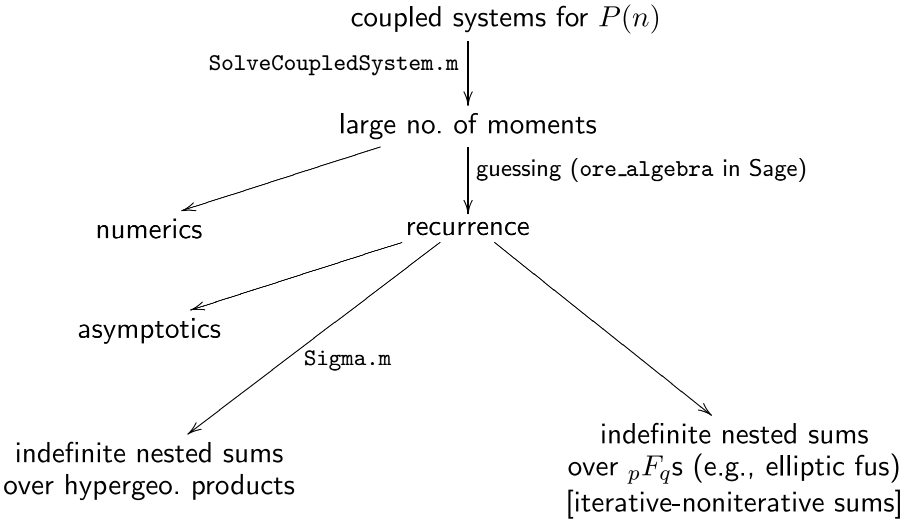
↙

numerics


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products


coupled systems for $P(n)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]

coupled systems for $P(n)$

↓ `SolveCoupledSystem.m`

large no. of moments

↓ guessing (`ore_algebra` in Sage)

recurrence

↓ `Sigma.m`

indefinite nested sums
over hypergeo. products

General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$\downarrow$ solver for $\hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$

$$I_i(n) = \underbrace{\varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_{0}(n) + \dots}_{\text{only numbers}}$$

$n = 0, 1, \dots, 8000$

$\downarrow$ plug into $\hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{numbers}} + \underbrace{\varepsilon^{0} P_{0}(n)}_{\text{numbers}} + \dots$$

$n = 0, 1, \dots, 8000$

General strategy: physical problem $\hat{P}(x)$

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{P}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$\downarrow$ solver for $\hat{I}_i(x) = \sum_{n=0}^{\infty} I_i(n) x^n$

$$I_i(n) = \underbrace{\varepsilon^{-3} F_{-3}(n) + \varepsilon^{-2} F_{-2}(n) + \varepsilon^{-1} F_{-1}(n) + \varepsilon^{0} F_0(n) + \dots}_{\text{only numbers}}$$

$\downarrow$ plug into $\hat{P}(x) = \sum_{n=0}^{\infty} P(n) x^n$

$$P(n) = \underbrace{\varepsilon^{-3} P_{-3}(n) + \varepsilon^{-2} P_{-2}(n) + \varepsilon^{-1} P_{-1}(n)}_{\text{nice}} + \underbrace{\varepsilon^{0} P_0(n)}_{\text{partially nice}} + \dots$$

all n solution

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[6]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= **initial =<< iFile16**

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[6]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= **initial =<< iFile16**

Out[7]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,

475840076183/7501410000, 1432950323678333/21965628762000,

21648380901382517/328583783127600,

5286978432377857 6751/802218994536960000,

49422862094045523994231/7537739922306161568 00,

331318798329079359207261 13/509557943985299969760000,

52092747218367551684487 77/80949984111854180459136,

561437119973447690210411 45213/88258926638358645638435 3664,

4535004333538456281947900 25124807/721722804887946855688695 0000000,

1406154337412047988611015 9898869387/226643167590350326435656 036000000,

71558652266649190332490578517861993657116837030770022280781149589503 0000000,

16286729046359273892841271257418854056836413/26939658805548039040134334473694310 4000000,

14287296426323024679514269058446918378052 99/23940759575034122827861315961573673600000,

49893869021959529450510280919915455078308 0767/84688836678529798131712623040540027 20000000, In[8]:= **rec =<< rFile16**

Out[8]=

$$(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} + 2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + 873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + 1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + 575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + 69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + 901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + 7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + 32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + 86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} + 124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} + 87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} + 25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} + 1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 P[n]$$

$$
\begin{aligned}
&-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 161315106167193 6n^{37} + \\
&101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
&2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
&9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
&7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
&356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
&9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
&170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
&1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} + \\
&13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
&59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
&167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
&284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
&279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} + \\
&149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} + \\
&38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} + \\
&3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} + \\
&101701393436276172443717692853400n + 6204709909986751913151675960000\Big)\mathrm{P}[n+1]
\end{aligned}
$$

$$+2(n+3)^2(2n+5)^3(2n+9)(2n+11)\Big(24739011624960n^{40}+3317836466356224n^{39}+215508170284466176n^{38}+9032884062187945984n^{37}+$$
$$274636134389959884800n^{36}+6455501959255126179840n^{35}+122094572934385260036096n^{34}+1909387225793663151898624n^{33}+$$
$$25180108291969215434326016n^{32}+284171960705270647479074816n^{31}+2775794400720227034854326272n^{30}+$$
$$23677622163992853854566219776n^{29}+177624312783583749157935120384n^{28}+1178515602115604757944201871360n^{27}+$$
$$6947091965313419323781358354432n^{26}+36515023100308314818702129258496n^{25}+171621148571344894953594594017280n^{24}+$$
$$722837793013976317556258102507520n^{23}+2732534027077907914497042720534528n^{22}+9281028665970648470895368668485120n^{21}+$$
$$28337819215557708948254385336117248n^{20}+77786125749274632150536464583130752n^{19}+191877161455672780973502244537632256n^{18}+$$
$$424953221702140663089937921965135648n^{17}+843818276409975584824720931649555264n^{16}+$$
$$1499359936674956711935311062995422344n^{15}+2378007025570977662661938772843220240n^{14}+$$
$$3355671771434535852147325502571953770n^{13}+4196375762867184563407432891655585484n^{12}+$$
$$4627675779563752366067861596232781096n^{11}+4473175960511956000526499430851993603n^{10}+$$
$$3761696365025837909581516781307249585n^{9}+2726553473467254373993685951699145492n^{8}+$$
$$1683383212304999468664293798012773485n^{7}+871926653651504419744271839781064837n^{6}+$$
$$371307437598003570058538796122994147n^{5}+126427972742886389602285855482966072n^{4}+33048762330145623969058704448697313n^{3}+$$
$$6217924746857741077419160100404560n^{2}+748298077423337427195946099994100n+43181089548034246077698611794000\Big)P[n+2]$$

$$
\begin{aligned}
&-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40}+3322784268681216n^{39}+216160919414112256n^{38}+9074528155284275200n^{37}+\\
&276348048819456311296n^{36}+6506479077331107315712n^{35}+123266585640616142569472n^{34}+1931040885785102661976064n^{33}+\\
&25510503383281445462081536n^{32}+288418124175428279391485952n^{31}+2822442799033603081019326464n^{30}+\\
&24120717233320712351821332480n^{29}+181295944719289040999116701696n^{28}+1205246297785423925076555694080n^{27}+\\
&7119049557560114436136213413888n^{26}+37496933571993839665392189775872n^{25}+176616172467048982234270428880896n^{24}+\\
&745539218875020737621728364206080n^{23}+2824909633156578132652259733712896n^{22}+9618101958268071244680677589035520n^{21}+\\
&29441860528446423517613263360742912n^{20}+81033563306363873505877563416477312n^{19}+200454769103641040142838133702338304n^{18}+\\
&445286624972461749049425309485328992n^{17}+887028447418790661018847407251573152n^{16}+\\
&1581538101499869694224895701784875304n^{15}+2517550244392724509968791166585362672n^{14}+\\
&3566593026520465155504695877897282630n^{13}+4479066125207404898722179511912639638n^{12}+\\
&4962006990874351800791769650243464872n^{11}+4819992643914265990647887896664485209n^{10}+\\
&4074895386694182240941538222230233221n^{9}+2970477229398746689186622534784613554n^{8}+\\
&1845274131994015990683957902602775337n^{7}+962091291302144537393228847830431614n^{6}+\\
&412595107814836563208757757032740146n^{5}+141540723940232563767779647013785485n^{4}+37292931812630561528276365992452010n^{3}+\\
&7074865777225416725452872895397100n^{2}+858794112392644074221312049837000n+49997386738260112603615104780000\Big)f[n+3]
\end{aligned}
$$

$$+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} +$$
$$3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} +$$
$$38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} +$$
$$69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} +$$
$$32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} +$$
$$1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} +$$
$$20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} +$$
$$260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} +$$
$$2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} +$$
$$10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} +$$
$$34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} +$$
$$69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} +$$
$$82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} +$$
$$55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} +$$
$$19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} +$$
$$3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} +$$
$$168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n +$$
$$127412073235174465112560388640 0\Big)P[n+4]$$

$$-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32} + 28361279668224n^{31} + 1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + 136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + 83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + 12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + 510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + 6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + 18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + 29346856353503020415409305704n^{10} + 29775859546803351930591002266n^{9} + 25770328899499991754425455738n^{8} + 18817114309842270306167785140n^{7} + 11424980760825630752861027739n^{6} + 5656051955667821083952617134n^{5} + 2221448212382554437709999491n^{4} + 664859653803075491350122060n^{3} + 142190920852333874895041748n^{2} + 19313175036907229252501700n + 1248723341516324359641600\Big) \mathrm{P}[n+5]==0$$

In[9]:= $\mathbf{recSol = SolveRecurrence[rec, P[n]]}$

In[9]:= **recSol = SolveRecurrence[rec, P[n]]**

Out[9]=

$$\{\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\}$$

$$\{0, -\frac{(3+2n)(-8-9n+2n^2)}{(1+n)^2(1+2n)^2}\}$$

$$\{0, -\frac{(3+2n)(-5+8n^2)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\}$$

$$\{0, \frac{(3+2n)(-513-2184n-2416n^2+768n^4)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \Big(-\frac{2(3+2n)(3+44n+48n^2)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} + \frac{12(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^2}{(1+n)(1+2n)} + \frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)(3+44n+48n^2)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\}$$

$$\{0, \frac{1}{16(1+n)^4(1+2n)^4}\big(72519 + 572343n + 1814716n^2 + 2918100n^3 + 2442240n^4 + 912896n^5 + 24576n^6 - 49152n^7\big) + \frac{16(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{3(1+n)(1+2n)} + \Big(-\frac{(3+2n)(29+307n+322n^2)}{4(1+n)^2(1+2n)^2} + \frac{44(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i^2} +$$

$$\Big(\frac{(3+2n)(91+259n+974n^2+1784n^3+1024n^4)}{4(1+n)^3(1+2n)^3} + \frac{22(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \frac{24(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)(-13-4n+16n^2)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{16(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} + \Big(-$$

$$\frac{(3+2n)(19+92n+80n^2)}{(1+n)^2(1+2n)^2} + \frac{40(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)^2 + \frac{20(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^3}{3(1+n)(1+2n)} +$$

$$\frac{64(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)} - \frac{3(3+2n)(63+209n+150n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2} +$$

$$\Big(\frac{(3+2n)(347+1795n+4302n^2+4856n^3+2048n^4)}{2(1+n)^3(1+2n)^3} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{-1+2i} -$$

$$\frac{4(3+2n)(19+92n+80n^2)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2} + \frac{32(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)} -$$

$$\frac{8(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)^2}{i}}{(1+n)(1+2n)} - \frac{16(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)^2}{-1+2i}}{(1+n)(1+2n)}$$

$$- \frac{32(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)} + \frac{64(3+2n)\displaystyle\sum_{i=1}^{n}\frac{\left(\displaystyle\sum_{j=1}^{i}\frac{1}{j}\right)\displaystyle\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)} +$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{-1+2j}\right)^2}{i}}{(1+n)(1+2n)} + \frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{-1+2j}\right)^2}{-1+2i}}{(1+n)(1+2n)}\}, \{1, 0\}\}$$

In[10]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

In[10]:= **sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]**

Out[10]=

$$\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-$$

$$\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+$$

$$\frac{44(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\frac{(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{i})^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^3}{9(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\frac{16(3+2n)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+$$

$$\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-$$

$$\frac{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\frac{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)^2(1+2n)^2}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{i})(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)(1+2n)}+\frac{64(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^3}{9(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{i}}{3(1+n)(1+2n)}-$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}3(1+n)(1+2n)}{+}\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)}$$

In[11]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[12]:= **sol = TransformToSSums[sol];**

In[13]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[11]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[12]:= **sol = TransformToSSums[sol];**

In[13]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

$$\text{Out[13]=}\ \frac{1}{3(1+n)^4(1+2n)^4}\big(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\big)+\frac{64(3+2n)^2S[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)S[1,n]^2}{3(1+n)(1+2n)^2}+\Big(-\frac{2(3+2n)\big(147+985n+1871n^2+1268n^3+212n^4\big)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)S[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)S[-2,2n]}{3(1+n)(1+2n)}\Big)S[1,2n]-\frac{4(3+2n)\big(23+123n+114n^2\big)S[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)S[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\big(53+229n+190n^2\big)S[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[3,2n]}{3(1+n)(1+2n)}+\Big(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)S[1,2n]}{3(1+n)(1+2n)^2}\Big)S[-1,2n]-\frac{64(3+2n)(2+3n)S[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\big(1+8n+8n^2\big)S[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)S[-2,1,2n]}{3(1+n)(1+2n)}$$

In[11]:= **<< HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[12]:= **sol = TransformToSSums[sol];**

In[13]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[14]:= **SExpansion[sol, n, 2]**

Out[14]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

Calculations based on Tactic 3:

- J. Blümlein, CS. The Method of Arbitrarily Large Moments to Calculate Single Scale Processes in Quantum Field Theory. Physics Letters B 771, pp. 31-36. 2017.
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Symbolic tools for special functions

Nested sums	**Nested integrals**	**Special numbers**
Harmonic Sums $\sum_{k=1}^{n}\frac{1}{k}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	Harmonic Polylogarithms $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{1+z}$	multiple zeta values $\int_0^1 dx\frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2)+\ldots$
gen. Harmonic Sums $\sum_{k=1}^{n}\frac{(1/2)^k}{k}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	gen. Harmonic Polylogarithms $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{z-3}$	gen. multiple zeta values $\int_0^1 dx\frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3)+\ldots$
Cycl. Harmonic Sums $\sum_{k=1}^{n}\frac{1}{(2k+1)}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	Cycl. Harmonic Polylogarithms $\int_0^x \frac{dy}{1+y^2}\int_0^y \frac{dz}{1-z+z^2}$	cycl. multiple zeta values $\mathbf{C} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$
Binomial Sums $\sum_{k=1}^{n}\frac{1}{k^2}\binom{2k}{k}(-1)^k$	root-valued iterated integrals $\int_0^x \frac{dy}{y}\int_0^y \frac{dz}{z\sqrt{1+z}}$	associated numbers $\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$
	iterated integrals on ${}_2F_1$'s $\int_0^z \frac{\ln(x)}{1+x}\,{}_2F_1\left[\begin{matrix}\frac{4}{3} & \frac{5}{3}\\ & 2\end{matrix};\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]dx$	associated numbers $\int_0^1 {}_2F_1\left[\begin{matrix}\frac{4}{3} & \frac{5}{3}\\ & 2\end{matrix};\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]dx$

Nested sums	**Nested integrals**	**Special numbers**
Harmonic Sums $\sum_{k=1}^{n}\frac{1}{k}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	Harmonic Polylogarithms $\int_0^x\frac{dy}{y}\int_0^y\frac{dz}{1+z}$	multiple zeta values $\int_0^1 dx\frac{\mathrm{Li}_3(x)}{1+x}=-2\mathrm{Li}_4(1/2)+\ldots$
gen. Harmonic Sums $\sum_{k=1}^{n}\frac{(1/2)^k}{k}\sum_{l=1}^{k}\frac{(-1)^l}{l^3}$	gen. Harmonic Polylogarithms $\int_0^x\frac{dy}{y}\int_0^y\frac{dz}{z-3}$	gen. multiple zeta values $\int_0^1 dx\frac{\ln(x+2)}{x-3/2}=\mathrm{Li}_2(1/3)+\ldots$
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Binomial Sums $\sum_{k=1}^{n}\frac{1}{k^2}\binom{2k}{k}(-1)^k$	root-valued iterated integrals $\int_0^x\frac{dy}{y}\int_0^y\frac{dz}{z\sqrt{1+z}}$	associated numbers $\mathrm{H}_{8,w_3}=2\mathrm{arccot}(\sqrt{7})^2$
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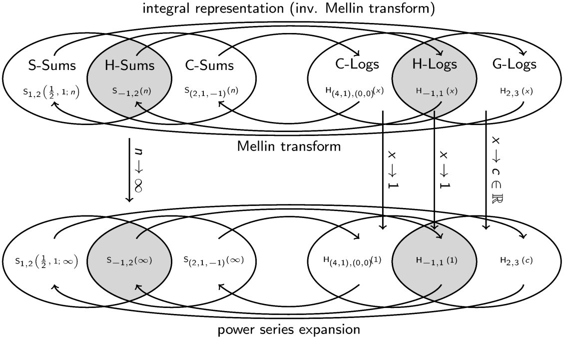

Symbolic tools for special functions

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,...)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

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Integral representation:

$$=\int_0^1 \frac{x^n-1}{1-x}\left(\int_0^x \frac{\int_0^y \frac{1}{1-z}dz}{y}dy-\zeta(2)\right)dx, \qquad \zeta(z):=\sum_{i=1}^{\infty}1/i^z$$

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Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$
$$- \tfrac{1}{100n^5} - \tfrac{1}{6n^4} + \tfrac{13}{36n^3} - \tfrac{1}{4n^2} - \tfrac{1}{n} + 2\zeta(3) + O(\tfrac{\ln(n)}{n^6}).$$

limit computations **numerical evaluation**

- Generalized algorithms for generalized harmonic sums

$$\sum_{k=1}^{n}\frac{2^k\sum_{i=1}^{k}\frac{2^{-i}\sum_{j=1}^{i}\frac{H_j}{j}}{i}}{k}=-\frac{21\zeta(2)^2}{20n}+\frac{1}{8n^2}+\frac{295}{216n^3}-\frac{1115}{96n^4}+O(n^{-5})$$
$$+\big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\big)\zeta(2)$$
$$+2^n\big(\frac{3}{2n}+\frac{3}{2n^2}+\frac{9}{2n^3}+\frac{39}{2n^4}+O(n^{-5})\big)\zeta(3)$$
$$+\big(\frac{1}{n}+\frac{3}{4n^2}-\frac{157}{36n^3}+\frac{19}{n^4}+O(n^{-5})\big)(\log(n)+\gamma)$$
$$+\big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\big)(\log(n)+\gamma)^2)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for cyclotomic harmonic sums

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{1+2i}}{j^2}}{(1+2k)^2} = \Big(-3+\frac{35\zeta(3)}{16}\Big)\zeta(2)-\frac{31\zeta(5)}{8}$$
$$+\frac{1}{n}-\frac{33}{32n^2}+\frac{17}{16n^3}-\frac{4795}{4608n^4}+O(n^{-5})$$
$$+\log(2)\Big(6\zeta(2)-\frac{1}{n}+\frac{9}{8n^2}-\frac{7}{6n^3}+\frac{209}{192n^4}+O(n^{-5})\Big)$$
$$+\Big(-\frac{7}{4}-\frac{7}{16n}+\frac{7}{16n^2}-\frac{77}{192n^3}+\frac{21}{64n^4}+O(n^{-5})\Big)\zeta(3)$$
$$+\Big(\frac{1}{16n^2}-\frac{1}{8n^3}+\frac{65}{384n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{n} \frac{4^j H_{j-1}}{\binom{2j}{j} j^2} = 7\zeta(3) + \sqrt{\pi}\sqrt{n}\Bigg\{\Bigg[-\frac{2}{n} + \frac{5}{12n^2} - \frac{21}{320n^3} - \frac{223}{10752n^4} + \frac{671}{49152n^5} + \frac{11635}{1441792n^6} - \frac{1196757}{136314880n^7} - \frac{376193}{50331648n^8} + \frac{201980317}{18253611008n^9} + O(n^{-10})\Bigg]\ln(\bar{n}) - \frac{4}{n} + \frac{5}{18n^2} - \frac{263}{2400n^3} + \frac{579}{12544n^4} + \frac{10123}{1105920n^5} - \frac{1705445}{71368704n^6} - \frac{27135463}{11164188672n^7} + \frac{197432563}{7927234560n^8} + \frac{405757489}{775778467840n^9} + O(n^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]
Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

Conclusion

Our calculations rely on

1. symbolic summation and integration methods to derive recurrences
2. flexible recurrence and DE solver
3. coupled systems solver
4. the large moment method

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6. stable and efficient software packages

Main CA-packages

In[15]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[16]:= << **MultiIntegrate.m**

MultiIntegrate by Jakob Ablinger © RISC-Linz

In[17]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[18]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[19]:= << **SumProduction.m**

SumProduction by Carsten Schneider © RISC-Linz

In[20]:= << **OreSys.m**

OreSys by Stefan Gerhold (optimized by Carsten Schneider) © RISC-Linz

In[21]:= << **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

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Within the RISC-DESY cooperation we expect that we will discover and explore many

new algorithms in CA and results in QFT!