Classical Scattering in GR from a Gauge-Invariant Double Copy

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with

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SAGEX CLOSING MEETING, 24-27/6/2022

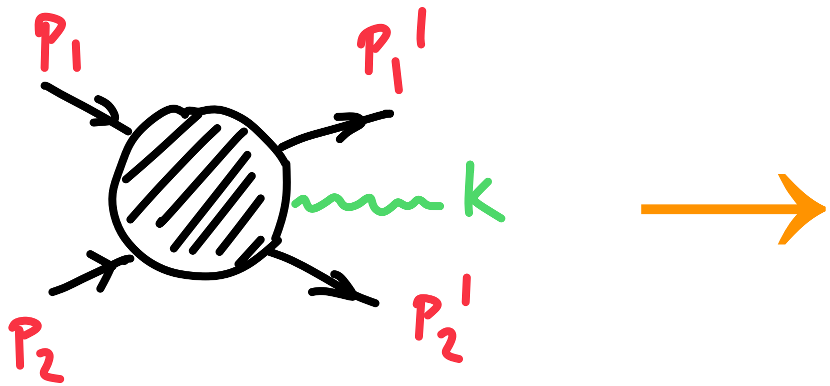
SAGEX Scattering Amplitudes: from Geometry to Experiment

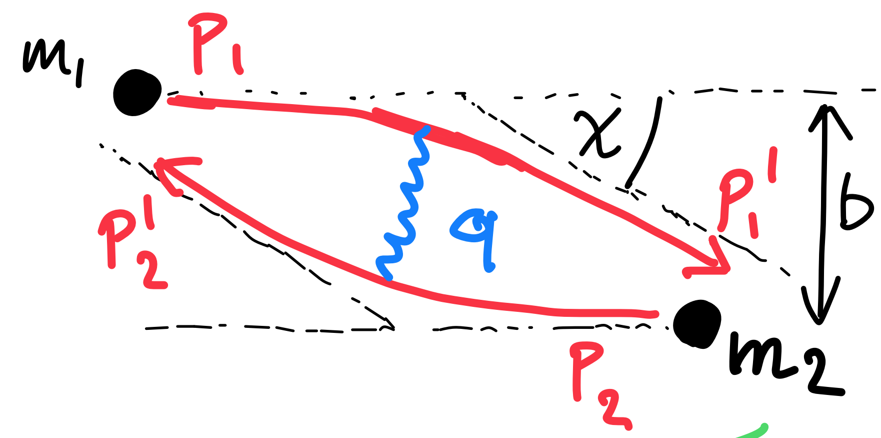
Queen Mary University of London

Outline

- Efficient amplitudes methods to compute classical observables in (effective field theories) of General Relativity
- Heavy Mass Effective Theory:
  - Refined Soft Expansion
  - Perform $\hbar \to 0$ as soon as possible
- A new gauge invariant double copy:
  - new version of Gravity = (Yang-Mills)^2
  - BCJ numerators for any multiplicity
  - Kinematic Algebra = Hopf algebra

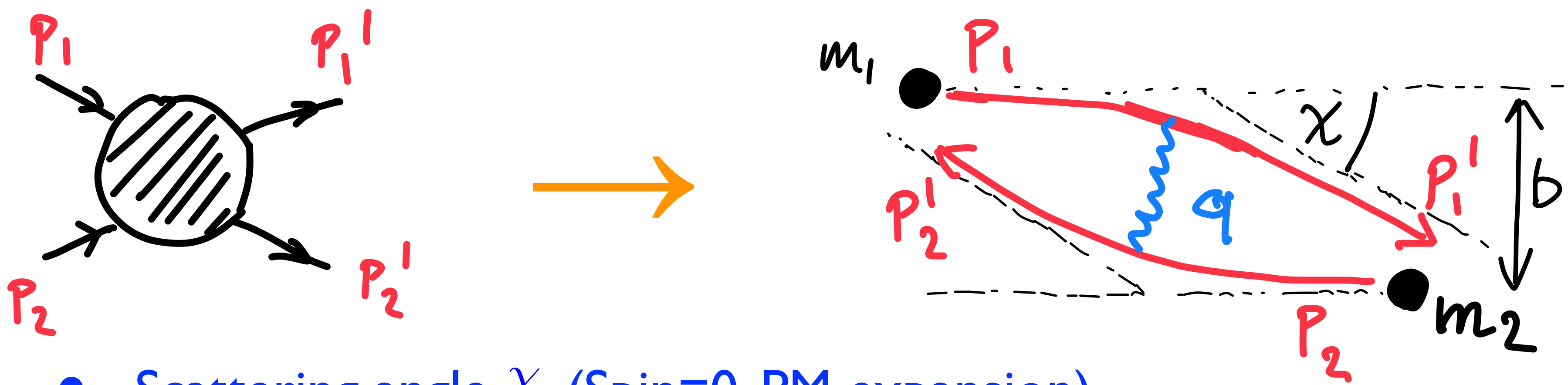
From amplitudes to (pre)observables





- Effective potentials/one-body Hamiltonians
  - Fourier Transform of $2 \rightarrow 2$ amplitudes
  - EFT matching (talk by Zvi)
  - Lippmann-Schwinger (talk by Emil)
- Observables: bending angle, wave forms…
  - Eikonal Approach & Radial Action (talk by Emil & Zvi)
  - KMOC and radiation (talks by David and Ruth)

Black hole scattering from HEFT



- Scattering angle χ (Spin=0, PM expansion)
  - Depends on: $m_1, \quad m_2, \quad y = v_1 \cdot v_2, \quad b \overset{F.T.}{\longleftrightarrow} q$

    $p_1 = m_1 v_1, \quad p_2 = m_2 v_2$

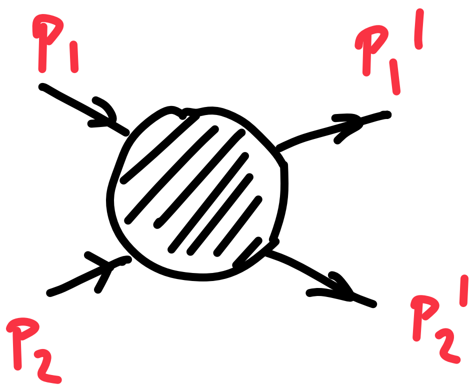
- Note $\quad m_{1,2} \ , \ p_{1,2} \ >> \ q \ , \ l_i$ (momenta of exchanged gravitons)
  - Classical limit: $q \ \rightarrow \ \hbar\, q \ , \ l_i \ \rightarrow \hbar\, l_i$ while $\hbar \rightarrow 0$
  - Same as heavy mass exp., drop quantum corrections
  - Reminiscent of Heavy-Quark Effective Theory (Georgi…)
- Heavy-Mass Effective Field Theory (HEFT)
  (Damgaard, Haddad, Helset; AB, Chen, Travaglini, Wen)

Summary of tools

- Modern, generalised unitarity + integration techniques/ differential equations.
  - Full soft region not only potential region!
  - Focus on non-analytic terms in q^2
    $\longrightarrow$ long-range physics
- Input for unitarity cuts:
  - trees with 2 massive scalars + n gravitons
    $\longrightarrow$ expand from the get-go $\longrightarrow$ HEFT amplitudes
  - pure graviton amplitudes
- Color-kinematic duality for HEFT amplitudes (more later)
  - gauge invariant, all multiplicity expressions
  - kinematic algebra = quasi-shuffle Hopf algebra

Kinematics: to bar or not to bar

- Barred variables: produce a clean expansion in $\hbar$
- Once we have defined a classical observable, removing quantum and hyper classical pieces we can switch to original variables



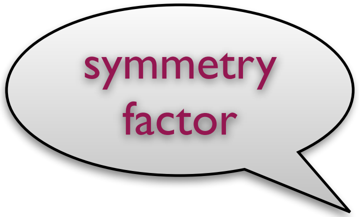
$$(p_1)^\mu := \bar{p}_1^\mu + \frac{q^\mu}{2} = (E_1, \vec{p} + \vec{q}/2)\,,$$
$$(p_1')^\mu := \bar{p}_1^\mu - \frac{q^\mu}{2} = (E_1, \vec{p} - \vec{q}/2)\,,$$
$$(p_2)^\mu := \bar{p}_2^\mu - \frac{q^\mu}{2} = (E_2, -\vec{p} - \vec{q}/2)\,,$$
$$(p_2')^\mu := \bar{p}_2^\mu + \frac{q^\mu}{2} = (E_2, -\vec{p} + \vec{q}/2)\,.$$

- Crucially: $\bar{p}_{1,2} \cdot q = 0$ while $p_i \cdot q = \pm q^2/2$
- Also: $(\bar{p}_{1,2})^2 = \bar{m}_i^2 = m_i^2 - q^2/4$

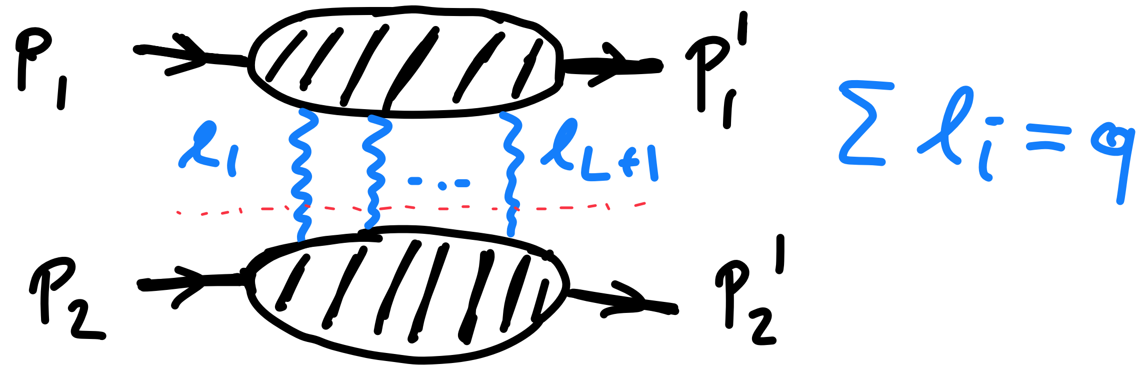
$$\bar{y} = \frac{\bar{p}_1 \cdot \bar{p}_2}{\bar{m}_1 \bar{m}_2} = \bar{v}_1 \cdot \bar{v}_2$$

$$J = |\vec{p}| b$$

On to the HEFT expansion



$$\frac{1}{(L+1)!}$$



- Consider an L-loop cut:
- Each blob is a tree amplitude M with 2 massive scalars and many gravitons
- Crucially, we can expand M cleanly in inverse powers of masses or equivalently in powers of $\hbar$
  - For this we need to expand the massive propagators

$$\frac{1}{(\bar{p} \pm q/2 + Q)^2 - m^2 + i\varepsilon} \simeq \frac{1}{2\bar{p}\cdot Q + i\varepsilon} + \text{higher powers } \hbar$$

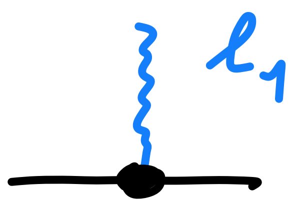
  - and in the sum of diagrams we always run into

$$\frac{1}{2\bar{p}\cdot Q + i\,\varepsilon} + \frac{1}{-2\bar{p}\cdot Q + i\,\varepsilon} = -i\pi\delta(\bar{p}\cdot Q)$$

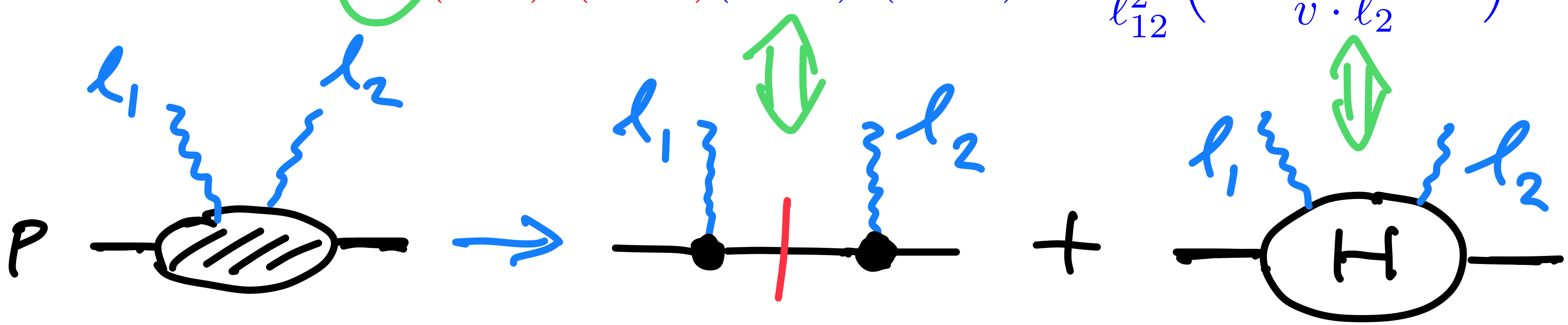
HEFT expansion cont'd

- In a nutshell we find that any M can be expanded in terms of HEFT amplitudes and delta functions!
- Best to see some examples

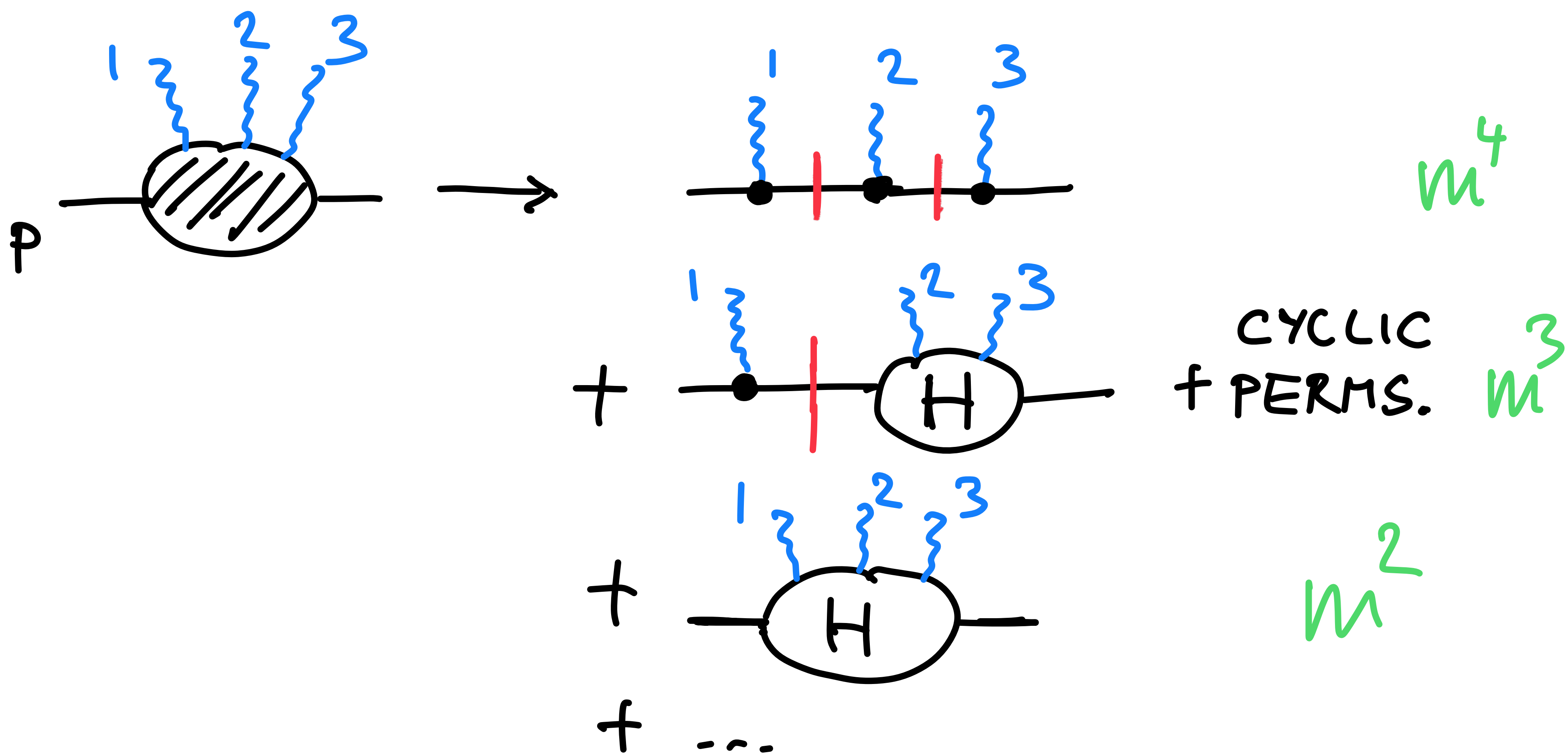
3 points: $M_3 \to \bar{m}^2(\bar{v}\cdot\varepsilon_1)^2$



4 points: $M_4 \to \bar{m}^3(-i\pi)\delta(\bar{v}\cdot\ell_1)(\bar{v}\cdot\varepsilon_1)^2(\bar{v}\cdot\varepsilon_2)^2 + \frac{\bar{m}^2}{\ell_{12}^2}\left(\frac{\bar{v}\cdot F_1\cdot F_2\cdot\bar{v}}{\bar{v}\cdot\ell_2}\right)^2 + \cdots$



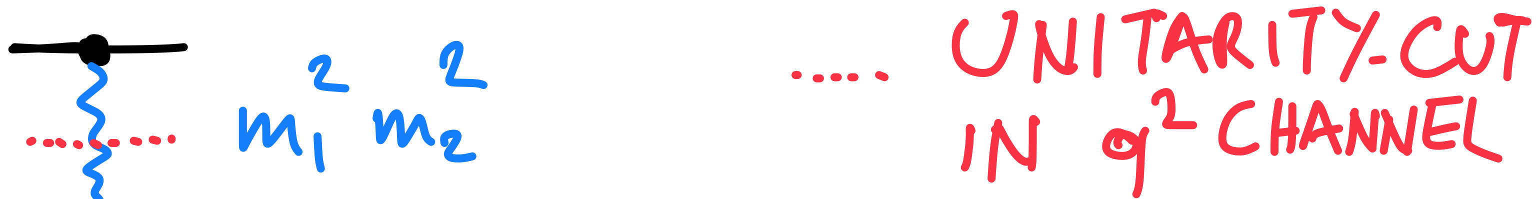
...and schematically for 5 point:



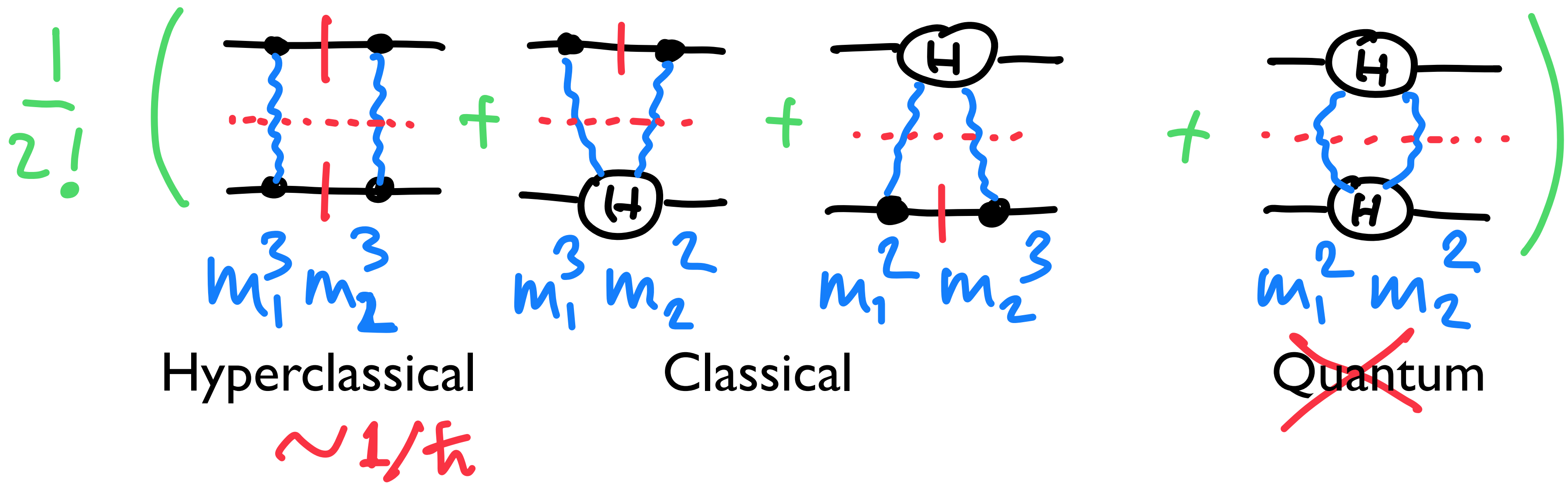
HEFT diagrams to the masses

- Expansion in masses = expansion in HEFT amplitudes

Tree level:



One loop:



Exponentiation

- Classical Amplitudes exponentiate in Impact Parameter Space (IPS) (Glauber, Levi+Sucher,… ,Amati+Ciafaloni+Veneziano, Kabat+Ortiz)

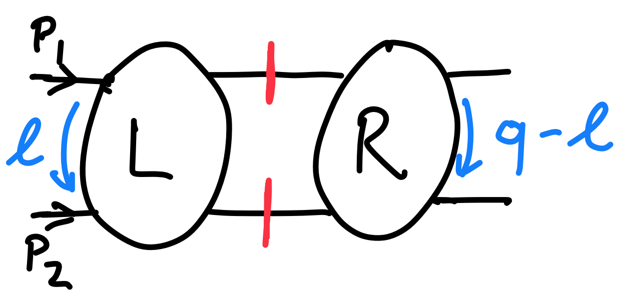
$$\widetilde{S} = 1 + i\widetilde{\mathcal{M}} = e^{i\delta_{\mathrm{HEFT}}}$$

Scattering angle: $\chi = -\frac{\partial}{\partial J}\,\mathrm{Re}\;\delta_{\mathrm{HEFT}}$

$$\widetilde{\mathcal{M}}(b) := \int \frac{d^D q}{(2\pi)^{D-2}}\delta(2\bar{p}_1\cdot q)\delta(2\bar{p}_2\cdot q)e^{iq\cdot b}\mathcal{M}(q) = \frac{1}{\bar{\mathcal{J}}}\int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}}e^{-i\vec{q}\cdot\vec{b}}\mathcal{M}(q)$$

- 2-massive-particle-reducible diagrams factorise in IPS
  $\longrightarrow$ Exponentiation

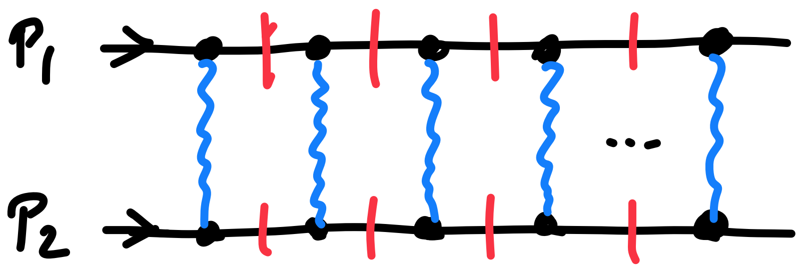
- Why? Convolution in momentum space = product in IPS



$$= \int \frac{d^D\ell}{(2\pi)^D}(2\pi)^2\delta(\bar{p}_1\cdot\ell)\delta(\bar{p}_2\cdot\ell)\mathcal{M}_{\mathrm{L}}(\ell)\mathcal{M}_{\mathrm{R}}(q-\ell)$$

$$\xrightarrow{\mathrm{IPS}} \widetilde{\mathcal{M}}_{\mathrm{L}}(b)\widetilde{\mathcal{M}}_{\mathrm{R}}(b)$$

Exponentiation cont'd



$$\xrightarrow{\text{IPS}} \frac{\left(i\widetilde{\mathcal{M}}_{\text{tree}}\right)^{L+1}}{(L+1)!} \xrightarrow{\sum_L} e^{i\widetilde{\mathcal{M}}_{\text{tree}}} - 1$$

Kabat-Ortiz

- HEFT expansion
  - Only compute 2-massive-particle-irreducible (2MPI) diagrams, ignore iterating diagrams
  - Only keep classical $m^{(L+2)}$ terms

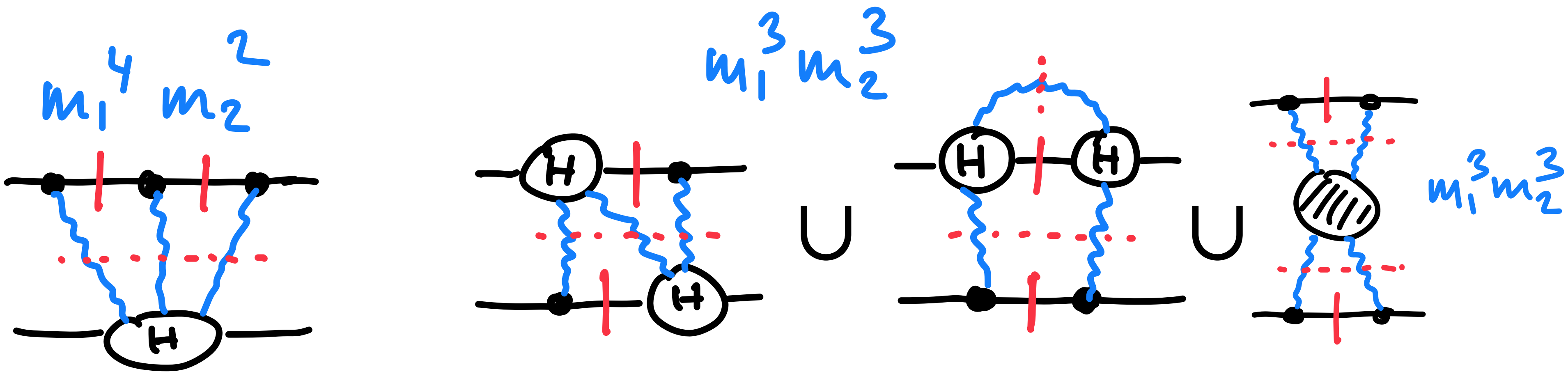
$$\delta_{\text{HEFT}} = \delta^{(0)} + \delta^{(1)} + \delta^{(2)} + \dots$$

$$\delta^{(0)} = \widetilde{\mathcal{M}}_{\text{tree}}$$

$$\delta^{(1)} = \widetilde{\mathcal{M}}^{\text{2MPI}}_{\text{1-loop}}$$

$$\delta^{(2)} = \widetilde{\mathcal{M}}^{\text{2MPI}}_{\text{2-loop}}$$

$$\delta^{(2)} = \widetilde{\mathcal{M}}^{\text{2MPI}}_{\text{2-loop}}$$



probe limit　　conservative　　radiation reaction　　washing machine

- Advantages of HEFT
  - Expansion done at the earliest stage
  - HEFT trees very compact, (squares of) linear propagators
  - Only keep diagrams that contribute to phase

Scattering angle up to 3PM

$$\chi = -\frac{\partial}{\partial J}\,\mathrm{Re}\,\delta_{\mathrm{HEFT}} \quad , \quad J = Pb$$

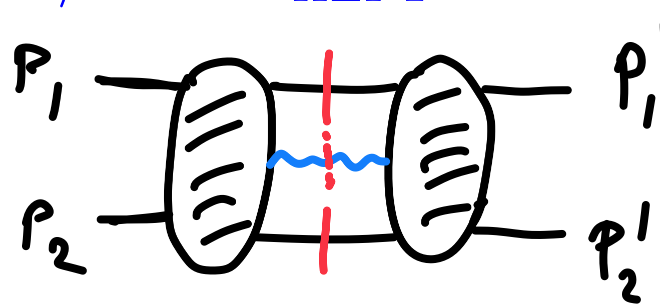
$$\chi = \frac{G}{J}\,\frac{2m_1m_2\left(2y^2-1\right)}{\sqrt{y^2-1}} + \frac{G^2}{J^2}\,\frac{3\pi}{4\sqrt{s}}m_1^2m_2^2(m_1+m_2)\left(5y^2-1\right)$$
$$+\frac{G^3}{J^3}\,\frac{m_1m_2\sqrt{y^2-1}}{\pi s}\Bigg\{m_1^2m_2^2(m_1^2+m_2^2)\frac{2\pi\left(64y^6-120y^4+60y^2-5\right)}{3\left(y^2-1\right)^2}$$
$$+m_1^3m_2^3(-8\pi)\left[\frac{\left(5y^2-8\right)\left(1-2y^2\right)^2}{6\left(y^2-1\right)^{3/2}} - \frac{y\left(2y^2-3\right)\left(1-2y^2\right)^2\operatorname{arccosh}(y)}{2\left(y^2-1\right)^2}\right.$$
$$\left.+\frac{y\left(55-6y^2\left(6y^4-19y^2+22\right)\right)}{6\left(y^2-1\right)^2} + \frac{\left(4y^4-12y^2-3\right)\operatorname{arccosh}(y)}{\sqrt{y^2-1}}\right]\Bigg\}$$

- Agrees with previous 3PM calculations:
  - Bern, Cheung, Roiban, Shen, Solon, Zeng (2019); Cheung, Solon (2020); Kaelin, Porto
  - Radiation reaction: Damour…; Di Vecchia, Heissenberg, Russo, Veneziano; Bjerrum-Bohr, Damgaard, Plante, Vanhove

Comments

- Easy to obtain integrands for higher orders; compact expression of HEFT amplitudes
  - Hard to get the relevant master integrals from differential equations & boundary conditions
  - Similarities with PM world-line approaches (Kaelin, Porto; Mogull, Plefka, Steinhoff)
- Up to 3PM $\mathrm{Re}\ \delta_{\mathrm{HEFT}}$ is equal to "radial action". Precise relation to approaches by Bern et al. and Damgaard, Vanhove,(+Bjerrum-Bohr, Plante)?

$$\hat{S} = e^{i\hat{N}} \quad , \quad \langle \hat{N} \rangle = \mathrm{Re}\ \delta_{\mathrm{HEFT}}$$

$\mathrm{Im}\ \delta_{\mathrm{HEFT}} \sim$ 

HEFT double copy & its kinematic algebra

- Compact, manifestly gauge-invariant, crossing symmetric numerators for all tree HEFT amplitudes:
  2 massive scalars + n-2 gluons/gravitons, leading term in large mass limit

- Mysterious kinematic algebra = quasi-shuffle Hopf algebra

- Kinematic algebra only know in few cases:
  self-dual Yang-Mills(O'Connell,Monteiro); 3D CS; (N)MHV sector of YM(Chen,Johansson,Teng,Wang)

- Can obtain numerators for pure YM/GR from HEFT:
  2 scalars ~ p = m v and (n-2) gluons/gravitons labelled {1,…,n-2}

  - decoupling limit

$$v \to \epsilon_{n-1} \quad , \quad (p_1 + p_2 + \ldots + p_{n-2})^2 \to 0$$

Simple examples

- 3-point

$$A(1;v) = m\,\epsilon_1 \cdot v \quad , \quad M(1;v) = A(1;v)^2 = (m\,\epsilon_1 \cdot v)^2$$

- 4-point

$$A(12;v) = \frac{\mathcal{N}([1,2],v)}{s_{12}} \quad , \quad M(12;v) = \frac{\mathcal{N}([1,2],v)^2}{s_{12}}$$

  - BCJ numerator

$$\mathcal{N}([1,2],v) = m\left(\frac{v \cdot F_1 \cdot F_2 \cdot v}{v \cdot p_1}\right) \quad , \quad F_i^{\mu\nu} = p_i^\mu \epsilon_i^\nu - p_i^\nu \epsilon_i^\mu$$

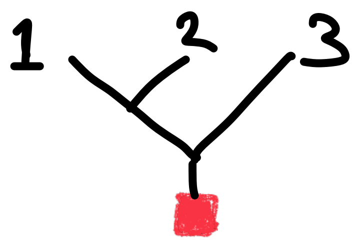
- Note numerator manifestly gauge invariant & YM/GR HEFT amplitudes are linear/quadratic in mass m

Double copy for YM and GR HEFT amplitudes

$$A(12\ldots n{-}2,v) = \sum_{\Gamma\in\rho}\frac{\mathcal{N}(\Gamma,v)}{d_\Gamma}\,,\quad M(12\ldots n{-}2,v) = \sum_{\Gamma\in\tilde\rho}\frac{\left[\mathcal{N}(\Gamma,v)\right]^2}{d_\Gamma}$$

- n=5 $\rho=\{[[1,2],3],[1,[2,3]]\}\quad,\quad\tilde\rho=\{[[1,2],3],[[1,3],2],[1,[2,3]]\}$
  - Ordered (YM) Unordered (GR) nested commutators

$$\mathcal{N}([[1,2],3],v)\,,\qquad d_{[[1,2],3]}=p_{12}^2p_{123}^2$$



Pre-numerator

Chen, Johansson, Teng, Wang

$$\mathcal{N}([[1,2],3]],v) = \mathcal{N}(123,v)-\mathcal{N}(213,v)-\mathcal{N}(312,v)+\mathcal{N}(321,v)$$

- Kinematic Jacobi relations manifest!

$$\mathcal{N}([1,[2,3]],v)=\mathcal{N}([[1,2],3],v)-\mathcal{N}([[1,3],2],v)$$

- Pre-numerators constructed by multiplying generators of an abstract algebra with a fusion product (Chen, Johansson, Teng, Wang)

$$\mathcal{N}(12\ldots n{-}2, v) := \langle T_{(1)} \star T_{(2)} \star \cdots \star T_{(n-2)} \rangle$$

- Step by step construction
  - 4 points $\langle T_{(1)} \star T_{(2)} \rangle = -\langle T_{(12)} \rangle$
  - 5 points $T_{(12)} \star T_{(3)} = T_{(12),(3)} + T_{(13),(2)} - T_{(123)}$
  - 6 points
    $$T_{(123)} \star T_{(4)} = T_{(123),(4)} + T_{(14),(23)} - T_{(1234)}$$
    $$T_{(12),(3)} \star T_{(4)} = T_{(12),(3),(4)} + T_{(12),(4),(3)} + T_{(14),(2),(3)} - T_{(12),(34)} - T_{(124),(3)}$$
    $$T_{(13),(2)} \star T_{(4)} = T_{(13),(2),(4)} + T_{(13),(4),(2)} + T_{(14),(3),(2)} - T_{(13),(24)} - T_{(134),(2)}$$
- Leg 1 is special, always in first position; quasi shuffle product
- Number of terms: 1, 3, 13,…

Examples (set m=1)

$$\mathcal{N}(1,v) = v\cdot\varepsilon_1\,,$$

$$V_\tau^{\mu\nu} := v^\mu \sum_{j\in\tau} p_j^\nu = v^\mu p_\tau^\nu$$

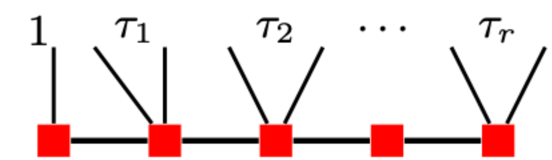
$$\mathcal{N}(12,v) = -\frac{v\cdot F_1\cdot F_2\cdot v}{2v\cdot p_1}\,,$$

$$\mathcal{N}(123,v) = \frac{v\cdot F_1\cdot F_2\cdot F_3\cdot v}{3v\cdot p_1} - \frac{v\cdot F_1\cdot F_2\cdot V_{12}\cdot F_3\cdot v}{3v\cdot p_1 v\cdot p_{12}} - \frac{v\cdot F_1\cdot F_3\cdot V_1\cdot F_2\cdot v}{3v\cdot p_1 v\cdot p_{13}}\,.$$

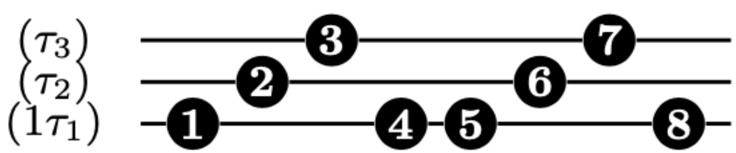
- General formula: $\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)}\rangle = \frac{v\cdot F_{1\tau_1}\cdot V_{\Theta(\tau_2)}\cdot F_{\tau_2}\cdots V_{\Theta(\tau_r)}\cdot F_{\tau_r}\cdot v}{(n-2)v\cdot p_1 v\cdot p_{1\tau_1}\cdots v\cdot p_{1\tau_1\tau_2\cdots\tau_{r-1}}}$

$$\mathcal{N}(1\ldots n-2,v) = \sum_{r=1}^{n-3}\sum_{\tau\in\mathbf{P}^{(r)}_{\{2,\cdots,n-2\}}} (-1)^{n+r}\langle T_{(1\tau_1),\ldots,(\tau_r)}\rangle$$

Sum over ordered partitions of {23…n-2} into r subsets



- “Musical diagrams”: e.g. (1458)(26)(37)



$$\Theta(26) = \{1\}\,,\quad \Theta(37) = \{1,2\}$$

Kinematic Hopf algebra

- Quasi-shuffle Hopf algebra that generates all ordered partitions of a given set (Hoffman; Fauvet, Foissy & Manchon)

Shuffle: $(A)(B) = (A,B) + (B,A)$

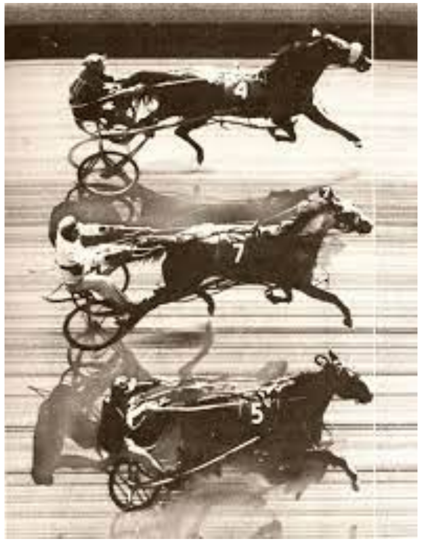
Quasi-shuffle $(A)(B) = (A,B) + (B,A) + (A \sqcup B)$

Extra terms

- Number of such sets = Fubini numbers!

n	3	4	5	6	7	8	9	10
F_{n-3}	1	1	3	13	75	541	4683	47293

  - Also counts possible outcomes of horse races
  - Counts number of permutahedron faces, Cayley trees, Cayley permutations…



Comments

- "Decoupling limit": all numerators for pure YM/GR from HEFT

$$\mathcal{N}([\ldots[1,2],3],\ldots n{-}2],v) = (n{-}2)\,\mathcal{N}(1\ldots n{-}2,v)$$

$$\mathcal{N}^{\mathrm{YM}}([1\ldots n{-}1]) = \mathcal{N}([1\ldots n{-}2],v)\Big|^{v\to\epsilon_{n-1}}_{p^2_{1\ldots n-2}\to 0}$$

  - On-shell limit: propagator matrix becomes degenerate $\longrightarrow$ generalised BCJ gauge invariance comes back
  - Numerators not (fully) gauge invariant anymore
- What can the Hopf algebra do for us?
  - e.g the (multi)co-product is related to multiple factorisation of numerators on massive poles
- Can we find an explicit realisation of the generators?

EFT's of general relativity

(Accettulli-Huber, AB, De Angelis, Travaglini)

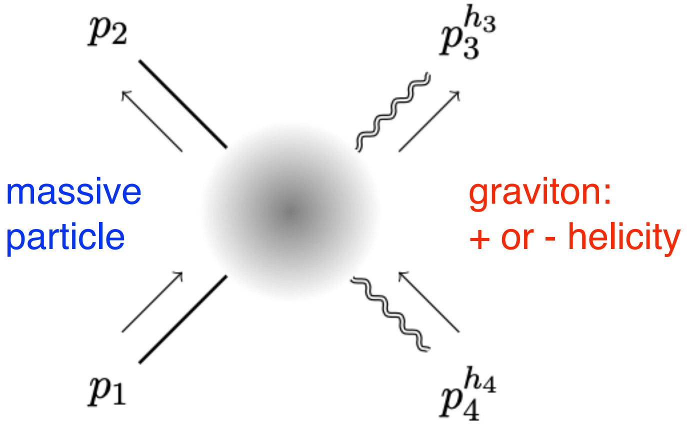
$$S = \int d^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}R - \frac{2}{\kappa^2}\left(\frac{\alpha_1}{48}I_1 + \frac{\alpha_2}{24}G_3\right) + \frac{2}{\kappa^2}(\beta_1\mathcal{C}^2 + \beta_2\widetilde{\mathcal{C}}^2)\right]$$

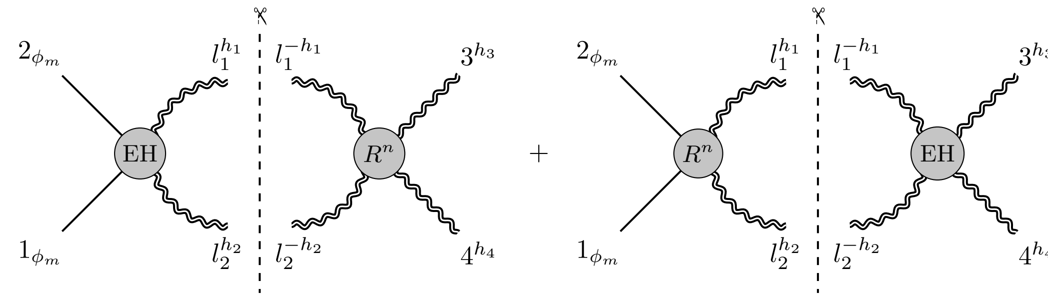
$$I_1 := R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}\ , \qquad G_3 := I_1 - 2R^{\mu\nu\alpha}{}_{\beta}R^{\beta\gamma}{}_{\nu\sigma}R^{\sigma}{}_{\mu\gamma\alpha}$$

$$\mathcal{C} := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\ , \qquad \widetilde{\mathcal{C}} := \frac{1}{2}R_{\mu\nu\alpha\beta}\,\epsilon^{\alpha\beta}{}_{\gamma\delta}\,R^{\gamma\delta\mu\nu}\ .$$

- Consider higher-derivative modifications of GR
- Can we get constraints on the coefficients $\alpha_i\ ,\ \beta_i$?
  - Causality constraints (cf. Camanho, Edelstein, Maldacena, Zhiboedov (CEMZ); De Rham, Tolley…)
  - Maybe from gravitational waves? (Buonanno, Gorbenko, Senatore…)
- Observables we looked at
  - Deflection angle + Shapiro time delay (causality)
  - Corrections to potential and quadrupole $\longrightarrow$ Waveforms

Shapiro time delay





$$S_{\rm eik} \sim e^{i\delta}\ ,\ \theta\ =\ \frac{1}{\omega}\frac{\partial\delta}{\partial b}\ ,\quad \text{Shapiro time delay:}\ \ t = \frac{\partial\delta}{\partial\omega}$$

- Note: 4 possible processes
- Eikonal phase is a matrix
- Results (using eigenvalues)

$$\text{helicity preserving:}\ \ \pm \to \pm$$
$$\text{helicity flipping:}\ \ \pm \to \mp$$

$$\delta \sim \begin{pmatrix} \tilde{A}(\phi,\phi,+,-) & \tilde{A}(\phi,\phi,+,+) \\ \tilde{A}(\phi,\phi,-,-) & \tilde{A}(\phi,\phi,-,+) \end{pmatrix}$$

- (Riem)^3 $\Delta t_{\mathrm{EH}-R^3} = 4Gm\left[\log\frac{b_0}{b} \pm \frac{3\alpha_1}{b^4} + \frac{\alpha_1\pi}{256}\left(-9 \pm 1365\right)\frac{Gm}{b^5}\right].$
  - Consistent with CEMZ, causality violation if b too small…
- (Riem)^4 $\Delta t^{(1,2)}_{R^4} = \beta_{1,2}\,(Gm)^2\,\frac{945\pi}{2}\,\frac{\omega^2}{b^5}\ \Rightarrow\ \beta_{1,2} > 0$ “positivity constraint”

Corrections to potential/quadrupole

- Focus on $(\text{Riem})^3$, dominant over $(\text{Riem})^4$, also α_2 not constrained by causality
- Potential $$V(\vec{r},\vec{p}) = -\frac{G m_1 m_2}{r} + \frac{3}{8}\frac{\alpha_1 G^2}{r^6}\frac{(m_1+m_2)^3}{m_1 m_2}\vec{p}^{\,2} - \frac{3}{4}\frac{\alpha_2 G^2}{r^6} m_1 m_2 (m_1+m_2)\left(1 - \frac{m_1^2+m_2^2}{2\, m_1^2 m_2^2}\,\vec{p}^{\,2}\right) + \cdots ,$$
- Quadrupole $$Q^{ij} = \left(1 + 3G\frac{\alpha_2\, m_1 m_2 - (\alpha_1 + 2\alpha_2)\, M \frac{\vec{p}^{\,2}}{\mu}}{\mu\, r^5}\right) Q_N^{ij}\ , \quad Q_N^{ij} = \mu\left(x^i x^j - \frac{1}{3} r^2 \delta^{ij}\right)$$
- The modified waveforms can be run against LIGO data (Sennett, Brito, Buonanno, Gorbenko, Senatore)
  - $(\text{Riem})^4$ terms: $L_{\text{eff}} < 100$km , $(\text{Riem})^3$ terms: $L_{\text{eff}} < 70$km
- Analysis of quasi normal modes (Silva, Ghosh, Buonanno)
  - $(\text{Riem})^4$ terms: $L_{\text{eff}} < 51$km , $(\text{Riem})^3$ terms: $L_{\text{eff}} < 38$km
- Maybe totally crazy, but still a fun possibility to consider!!

Conclusions

- HEFT is an efficient tool to compute classical observables in GR avoiding iterating/quantum contributions from the get-go
  - To-do: spin, radiation, higher PM, conceptual questions about extraction of classical…
- HEFT as a novel, gauge invariant version of the BCJ double copy, kinematic algebra = Hopf algebra
  - Future: understand full implications of Hopf algebra (co-product, antipod), explicit generators…
- Amplitudes useful to study EFTs of GR (higher derivative terms):
  - Modifications of potential & waveforms, causality constraints..
  - Can GW experiments constrain coefficients higher derivative terms?