The Dual Conformal Box Integral in Minkowski Space

Based on 2006.11292 with M Staudacher

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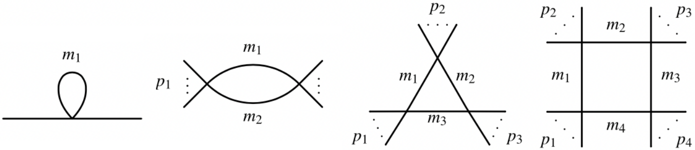
One-loop in Gauge Theory

One-loop Feynman integrals can be color-stripped/tensor reduced to a basis of scalar integrals.



Massless box (finite in four dimensions!) more constrained than expected - 'dual conformal' symmetry.

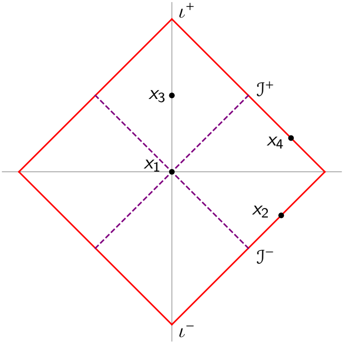
Fishnet theory allows us to study a whole spectrum of dual conformal diagrams, of which the box integral is simplest.

Minkowski Space

QFT calculations typically done after 'Wick rotation' to Euclidean space.

Study conformal symmetry directly in Minkowski space. Dual conformal symmetry of box integral is subtly broken.

Exploit some features of Minkowski space for calculation. Box integral is perfect testing ground.

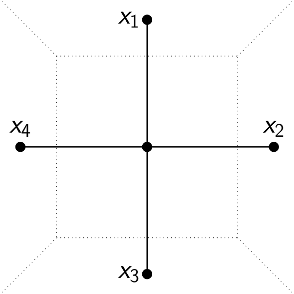


Box integral - x space

$$p_i = x_i - x_{i+1}, \qquad x_{ij} = x_i - x_j$$

$$I(\boldsymbol{x}) = \int d^4x_5 \frac{x_{13}^2 x_{24}^2}{(x_{15}^2 + i\epsilon)(x_{25}^2 + i\epsilon)(x_{35}^2 + i\epsilon)(x_{45}^2 + i\epsilon)}$$

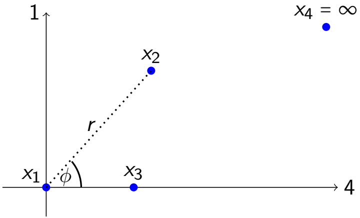
$=$



Euclidean Conformal Plane

Given a configuration $\boldsymbol{w} = (x_1, x_2, x_3, x_4)$ of four points in $\mathbb{R}^4$.

Can always find $A \in \mathrm{Conf}(\mathbb{R}^4)$ such that $A\boldsymbol{w} = \bar{\boldsymbol{w}}(r, \phi)$.



Conformal invariants of $\boldsymbol{w}$ are $z = re^{i\phi}$, $\bar{z} = z^*$.

Defined by $z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $\qquad (1 - z)(1 - \bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

All $\boldsymbol{w} \in V^E_{z,\bar{z}}$ are conformally equivalent!

Euclidean Conformal Box

Look at Euclidean box

$$I^E(\boldsymbol{w}) = \int d^4x_5 \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}.$$

Invariant under $\mathrm{Conf}(\mathbb{R}^4)$ i.e. translations, rotations, dilatations, SCTs.

$\longrightarrow$ Euclidean box is a function of two conformal invariants

$$I^E(z,\bar{z}) = \frac{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)}{z-\bar{z}}$$

How to calculate? Feynman/Mellin Barnes parametrisation!

Minkowski Subtleties

Minkowski box? Invariance under translations, rotations, dilatations ✓

However under SCTs $\boldsymbol{w} \to \mathrm{SCT}_b \boldsymbol{w}$ for $b \in \mathbb{R}^{1,3}$

$$I(C_b \boldsymbol{w}) = \int d^4 x_5 \frac{x_{13}^2 x_{24}^2}{(x_{15}^2 + i\epsilon_{15})(x_{25}^2 + i\epsilon_{25})(x_{35}^2 + i\epsilon_{35})(x_{45}^2 + i\epsilon_{45})},$$

Non-invariance leads to 'branch jumping' $I(z, \bar{z}, k) = I^E(z, \bar{z}) + \Delta_k(z, \bar{z})$.

How many branches can one reach with SCTs? How many branches are there for a given $z, \bar{z}$?

Branches for Fixed $z, \bar{z}$

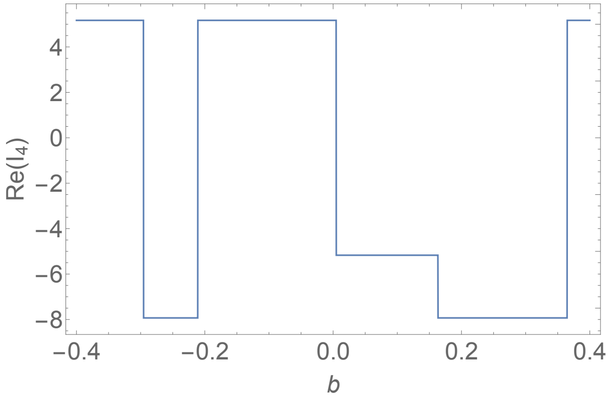


Figure: $I(\mathrm{SCT}_{(b,0,0,0)}\boldsymbol{w}), z = 0.72, \bar{z} = 20.92$

Minkowski Conformal Geometry

Let $V_{z,\bar{z}}$ be set of 4-tuples $\boldsymbol{w} = (x_1, x_2, x_3, x_4)$ of points in Minkowski space with conformal invariants $z, \bar{z}$.

$\mathrm{Conf}(\mathbb{R}^{1,3})$ does **not** act transitively on each $V_{z,\bar{z}}$. For each $z, \bar{z} \in \mathbb{R}$ there are **two** conformal equivalence classes $V_{i,z,\bar{z}}$ and $\bar{V}_{i,z,\bar{z}}$.

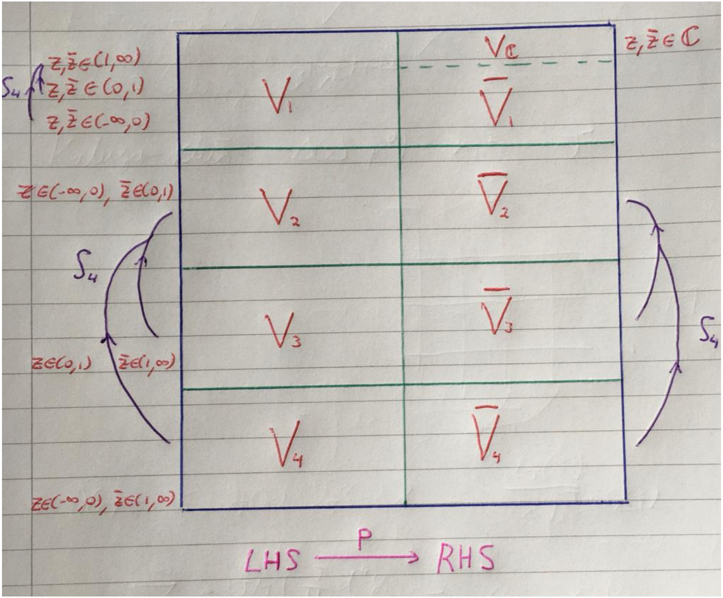
Everything depends on the signs of the *kinematics* of $\boldsymbol{w}$

$$k(\boldsymbol{w}) \equiv (x_{12}^2, x_{34}^2, x_{23}^2, x_{14}^2, x_{13}^2, x_{24}^2).$$

$2^6 = 64 = 8 \times 8$ possibilities for $\mathrm{sgn}(k(\boldsymbol{w}))$.

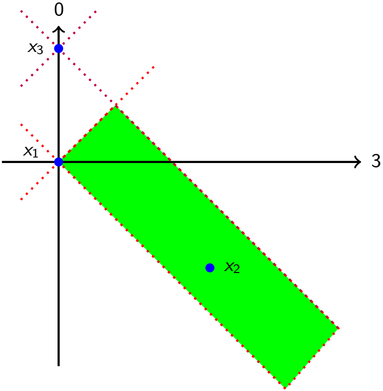
Box integral depends on $z, \bar{z}$ and $k \equiv \mathrm{sgn}(k(\boldsymbol{w}))$: *pseudo*-conformal invariance.

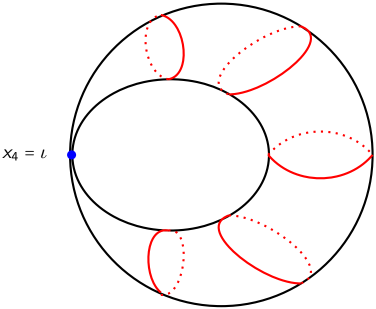
Decomposition of V



Minkowski Conformal Planes

Each $V_i, \bar{V}_i$ and $V_{\mathbb{C}}$ has their own *Minkowskian conformal plane*.





Use these to show that $\mathrm{Conf}(\mathbb{R}^{1,3})$ acts transitively on configurations in $V_i, \bar{V}_i, V_{\mathbb{C}}$ with a fixed $z, \bar{z}$.

Box Integral Branches

Box integral has been calculated in all kinematic regions before. We write it in a conformally invariant way in $V_1, \bar{V}_1, V_2, \bar{V}_2, V_{\mathbb{C}}$.

$$f_1(z,\bar{z}) = 1, \qquad f_2(z,\bar{z}) = \log(z/\bar{z}), \qquad f_3(z,\bar{z}) = \log\left(\frac{1-z}{1-\bar{z}}\right),$$

$$f_4(z,\bar{z}) = 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right),$$

e.g. for V_2

k	$I(z,\bar{z},k)$
$(-+++++), (---+-+), (--+-+-)$	$\frac{f_4}{z-\bar{z}}$
$(+++--+), (++-++-), (-+----)$	$\frac{f_4-2\pi i f_3}{z-\bar{z}}$
$(+---++)$	$\frac{f_4-2\pi i f_2}{z-\bar{z}}$
$(+-++--)$	$\frac{f_4+2\pi i(f_2-f_3-2\pi i f_1)}{z-\bar{z}}$

Find up to **four** branches of integral in each $V_i, \bar{V}_i, V_{\mathbb{C}}$.

Results

- Classified conformally equivalent configurations of four points in Minkowski space.
- Wrote Minkowskian box integral in a conformally invariant way, and found up to four branches of box integral in each conformal equivalence class.
- Studied novel set of higher infinity configurations in Minkowski space, and calculated box integral for these directly in Minkowski space.