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Non-Planar Anomalous Dimensions in Super Yang-Mills

based on arXiv:2005.14254 with T. McLoughlin and R. Pereira

3rd SAGEX Workshop

Anne Spiering

Motivation: Scaling Dimensions & Correlation Functions

- Two-point correlation function in a CFT

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- A lot is known about anomalous dimensions in the planar limit of $\mathcal{N}=4$ SYM
  Here: non-planar corrections to one-loop anomalous dimensions

SU(2) Sector and Dilatation Operator

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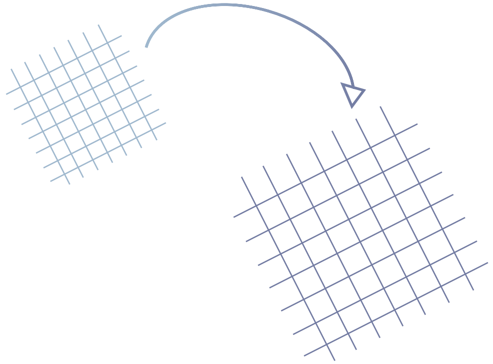

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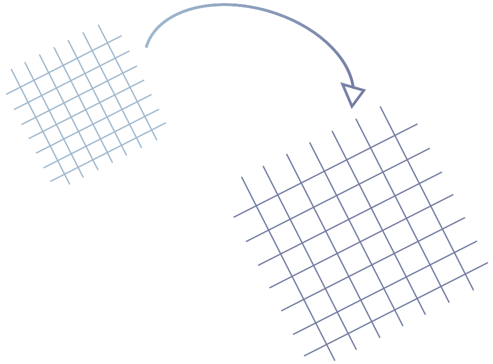

- In perturbation theory: $\mathcal{D} = \mathcal{D}_0 + \frac{\lambda}{16\pi^2}\mathcal{D}_2 + \mathcal{O}(\lambda^2)$

$\mathcal{D}_0$: classical scaling dimension

$\mathcal{D}_2$: one-loop anomalous dimension

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$$\mathcal{D}_2 = -\frac{2}{N} : \mathrm{tr}([X, Z][\check{X}, \check{Z}]) :$$

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⚠ Operator mixing: need to diagonalize mixing matrix, su(2) sector closed

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- For short operators: direct diagonalization to solve mixing & find dimensions
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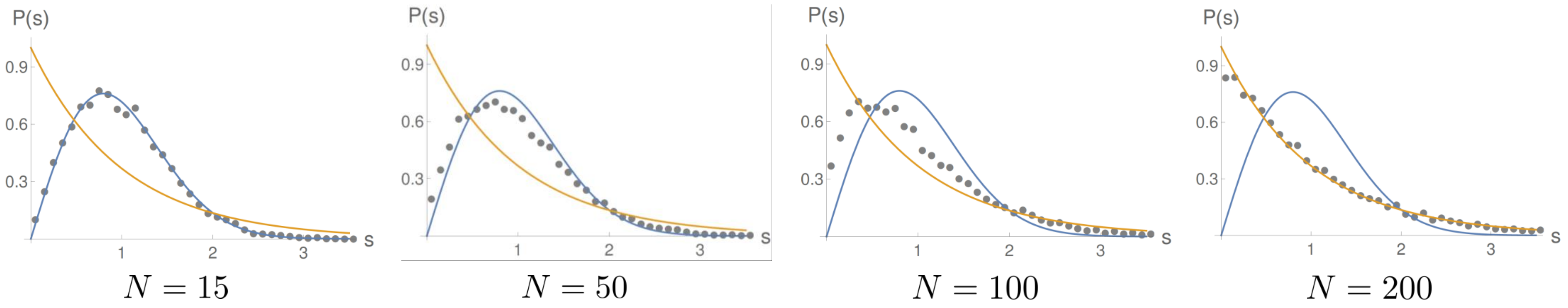
Level-spacing analysis for one-loop scaling dimensions



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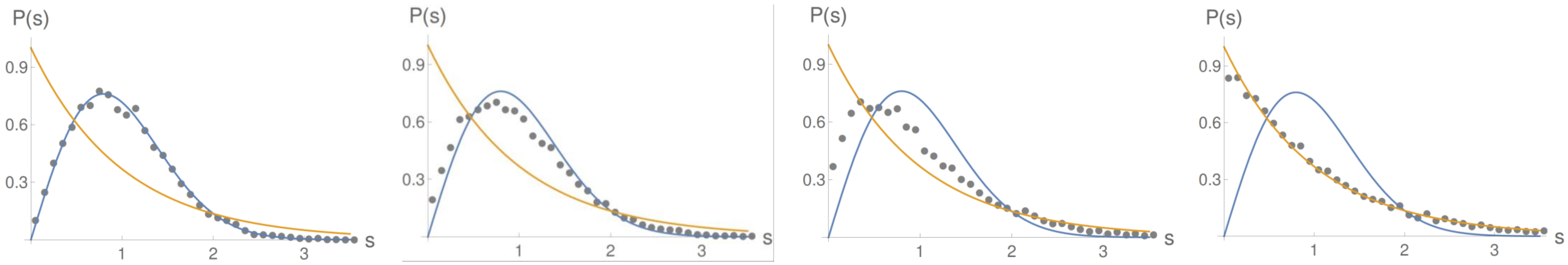

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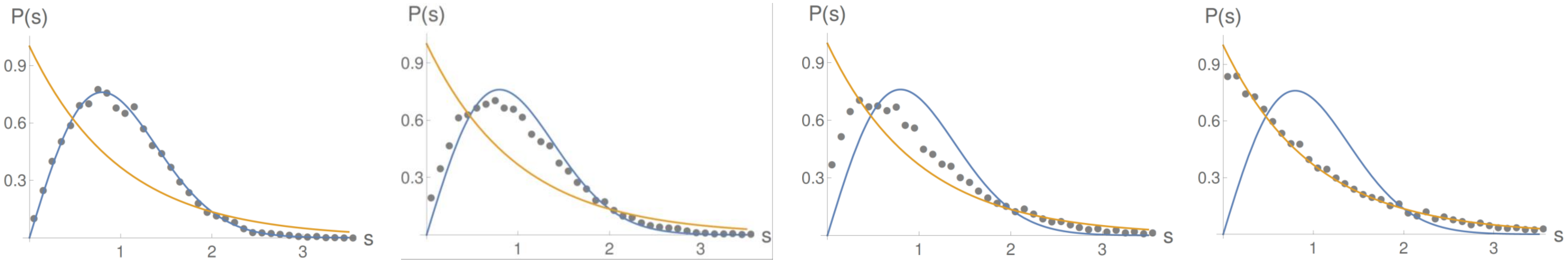

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Wigner-Dyson statistics for finite N
characteristic for quantum chaos (GOE)

Poisson statistics for $N \to \infty$
characteristic for integrability

Planar Integrability

- The planar limit is integrable! [Minahan, Zarembo '02]

$$\mathcal{D}_2 = -\frac{2}{N} : \mathrm{tr}([X,Z][\check{X},\check{Z}]) : \xrightarrow{N \to \infty} \mathcal{H}_{XXX} = 2\sum_{i=1}^{L} (\mathbb{1}_{i,i+1} - \mathbb{P}_{i,i+1})$$

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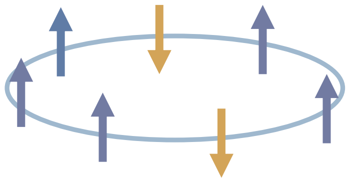

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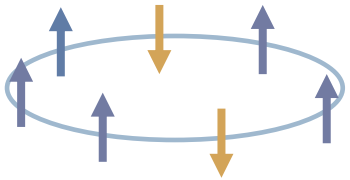

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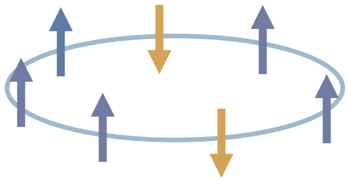

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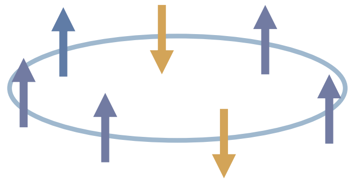

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- At finite N: no traces of integrability,

  level statistics characteristic for a quantum-chaotic model

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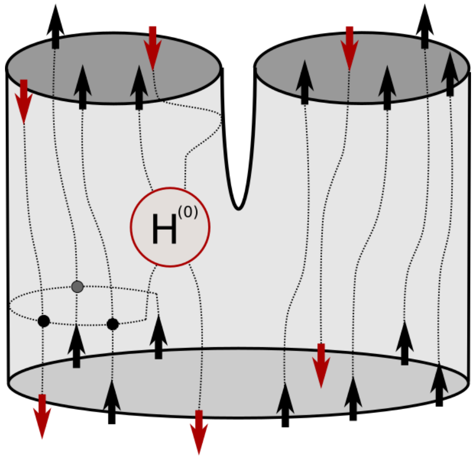
- Non-degenerate quantum-mechanical perturbation theory to obtain first non-planar correction

$$E^{(2)}(\{p\}) = \sum_{\{I\}} \frac{\langle\{p\}|\mathcal{H}^{-}|I\rangle\langle I|\mathcal{H}^{+}|\{p\}\rangle}{E^{(0)}(\{p\}) - E^{(0)}(I)}$$



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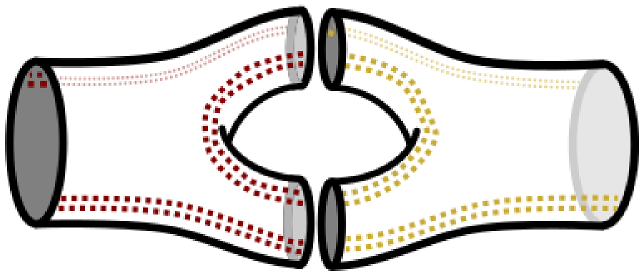

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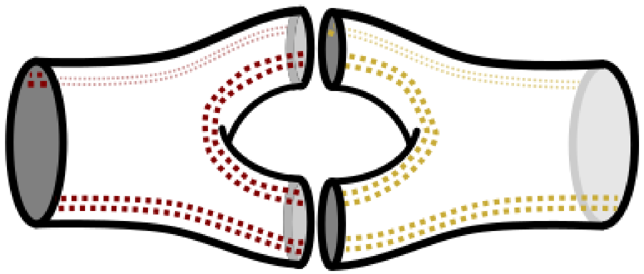

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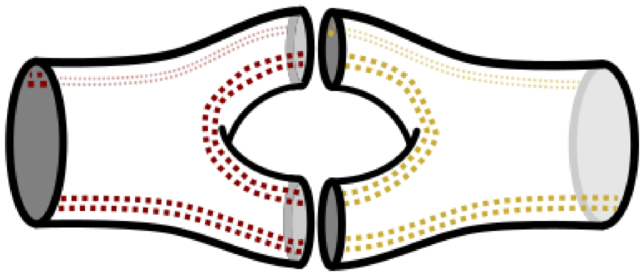

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- Perform intermediate sum?
- Higher loops? Other sectors?