# From quantum to classical scattering in post-Minkowskian gravity

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- Scattering amplitudes naturally provides this observable: can we improve the method to high PM order?
- Formula connecting $\mathcal{M}$ and $\theta_{PM}$ to all orders (no potentials)

**Main result**

$$\theta_{PM} = \sum_{k=1}^{\infty} \frac{2b}{k!} \int_0^{\infty} du \, (\partial_{b^2})^k \left( \frac{\tilde{\mathcal{M}}^{cl.}(r, p_\infty) r^2}{p_\infty^2} \right)^k \frac{1}{r^2}$$

$$r = \sqrt{u^2 + b^2}$$

- The two-body problem in General Relativity is given by

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \frac{8\pi G_N}{c^4} \mathcal{T}_{\mu\nu} \qquad , \qquad \dot{u}_a^\mu = -\Gamma^\mu_{\alpha\beta}(g_{\mu\nu}) u_a^\alpha u_a^\beta$$

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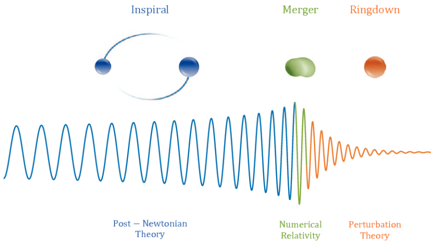

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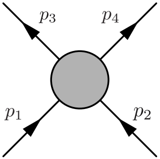
## State of the art

*Bern et al.* has computed $\theta_{PM} \sim G_N^3$ with a $\mathcal{V}_{PM}$ based approach, but the method is hard to implement at higher PM orders

1. Compute the scattering amplitude of a $2 \to 2$ process between two scalar massive particles exchanging gravitons (C.M. frame)

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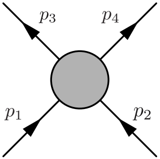


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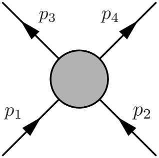


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$$\tilde{\mathcal{V}}_{PM}(\vec{p}, \vec{p}\,') = \mathcal{M}(\vec{p}, \vec{p}\,') - \int_{\vec{n}} \frac{\mathcal{M}(\vec{p}, \vec{n})\, \tilde{\mathcal{V}}_{PM}(\vec{n}, \vec{p}\,')}{E_p - E_n + i\epsilon}$$

3. The *fully relativistic scattering angle* $\theta_{PM}$ is given by

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**Issues with $p^2(r)$ and $r_{min}$**

- The computation of $p^2(r)$ seems to follow no specific rule

$$\frac{p^2}{2\mu} + \frac{G_N \mu M}{r} = E \quad vs. \quad \sum_{i=1}^{2}\sqrt{p^2 + m_i^2} + \mathcal{V}_{PM}(r, p) = E$$

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- $r_{min}$ solves a polynomial equation (in $\theta_{PM}$ divergences appear)

# A new map from $\mathcal{M}_{PM}$ to $\theta_{PM}$

- We can apply the implicit function theorem,

$$p^2(r) = p_\infty^2 + \mathcal{V}_{PM}(r, p_\infty) - 2E\xi\mathcal{V}_{PM}(r, p_\infty)\partial_{p^2}\mathcal{V}_{PM}(r, p_\infty) + \ldots$$

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- The L-S equations in position space can be rewritten as a differential equation for a fully relativistic potential and $\mathcal{M}$,

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**QCD meets gravity 2016, Damour**

$$p^2(r) = p_\infty^2 + \tilde{\mathcal{M}}^{cl.}_{tree}(r, p_\infty) + \tilde{\mathcal{M}}^{cl.}_{1-loop}(r, p_\infty) + \ldots$$

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- $p^2 = p_\infty^2 + \frac{G_N f_1(E)}{r} + ....$
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$$\begin{cases} \hat{\mathcal{H}} = \hat{p}^2 - \mathcal{V}(r, E) \\ \mathcal{V}(r, E) = \frac{G_N f_1(E)}{r} + ... \end{cases}$$

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## Main result (see also Khalin and Porto)

$$\mathcal{V} = \tilde{\mathcal{M}}^{cl.} \quad \Rightarrow \quad p^2(r) = p_\infty^2 + \tilde{\mathcal{M}}^{cl.}(r)\,,\ \forall G_N^n$$

- The post-Minkowskian scattering angle $\theta_{PM}$ becomes

$$\frac{\theta_{PM}}{2} = -\frac{\partial}{\partial L} \int_{r_{min}}^{+\infty} dr \sqrt{p_\infty^2 + \tilde{\mathcal{M}}^{cl.}(r, p_\infty) - \frac{L^2}{r^2}} - \frac{\pi}{2}$$

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$$r = \sqrt{u^2 + b^2}$$

- This series corrects the known Bohm's formula

$$\theta_{PM} = \theta^{(1)} + \ldots \quad , \quad \theta^{(1)} = \frac{2b}{p_\infty^2} \int_0^\infty du \, \partial_{b^2} \tilde{\mathcal{M}}^{cl.}(r, p_\infty)$$

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$$= \frac{G_N}{b} c_1 + \left(\frac{G_N}{b}\right)^2 \frac{\pi c_2}{4} + \left(\frac{G_N}{b}\right)^3 \left(c_3 + \frac{c_1 c_2}{2} - \frac{c_1^3}{4}\right) + \ldots$$

# Conclusions

## $\mathcal{M}_{PM} \to \mathcal{V}_{PM} \to \theta_{PM}$

- Presence of ***gauge dependent quantities*** as $\mathcal{V}_{PM}$ and $r_{min}$
- The quantity $p^2(r)$ required an additional calculation
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### Future directions

- Massless limit (work in progress with O'Connell and Gonzo)
- Arbitrary dimensions $D$ (Damgaard, Di Vecchia, Heissenberg)